Lógica en la Informática / Logic in Computer Science

Friday November 10th, 2019

Time: 1h30min. No books, lecture notes or formula sheets allowed.

1) (4 points)
Consider the following statement. For all propositional formulas F, G, H,
$(F \to G) \land (H \to G)$ is satisfiable iff $\neg G \models \neg F \land \neg H$.
Prove the following using only the definitions of propositional logic.
1a) Is the $\Longrightarrow$ implication of this iff statement true?
1b) Is the $\Longleftarrow$ implication of this iff statement true?
1c) Is it true that if $\neg G \models \neg F \land \neg H$, then $(F \to G) \land (H \to G)$ is a tautology?
(hint for 1c: use what you did in 1b).

Answer:
1a is not true.
Counter example: Let $F = G = p$ and $H = q$. Then $(F \to G) \land (H \to G)$ is satisfiable (any interpretation where p is true is a model), but $\neg G \not\models \neg F \land \neg H$: if $I(p) = 0$ and $I(q) = 1$ then $I \models \neg G$ but $I \not\models \neg F \land \neg H$.

1b and 1c are true:
$\neg G \models \neg F \land \neg H$ $\Longrightarrow$ (by def. of logical consequence)
for all I, either $I \not\models \neg G$ or $I \models \neg F \land \neg H$ $\Longrightarrow$ (by def of $\models$)
for all I, either $eval_I(\neg G) = 0$ or $eval_I(\neg F \land \neg H) = 1$ $\Longrightarrow$ (by def of $eval \neg, \land$)
for all I, either $1 - eval_I(G) = 0$ or $min(eval_I(\neg F), eval_I(\neg H)) = 1$ $\Longrightarrow$ (by def of $eval$ and min)
for all I, either $eval_I(G) = 1$ or $eval_I(\neg F) = eval_I(\neg H) = 1$ $\Longrightarrow$ (by def of max)
for all I, $max(eval_I(\neg F), eval_I(G)) = 1$ and $max(eval_I(\neg H), eval_I(G))) = 1$ $\Longrightarrow$ (by def of eval $\lor$)
for all I, $eval_I(\neg F \lor G) = 1$ and $eval_I(\neg H \lor G) = 1$ $\Longrightarrow$ (by def of min)
for all I, $min(eval_I(\neg F \lor G), eval_I(\neg H \lor G)) = 1$ $\Longrightarrow$ (by def of eval $\land$)
for all I, $eval_I(\neg F \lor G) \land (\neg H \lor G)) = 1$ $\Longrightarrow$ (by def of $\to$)
for all I, $eval_I(F \to G) \land (H \to G)) = 1$ $\Longrightarrow$ (by def of $\models$)
for all I, $I \models (F \to G) \land (H \to G)$ $\Longrightarrow$ (by def of satisfiable and tautology)
$(F \to G) \land (H \to G)$ is satisfiable, and, in fact, it is a tautology.

2) (4 points) Let S_1, S_2 be the two sets of clauses given below. How many models does each one of them have? Give a very short and simple answer, based on what these sets encode.

$$S_1 = \{ \neg x_0 \lor \neg x_1,\quad \neg x_0 \lor \neg x_2,\quad \neg x_0 \lor \neg a_1,\quad \neg x_1 \lor \neg x_2,\quad \neg x_1 \lor \neg a_1,\quad \neg x_2 \lor \neg a_1,\quad a_1 \lor \neg x_3,\quad a_1 \lor \neg x_4,\quad \neg x_3 \lor \neg x_4 \}$$

$$\begin{aligned} S_2 = \{ \quad & \neg x_0 \lor \neg a_4, & \neg x_0 \lor \neg a_2, & \quad \neg x_0 \lor \neg a_1 \\ & \neg x_1 \lor \neg a_4, & \neg x_1 \lor \neg a_2, & \quad \neg x_1 \lor a_1 \\ & \neg x_2 \lor \neg a_4, & \neg x_2 \lor a_2, & \quad \neg x_2 \lor \neg a_1 \\ & \neg x_3 \lor \neg a_4, & \neg x_3 \lor a_2, & \quad \neg x_3 \lor a_1 \\ & \neg x_4 \lor a_4, & \neg x_4 \lor \neg a_2, & \quad \neg x_4 \lor \neg a_1 \quad \} \end{aligned}$$

Answer: S_1 and S_2 are the *Heule-3* and *logarithmic* encodings of $x_0 + \ldots + x_4 \leq 1$, respectively.

S_1 has 7 models: if some x_i is true then all other x_j become false and also a_1 has only one possible value (5 models); if *all* x_i are false then a_1 can take either value (2 more models).

S_2 has 13 models: if some x_i is true then all other x_j become false and also the a_4, a_2, a_1 have only one possible value (5 models); if *all* x_i are false then the a_4, a_2, a_1 can take all $2^3 = 8$ possible values.

3) (2 points) Given a graph, we want to decide whether it is 2-colorable, that is, if we can assign one of 2 colors to each node such that, for every edge (u, v), nodes u and v get different colors. Give a short and simple answer *based on propositional logic* of the following: what is the computational complexity of this problem? Is it polynomial, NP-complete?

Answer: We can solve it with 2-SAT, so it is polynomial, in fact, linear. For each node i we introduce a variable x_i meaning "node i has color 1" (if x_i is false it means node i has the other color). Moreover, there will be two binary clauses $x_u \vee x_v$ and $\neg x_u \vee \neg x_v$ for each edge (u, v).