Convergence theorems for some layout measures on random lattice and random geometric graphs *

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Abstract

This work deals with convergence theorems and bounds on the cost of several layout measures for lattice graphs, random lattice graphs and sparse random geometric graphs. Specifically, we consider the following problems: *Minimum Linear Arrangement*, *Cutwidth*, *Sum Cut*, *Vertex Separation*, *Edge Bisection* and *Vertex Bisection*. For full square lattices, we give optimal layouts for the problems still open. For arbitrary lattice graphs, we present best possible bounds disregarding a constant factor. We apply percolation theory to the study of lattice graphs in a probabilistic setting. In particular, we deal with the subcritical regime that this class of graphs exhibits and characterize the behavior of several layout measures in this space of probability. We extend the results on random lattice graphs to random geometric graphs, which are graphs whose nodes are spread at random in the unit square and whose edges connect pairs of points which are within a given distance. We also characterize the behavior of several layout measures on random geometric graphs in their subcritical regime. Our main results are convergence theorems that can be viewed as an analog of the Beardwood, Halton and Hammersley theorem for the Euclidean TSP on random points in the unit square.

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1 Introduction

Several well-known optimization problems on graphs can be formulated as *graph layout problems*. A *(linear) layout* of a a graph with n vertices is a bijection between the vertex set and the set of naturals from 1 to n. Graph layout problems, also denoted in the literature as *linear ordering problems* or *linear arrangement problems*, seek for a layout that minimizes a cost associated with each problem. The particular layout problems that we consider include *Minimum Linear Arrangement*, *Cutwidth*, *Sum Cut* and *Vertex Separation*. We also consider the *Edge Bisection* and the *Vertex Bisection* problems, which are partitioning problems, but can also be formulated as layout problems. The Sum Cut problem is equivalent to the *Profile* problem and the Vertex Separation problem to the *Pathwidth* problem. For general graphs, all these problems are **NP**-hard. Moreover, the decisional version of Cutwidth and Vertex Separation problems are **NP**-complete even when restricted to lattice graphs and unit disk graphs [10]. All of them have a long history behind, owing to their practical relevance in different applications [36, 15, 1, 33, 24, 23, 2, 26, 7, 17, 19, 33, 18, 5, 22, 27, 34, 35].

Our layout problems are formally defined as follows. A *layout* φ of a graph $G = (V, E)$ is a one-to-one function $\varphi : V \to \{1, \ldots, n\}$ with $n = |V|$. Given a graph G, a layout φ of G and an integer i, let us define the sets

$$L(i, \varphi, G) = \{u \in V(G) \ : \ \varphi(u) \le i\} \quad \text{and} \quad R(i, \varphi, G) = \{u \in V(G) \ : \ \varphi(u) > i\},$$

the measures

$$\begin{aligned} \theta(i, \varphi, G) &= |\{uv \in E(G) \ : \ u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|, \\ \delta(i, \varphi, G) &= |\{u \in L(i, \varphi, G) \ : \ \exists v \in R(i, \varphi, G) : uv \in E(G)\}|, \\ \lambda(uv, \varphi, G) &= |\varphi(u) - \varphi(v)| \qquad \text{where } uv \in E(G), \end{aligned}$$

and the following graph layout problems:

- Minimum Linear Arrangement (MinLA): Given a graph $G = (V, E)$, find $\textsc{minla}(G) = \min_\varphi \sum_{uv \in E} \lambda(uv, \varphi, G) = \min_\varphi \sum_{i=1}^{n} \theta(i, \varphi, G)$.
- Cutwidth (CutWidth): Given a graph $G = (V, E)$, find $\textsc{mincw}(G) = \min_\varphi \max_{i=1}^{n} \theta(i, \varphi, G)$.
- Vertex Separation (VertSep): Given a graph $G = (V, E)$, find $\textsc{minvs}(G) = \min_\varphi \max_{i=1}^{n} \delta(i, \varphi, G)$.
- Sum Cut (SumCut): Given a graph $G = (V, E)$, find $\textsc{minsc}(G) = \min_\varphi \sum_{i=1}^{n} \delta(i, \varphi, G)$.
- Edge Bisection (EdgeBis): Given a graph $G = (V, E)$, find $\textsc{mineb}(G) = \min_\varphi \theta(\lfloor n/2 \rfloor, \varphi, G)$.
- Vertex Bisection (VertBis): Given a graph $G = (V, E)$, find $\textsc{minvb}(G) = \min_\varphi \delta(\lfloor n/2 \rfloor, \varphi, G)$.

Graphs encoding circuits or grids are typical instances of linear arrangement problems. We consider these instances as sparse graphs that have clustering and geometric properties. For these classes of graphs, not much is known. In this paper, we are concerned with *lattice graphs*, *random lattice graphs* and *random geometric graphs*. The study of these classes of graphs is an alternative to the standard $\mathcal{G}_{n,p}$ model of random graphs, which does not provide an informative framework to compare heuristics for layout problems [12]. In the rest of this section, we define these classes of graphs and outline and comment our main results.

Lattice graphs. A graph is said to be a *lattice graph* if it is a finite node-induced subgraph of the infinite lattice, that is, its vertex set is a finite subset of $\mathbb{Z}^2$ and two vertices are connected whenever they are at distance one.

The most simple lattice graph is L_m with node set $\{0,\dots,m-1\}^2$. The optimal layouts for the MINLA, CUTWIDTH and EDGEBIS problems on L_m are already known [25, 24, 22]: $\text{MINEB}(L_m) = \text{MINCW}(L_m) = m + (\text{odd } m)$ and $\text{MINLA}(L_m) = \frac{1}{3}(4-\sqrt{2})m^3 + o(m^3)$. In the current paper we show that the *diagonal layout* (where $\mathbf{x} = (x, y)$ precedes $\mathbf{x}' = (x', y')$ whenever $x + y < x' + y'$, and whenever $x + y = x' + y'$ and $x < x'$) is optimal for the VERTSEP, SUMCUT and VERTBIS problems on L_m.

Theorem 1. For all m, the diagonal layout is optimal for the VERTSEP, VERTBIS and SUMCUT problems on L_m. Moreover,

$$\begin{aligned} \text{MINVS}(L_m) &= m, \\ \text{MINVB}(L_m) &= m, \\ \text{MINSC}(L_m) &= \tfrac{2}{3}m^3 + \tfrac{1}{2}m^2 - \tfrac{7}{6}m. \end{aligned}$$

In the general case where the lattice graph is arbitrary, in this paper we provide the upper bounds stated in the following theorem:

Theorem 2. For any lattice graph L with n vertices,

$$\begin{aligned} \text{MINEB}(L) &\le 2\sqrt{2n} + 1, & (1)\\ \text{MINVB}(L) &\le 2\sqrt{2n} + 1, & (2)\\ \text{MINCW}(L) &\le 14\sqrt{n}, & (3)\\ \text{MINVS}(L) &\le 14\sqrt{n}, & (4)\\ \text{MINLA}(L) &\le 14n\sqrt{n}, & (5)\\ \text{MINSC}(L) &\le 14n\sqrt{n}. & (6) \end{aligned}$$

These bounds can be achieved algorithmically. Observe that in the case of L_m (where $n = m^2$), the above upper bounds are within a constant of their optimal costs (Theorem 1 and [25, 24, 22]). This shows that the bounds in Theorem 2 are best possible disregarding a constant factor. The only previous known result for lattice graphs was an exact polynomial time algorithm for the Edge Bisection problem on lattice graphs *without* holes [28].

Random lattice graphs. Percolation theory provides a framework to study lattice graphs in a probabilistic setting. We consider a *site percolation* process, where nodes from the infinite lattice ($\mathbb{Z}^2$) are selected with some probability p (selected nodes are called "open"). Let C_0 be the connected component where the origin belongs. A basic question in percolation theory is whether or not C_0 can be infinite. Let $\vartheta(p)$ denote the probability that $|C_0| = \infty$, and set $p_c = \inf\{p : \vartheta(p) > 0\}$, the *critical value* of p. It is well-known that $p_c \in (0.5, 1)$ [14]. In this paper, we consider only *subcritical* limiting regimes $p \in (0, p_c)$ in which all components are almost surely finite (results for *supercritical* regimes are derived in [9, 30]). In order to deal with finite graphs, we introduce the class of *random lattice graphs* with parameters m and p, denoted by $\mathcal{L}_{m,p}$, the lattice graphs whose set of vertices is obtained through the random selection of each element from $\{0,\dots,m-1\}^2$ chosen independently with probability p.

Regarding the Vertex Separation and the Cutwidth problem, in this paper we show that, with high probability, their optima are $\Theta(\sqrt{\log m})$.

Theorem 3. Let $p \in (0, p_c)$; then there exist constants $0 < c_1 < c_2$ such that

$$\lim_{m\to\infty} \mathbf{Pr}\left[c_1 \leq \frac{\text{MINVS}(\mathcal{L}_{m,p})}{\sqrt{\log m}} \leq \frac{\text{MINCW}(\mathcal{L}_{m,p})}{\sqrt{\log m}} \leq c_2\right] = 1.$$

Recall (see for example [8]), that if $(X_n)_{n\geq 1}$ is a sequence of random variables and X is a random variable, then X_n converges *in probability* to X ($X_n \xrightarrow{\Pr} X$) if, for every $\epsilon > 0$, it is the case that $\lim_{n\to\infty} \mathbf{Pr}\left[|X_n - X| > \epsilon\right] = 0$. One of the main results of this paper is stated in the following convergence theorem regarding the Minimum Linear Arrangement and Sum Cut problems:

Theorem 4. Let $p \in (0, p_c)$; then, there exist two constants $\beta_{\text{LA}}(p) > 0$ and $\beta_{\text{SC}}(p) > 0$ such that, as $m \to \infty$,

$$\frac{\text{MINLA}(\mathcal{L}_{m,p})}{m^2} \xrightarrow{\Pr} \beta_{\text{LA}}(p) \qquad \text{and} \qquad \frac{\text{MINSC}(\mathcal{L}_{m,p})}{m^2} \xrightarrow{\Pr} \beta_{\text{SC}}(p).$$

In the proofs, we characterize $\beta_{\text{LA}}(p)$ and $\beta_{\text{SC}}(p)$, but do not give their exact value.

Random geometric graphs. Instead of having nodes fixed at regular points in the plane, one can think about spreading nodes in the plane and connecting nodes that are not two far away. To formalize this concept, let r be a number such that $0 < r < 1$, let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and let V be any set of n points in the unit square ($[0,1]^2$). A *geometric graph* $G(V; r)$ with vertex set V and radius r is the graph $G = (V, E)$ where $E = \{uv \mid u, v \in V \wedge 0 < \|u - v\| \leq r\}$. Let $(r_i)_{i\geq 1}$ be a sequence of positive numbers and let $X = (X_i)_{i\geq 1}$ be a sequence of independently and uniformly distributed (i.u.d.) random points in $[0,1]^2$. For any natural n, we write $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ and denote by $\mathcal{G}(\mathcal{X}_n; r_n)$ the *random geometric graph* of n nodes on $\mathcal{X}_n$ and radius r_n.

Many empirical studies have used random geometric graphs as a basis to benchmark heuristics for layout or partitioning problems [16, 4, 21, 31]; however, their theoretical study is still incipient (see [11] for a survey).

Similarly to site percolation, continuum percolation and random geometric graphs exhibit a phase transition [29]: Suppose $\lim_{n\to\infty} nr_n^2 = \lambda$; then there exists a critical parameter λ_c such that when $\lambda < \lambda_c$, graphs $\mathcal{G}(\mathcal{X}_n; r_n)$ are likely to have at most $O(\log n)$ points in each connected component, while when $\lambda > \lambda_c$, there is likely to be a component with $\Theta(n)$ vertices. In this paper, we consider only *subcritical* limiting regimes $0 < \lambda < \lambda_c$ (results for *supercritical* regimes are derived in [10, 30]).

The behavior of the bisection problems on subcritical random geometric graphs is characterized by the following result, where λ'_c is a parameter that will be subsequently defined.

Theorem 5. Suppose $\lim_{n\to\infty} nr_n^2 = \lambda \in (0, \lambda'_c)$. Then, as $n \to \infty$,

$$\text{MINEB}(\mathcal{G}(\mathcal{X}_n; r_n)) \xrightarrow{\Pr} 0 \quad \text{and} \quad \text{MINVB}(\mathcal{G}(\mathcal{X}_n; r_n)) \xrightarrow{\Pr} 0.$$

In the case of the Cutwidth and Vertex Separation problems for subcritical random geometric graphs, this paper determines that, with high probability, their order of magnitude are $\Theta(\log n / \log\log n)$ and $\Theta((\log n / \log\log n)^2)$ respectively.

Theorem 6. Suppose $\lim_{n\to\infty} nr_n^2 = \lambda \in (0, \lambda_c)$. Then, there exist constants $0 < c_3 < c_4$ and $0 < c_5 < c_6$ such that

$$\lim_{n\to\infty} \mathbf{Pr}\left[c_3 \leq \frac{\text{MINVS}(\mathcal{G}(\mathcal{X}_n; r_n))}{\log n / \log\log n} \leq c_4\right] = 1 = \lim_{n\to\infty} \mathbf{Pr}\left[c_5 \leq \frac{\text{MINCW}(\mathcal{G}(\mathcal{X}_n; r_n))}{(\log n / \log\log n)^2} \leq c_6\right].$$

In the case of the Minimum Linear Arrangement and Sum Cut problems, we prove in this paper another convergence result:

Theorem 7. Suppose $\lim_{n\to\infty} nr_n^2 = \lambda \in (0, \lambda_c)$. Then, there exist two constants $\tilde{\beta}_{\text{LA}}(\lambda) > 0$ and $\tilde{\beta}_{\text{SC}}(\lambda) > 0$ such that, as $n \to \infty$,

$$\frac{\text{MINLA}(\mathcal{G}(\mathcal{X}_n; r_n))}{n} \xrightarrow{\text{Pr}} \tilde{\beta}_{\text{LA}}(\lambda) \qquad \text{and} \qquad \frac{\text{MINSC}(\mathcal{G}(\mathcal{X}_n; r_n))}{n} \xrightarrow{\text{Pr}} \tilde{\beta}_{\text{SC}}(\lambda).$$

As in the discrete case, we are able to characterize $\tilde{\beta}_{\text{LA}}(\lambda)$ and $\tilde{\beta}_{\text{SC}}(\lambda)$ as the expectation of some related quantity, but do not give their exact value. This last theorem can be viewed as analog to the Beardwood, Halton and Hammersley theorem for the TSP:

The BHH Theorem [3]. Let $X = (X_i)_{i\geq 1}$ be a sequence of independent and uniformly distributed points in $[0,1]^d$. Let $\text{MINTSP}(n)$ denote the length of the optimal solution of the TSP among the first n points of X. Then, there exists a constant $\beta_{\text{TSP}}(d)$ such that $\text{MINTSP}(n)/n^{(d-1)/d}$ converges to $\beta_{\text{TSP}}(d)$ almost surely as $n \to \infty$.

Indeed, the search of BHH-like results was one of the initial motivations to pursue this research (see [11]). A key property to prove BHH-like results is geometric sub-additivity [37, Chap. 3]. It is important to stress that geometric sub-additivity *does not hold* for our layout problems, therefore we take a completely different approach using percolation theory. In passing, we also remark that determining the exact value of $\beta_{\text{TSP}}(2)$ is still an open problem.

Organization of the paper. In Section 2, we prove our results for lattice graphs. These are subsequently used in Section 3, were we present results for random lattice graphs. In Section 4, we give the proofs of our results for random geometric graphs, which built on the theorems for random lattice graphs. We close the paper with a summary of our work and conclusions.

2 Lattice graphs

We begin this section by presenting a vertex isoperimetric inequality that will help us to characterize the optimal layouts for some of the problems defined in the previous section on square lattice graphs L_m. To attain this result, let φ_D denote the diagonal layout. The next Lemma is a special case of Corollary 9 in [6], who in fact prove the d-dimensional version for arbitrary d:

Lemma 1 (Vertex isoperimetric inequality). For any layout φ on L_m and any $k \in \{1, \ldots, m^2\}$, it is the case that $\delta(k, \varphi, L_m) \geq \delta(k, \varphi_D, L_m)$.

We are now ready to prove our first theorem.

Proof of Theorem 1. The previous isoperimetric inequality yields the optimality of φ_D for the costs of MINVS, MINSC and MINVB on L_m. Thus we get that $\text{MINVS}(L_m) = \text{MINVB}(L_m) = m$. To compute the sum of the cuts for φ_D, consider for each point in the lattice the value of the vertex cut produced by the diagonal ordering (see Figure 1 on the following page); arranging the sum by points with the same vertex cut, we get

$$\text{MINSC}(L_m) = \sum_{i=1}^{m^2} \delta(i, \varphi_D, L_m) = \sum_{i=1}^{m} \sum_{j=i}^{m} j + \sum_{i=3}^{m} i(i-2) + \sum_{i=2}^{m-1} i = \tfrac{2}{3}m^3 + \tfrac{1}{2}m^2 - \tfrac{7}{6}m.$$

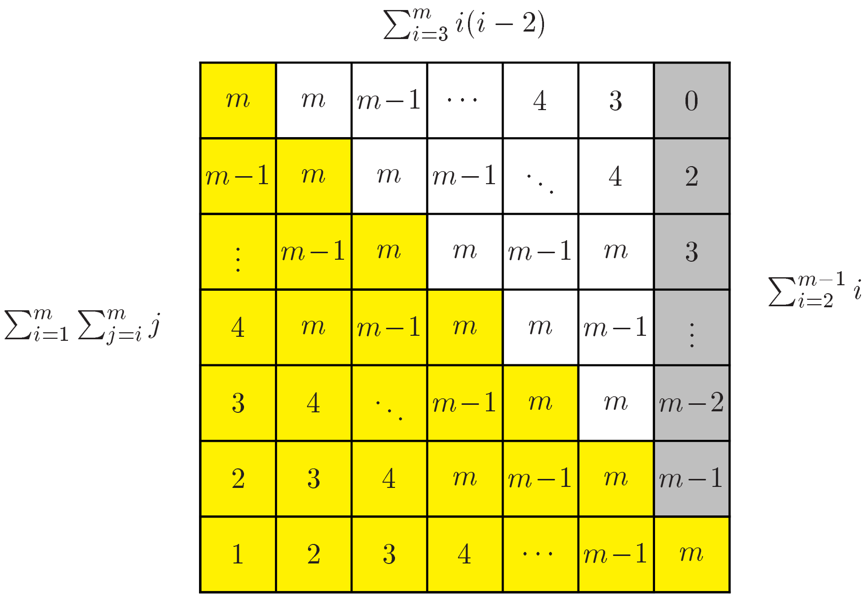


Figure 1: Values of the vertex cut in the diagonal ordering φ_D.

$\square$

We move now to general lattice graphs with n nodes.

Lemma 2. For any lattice graph L with n vertices, and any $k \in \{1, \ldots, n\}$, there is a layout φ on L such that $\theta(k, \varphi, L) \leq 2^{3/2}\sqrt{n} + 1$.

Proof. We are looking for a subset S of L consisting of k vertices, such that there are at most $2^{3/2}\sqrt{n} + 1$ edges between S and $L \setminus S$. Figure 2 on the next page illustrates this proof.

Let $\alpha > 0$ be a constant, to be chosen later. For $x \in \mathbb{Z}$ let $S_x = \{y \in \mathbb{Z} : (x, y) \in L\}$ and let $V = \{x \in \mathbb{Z} : |S_x| \geq \alpha\sqrt{n}\}$. For $i \in \mathbb{Z}$, let H_i denote the half-space $(-\infty, i] \times \mathbb{R}$. Set

$$i_0 = \min\{i \in \mathbb{Z} : |L \cap H_i| \geq k\}.$$

Consider the case $i_0 \notin V$. Then define S to be a set of the form

$$S = L \cap (H_{i_0 - 1} \cup (\{i_0\} \times (-\infty, j]))$$

with j chosen so that S has precisely k elements.

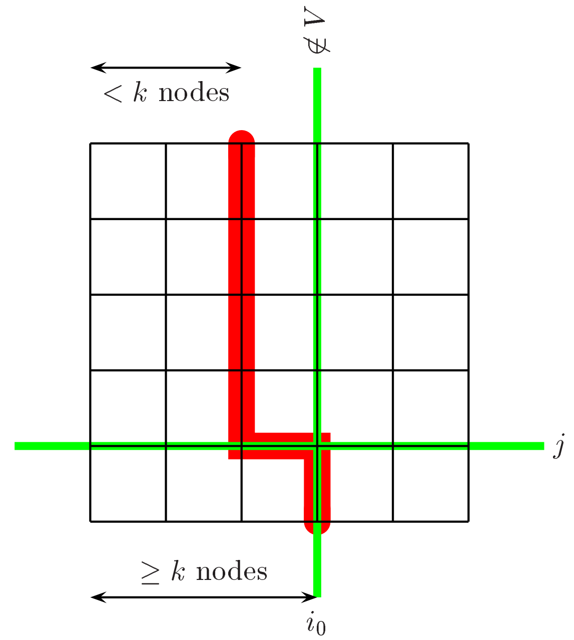
With this definition of S for $i_0 \notin V$, the number of horizontal edges between S and $L \backslash S$ is at most $|S_{i_0}|$, and hence is at most $\alpha\sqrt{n}$. There is at most one vertical edge between S and $L \backslash S$, so the number of edges from S to $L \backslash S$ is at most $\alpha\sqrt{n} + 1$ when $i_0 \notin V$.

Now consider the other case $i_0 \in V$. Let $I = [i_1, i_2]$ be the largest integer interval which includes i_0 and is contained in V. Then $i_1 - 1 \notin V$, and $i_2 + 1 \notin V$. Also, as $|V| \leq \alpha^{-1}\sqrt{n}$, $i_2 - i_1 + 1 \leq \alpha^{-1}\sqrt{n}$. We have

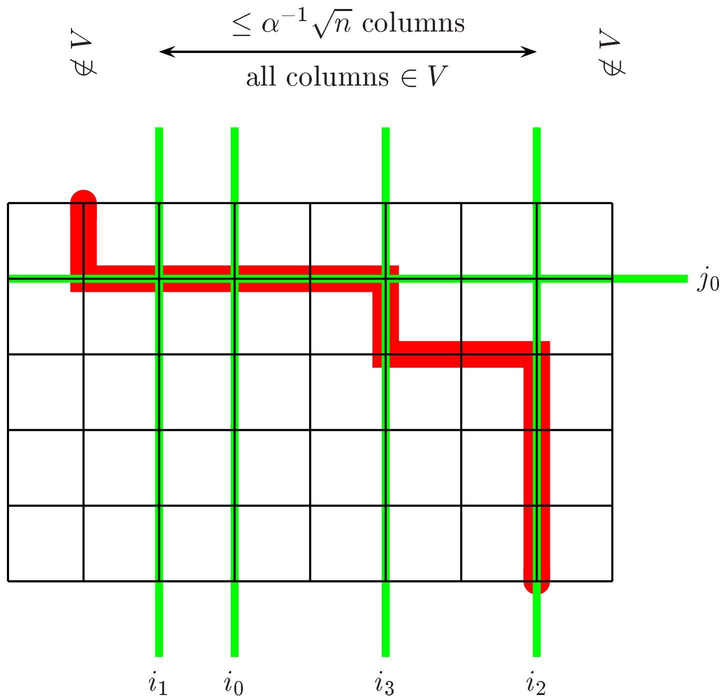
$$|L \cap H_{i_1 - 1}| < k \leq |L \cap H_{i_2}|.$$

For $j \in \mathbb{Z}$ let $T_j = [i_1, i_2] \times (-\infty, j]$. Choose j_0 so that

$$|L \cap (H_{i_1 - 1} \cup T_{j_0 - 1})| < k \leq |L \cap (H_{i_1 - 1} \cup T_{j_0})|,$$



(a) Case where $i_0 \notin V$



(b) Case where $i_0 \in V$

Figure 2: Illustration of the proof of Theorem 2. The thick line marks the potential nodes in the cut.

and let S be $L \cap (H_{i_1-1} \cup T_{j_0-1} \cup ([i_1, i_3] \times \{j_0\}))$, with $i_3 \in [i_1, i_2]$ chosen so that S has precisely k elements.

We estimate the number of edges between S and $L \backslash S$ for the case $i_0 \in V$. Since $i_1 - 1 \notin V$, and $i_2 + 1 \notin V$, the number of horizontal edges between S and $L \backslash S$ is at most $2\alpha\sqrt{n}+1$. Also, since $i_2 - i_1 + 1 \leq \alpha^{-1}\sqrt{n}$, the number of vertical edges between S and $L \backslash S$ is at most $\alpha^{-1}\sqrt{n}$. Combining these estimates we find that there are at most $(2\alpha+\alpha^{-1})\sqrt{n}+1$ edges between S and $L \backslash S$, whether or not $i_0 \in V$.

The minimum value of $2\alpha + \alpha^{-1}$ (achieved at $\alpha = 2^{-1/2}$) is $2\sqrt{2}$. Setting $\alpha = 2^{-1/2}$ in the above definition, we have the required partition. □

The previous lemma gives the key to prove our second theorem.

Proof of Theorem 2. Using Lemma 2, taking $k = \lfloor n/2 \rfloor$ and the fact that $\textsc{minvb}(L) \leq \textsc{mineb}(L)$, we get immediately equations (1) and (2).

In order to prove (3), first suppose we have $n = 2^k$ for an integer k. The proof is based on recursive bisection, with the cut size guaranteed by Lemma 2. Let $f(k)$ denote the maximum $\textsc{mincw}$ cost of all lattice graphs with 2^k vertices; then $f(k)$ satisfies the following recurrence:

$$f(k) \leq \begin{cases} 0 & \text{if } k = 0, \\ 2^{3/2}2^{k/2} + 1 + f(k-1) & \text{otherwise.} \end{cases}$$

Then, solving the recurrence, we get

$$f(k) \leq \sum_{j=1}^{k} (2^{3/2}2^{j/2} + 1) = 4(2^{1/2} + 1)(2^{k/2} - 1) + k.$$

We can drop the assumption that $n = 2^k$, by taking k so that $n \leq 2^k < 2n$, and adding extra points until one has a set of size 2^k. By monotonicity this process does not reduce the $\textsc{mincw}$ cost, so

$$\begin{aligned} \textsc{mincw}(L) &\leq 2^{5/2}(2^{1/2} + 1)\sqrt{n} + (\log_2(n) + 1) - 4(2^{1/2} + 1) \\ &\leq 13.657\sqrt{n} + \log_2(n) - 8. \end{aligned}$$

But notice that for any $x > 0$ we have $(\log_2(x) - 8)/\sqrt{x} < 0.067$; therefore the above bound for $\textsc{mincw}(L)$ is at most $14\sqrt{n}$ for all n.

In order to prove equations (4), (5) and (6), observe that for any graph G, it holds that $\textsc{minla}(G) \leq n \cdot \textsc{mincw}(G)$, $\textsc{minsc}(G) \leq n \cdot \textsc{minvs}(G)$, and $\textsc{minvs}(G) \leq \textsc{mincw}(G)$. □

3 Random lattice graphs

Let us describe some basic concepts of site percolation for a lattice L with vertex set $V_m = \{0, \ldots, m-1\}^2$. Given $p \in (0, 1)$, *site percolation* with parameter p on L is obtained by taking a random set of *open* vertices of V_m with each vertex being open with probability p independently of the others. Let $\mathcal{L}_{m,p}$ be the subgraph of L obtained by taking all edges between open vertices. We say that $\mathcal{L}_{m,p}$ is a *random lattice graph*. Denote by $\mathbf{Pr}_p$ and $\mathbf{E}_p$ the probability and expectation with respect to the described process of site percolation with parameter p. By a *cluster* we mean the set of vertices in any connected component of $\mathcal{L}_{m,p}$.

Let $\widetilde{C}_0$ denote the cluster in $\mathcal{L}_{m,p}$ that includes $(0,0)$ (possibly the empty set) and let $\widetilde{C}_x$ denote the cluster in $\mathcal{L}_{m,p}$ that includes the point $x \in V_m$.

A similar site percolation process can be generated analogously on the infinite lattice with vertex set $\mathbb{Z}^2$ and edges between nearest neighbors. In the same way we can extend $\mathbf{Pr}_p$ and $\mathbf{E}_p$ to this infinite process. Let C_0 denote the cluster in $\mathcal{L}_{m,p}$ that includes $(0,0)$ (possibly the empty set) and let C_x denote the cluster in $\mathcal{L}_{m,p}$ that includes the point x, both for site percolation on $\mathbb{Z}^2$. Notice that we can view the random lattice graph $\mathcal{L}_{m,p}$ as generated by a site percolation process on $\mathbb{Z}^2$ and taking the open vertices in V_m.

All through this section we consider random lattice graphs generated by *subcritical* limiting regimes ($p < p_c$), in which all clusters in the infinite process are almost surely finite. We begin by proving that, with high probability, the optimal costs for the CutWidth and VertSep problems on the subcritical percolation process lattice $\mathcal{L}_{m,p}$ are $\Theta(\sqrt{\log m})$.

Proof of Theorem 3. Recall that for any graph G, $\textsc{minvs}(G) \leq \textsc{mincw}(G)$. The $\textsc{mincw}$ cost of a disconnected graph is the maximum of the $\textsc{mincw}$ costs of its connected components. Hence, for any positive constant c_2,

$$\mathbf{Pr}\left[\textsc{mincw}(\mathcal{L}_{m,p}) \geq c_2\sqrt{\log m}\right] = \mathbf{Pr}\left[\cup_{x\in V_m}\left\{\textsc{mincw}(\widetilde{C}_x) \geq c_2\sqrt{\log m}\right\}\right].$$

By the site percolation version of Equation 5.7 in [14], there exists a constant $\alpha > 0$ such that $\mathbf{Pr}\left[|C_0| \geq k\right] \leq e^{-\alpha k}$. Therefore, by equation (3) in Theorem 2,

$$\begin{aligned}\mathbf{Pr}\left[\textsc{mincw}(\mathcal{L}_{m,p}) \geq c_2\sqrt{\log m}\right] &\leq \mathbf{Pr}\left[\cup_{x\in V_m}\{|\widetilde{C}_x| \geq (c_2/14)^2 \log m\}\right] \\ &\leq m^2 \exp(-\alpha(c_2/14)^2 \log m).\end{aligned}$$

Choosing $c_2 > 14\sqrt{2/\alpha}$ we get $\mathbf{Pr}\left[\textsc{mincw}(\mathcal{L}_{m,p}) \geq c_2\sqrt{\log m}\right] \to 0$.

To get a lower bound for $\textsc{minvs}(\mathcal{L}_{m,p})$, let $\delta > 0$ and let $T_1, \dots, T_{j(m)}$ be disjoint lattice subsquares of L_m, each of side $\lfloor(\delta \log m)^{1/2}\rfloor$, where $j(m) = \lfloor m/\lfloor(\delta \log m)^{1/2}\rfloor\rfloor^2$. Set $\gamma = \log(1/p)$ so that $p = e^{-\gamma}$. Let A_j be the event that all sites in T_j are open. Then

$$\mathbf{Pr}\left[A_j\right] = \exp(-\gamma\lfloor(\delta \log m)^{1/2}\rfloor^2) \geq m^{-\gamma\delta}.$$

Hence, $\mathbf{Pr}\left[\cap_{i=1}^{j(m)} A_i^c\right] \leq (1 - m^{-\gamma\delta})^{j(m)} \leq \exp(-m^{-\gamma\delta} j(m))$, which tends to zero provided δ is chosen so that $\gamma\delta < 2$. As $\textsc{minvs}(L_m) = m$, by Theorem 1, we get

$$\cup_{i=1}^{j(m)} A_j \subset \{\textsc{minvs}(\mathcal{L}_{m,p}) \geq (\delta \log m)^{1/2}\}.$$

Taking $c_1 = \sqrt{\delta}$ we obtain the lower bound. □

In the next lemma we prove that for subcritical site percolation with parameter p, the expected ratio between $\textsc{minla}(C_0)$ and $|C_0|$ is finite. We also give a similar result for the $\textsc{minsc}$ cost. To cover the case $C_0 = \emptyset$, we use the convention $0/0 = 0$, throughout the remainder of the paper.

Lemma 3. For any $p \in (0, p_c)$,

$$\mathbf{E}_p\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right] \in (0,\infty) \quad \text{and} \quad \mathbf{E}_p\left[\frac{\textsc{minsc}(C_0)}{|C_0|}\right] \in (0,\infty).$$

Proof. Let $R_0 = \min\{k : C_0 \subset [-k,k]^2\}$; then by considering the lexicographic ordering of vertices one sees that $\textsc{minsc}(L_m) \leq m^3$ and $\textsc{minla}(L_m) \leq m^3$, which together with monotonicity gives us that $\textsc{minsc}(C_0) \leq (2R_0+1)^3$ and $\textsc{minla}(C_0) \leq (2R_0+1)^3$. The statement of the lemma follows from the fact that $\mathbf{Pr}_p[R_0 > k]$ decays exponentially in k (again, see [14, Chap. 5]). □

We use the previous lemma to state one of our main results on random lattice graphs, namely that the values of MINLA and MINSC, divided by m^2, converge in probability to a constant.

Proof of Theorem 4. Consider $\mathcal{L}_{m,p}$ as being embedded in a site percolation process on the infinite lattice $\mathbb{Z}^2$. Then,

$$\begin{aligned}\frac{\textsc{minla}(\mathcal{L}_{m,p})}{m^2} &= m^{-2}\sum_{x\in V_m}\frac{\textsc{minla}(\widetilde{C}_x)}{|\widetilde{C}_x|} \\ &= m^{-2}\sum_{x\in V_m}\frac{\textsc{minla}(C_x)}{|C_x|} + m^{-2}\sum_{x\in V_m}\left(\frac{\textsc{minla}(\widetilde{C}_x)}{|\widetilde{C}_x|} - \frac{\textsc{minla}(C_x)}{|C_x|}\right). \qquad (7)\end{aligned}$$

Using Theorem VII.6.9 from [13] and the Kolmogorov zero-one law,

$$m^{-2}\sum_{x\in V_m}\frac{\textsc{minla}(C_x)}{|C_x|} \xrightarrow{\Pr} \mathbf{E}_p\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right].$$

Writing ∂V_m for the set of $x \in V_m$ with lattice neighbors in $\mathbb{Z}^2 \setminus V_m$, we get

$$\begin{aligned} m^{-2}\sum_{x\in V_m}\left|\frac{\textsc{minla}(\widetilde{C}_x)}{|\widetilde{C}_x|} - \frac{\textsc{minla}(C_x)}{|C_x|}\right| &\leq 2m^{-2}\sum_{x\in V_m, C_x\neq\widetilde{C}_x}\frac{\textsc{minla}(C_x)}{|\widetilde{C}_x|} \\ &\leq 2m^{-2}\sum_{y\in\partial V_m}\textsc{minla}(C_y),\end{aligned}$$

By the proof of Lemma 3, $\mathbf{E}_p[\textsc{minla}(C_y)]$ is finite and does not depend on y. Hence the mean of the above expression tends to zero. The result for MINLA then follows from (7) taking

$$\beta_{\text{LA}}(p) = \mathbf{E}_p\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right].$$

The proof for MINSC is just the same taking

$$\beta_{\text{SC}}(p) = \mathbf{E}_p\left[\frac{\textsc{minsc}(C_0)}{|C_0|}\right].$$

□

4 Random geometric graphs

Recall that, given a set of points V in the plane and a positive number r, $\mathcal{G}(V;r)$ denotes the geometric graph with vertex set V and radius r. In the following, $(r_i)_{i\geq 1}$ is a sequence of positive numbers and $X = (X_i)_{i\geq 1}$ is a sequence of independently and uniformly distributed (i.u.d.) random points in $[0,1]^2$. Also, $\mathcal{X}_n$ is the set of the first n points of X.

For an infinite-volume analog, let $\mathcal{P}_\lambda$ denote a homogeneous Poisson process on $\mathbb{R}^2$ of intensity λ, and set $\mathcal{P}_{\lambda,0} = \mathcal{P}_\lambda \cup \{(0,0)\}$. For n large, after appropriate scaling and centering at a randomly chosen point of $\mathcal{X}_n$, the graph $\mathcal{G}(\mathcal{X}_n; r_n)$ looks locally like $\mathcal{G}(\mathcal{P}_{\lambda,0}; 1)$. We consider a continuum site percolation process based on the Poisson process; let $\widetilde{\vartheta}(\lambda)$ be the probability that the added point at the origin lies in an infinite component of $\mathcal{G}(\mathcal{P}_{\lambda,0}; 1)$. Then define the critical percolation λ_c as the infimum of $\{\lambda > 0 : \widetilde{\vartheta}(\lambda) > 0\}$. It is well known [14] that $\lambda_c \in (0, \infty)$. In this section we shall deal with random geometric graphs satisfying the condition $\lim_{n\to\infty} nr_n^2 = \lambda$, for the subcritical regime $\lambda < \lambda_c$.

In the following, B_m denotes the box $[0, m)^2$.

We first deal with the behavior of the bisection related problems. Setting $\lambda_c' = \inf\{\lambda > 0 : \widetilde{\vartheta}(\lambda) \geq \frac{1}{2}\}$, the subcritical regime for EdgeBis and VertBis is given by $\lambda < \lambda_c'$.

Proof of Theorem 5. We need to show that with high probability, there is a subset W of $\mathcal{X}_n$, of cardinality $\lfloor \frac{n}{2} \rfloor$, with no edges between W and $\mathcal{X}_n \backslash W$. By hypothesis, $\widetilde{\vartheta}(\lambda) < \frac{1}{2}$.

For $k \in \mathbb{N}$, set $\pi_k = \mathbf{Pr}_\lambda[|C_0| = k]$, and note $\pi_k > 0$. Let $N_n(k)$ denote the number of points of $G(\mathcal{X}_n; r_n)$ lying in clusters of size k.

Let $p_k(x)$ denote the probability that when adding a point x to a set of $n-1$ uniformly distributed points, the new point will be in a cluster of size k. Then,

$$\mathbf{E}[N_n(k)] = n \int_{[0,1]^2} p_k(x) \mathrm{d}x.$$

For x not on the boundary of $[0,1]^2$, we have that $p_k(x) \to \pi_k$, so by the dominated convergence theorem, $\mathbf{E}[N_n(k)]/n \to \pi_k$. To look at the variance, notice that since $N_n(k)(N_n(k)-1)$ is twice the number of pairs of points both in clusters of size k, if we denote by $p_{k,k}(x,y)$ the probability that when inserting points at x and y into a set of $n-2$ uniform points they will both be in a cluster of size k, then

$$\mathbf{E}[N_n(k)(N_n(k)-1)] = n(n-1) \int_{[0,1]^2} \int_{[0,1]^2} p_{k,k}(x,y) \mathrm{d}x \mathrm{d}y.$$

For points x and y not on the boundary with $x \neq y$, we have that $p_{k,k}(x,y) \to (\pi_k)^2$, hence using again the dominated convergence theorem $\mathbf{E}[(N_n(k)/n)^2] \to \pi_k^2$. So $\mathbf{Var}[N_n(k)/n] \to 0$, and by Chebyschev's inequality we can conclude

$$n^{-1} N_n(k) \xrightarrow{\mathrm{Pr}} \pi_k. \tag{8}$$

As $1 - \sum_k \pi_k = \widetilde{\vartheta}(\lambda) < 1/2$, we can choose k_1 such that $\sum_{k \leq k_1} \pi_k > 1/2$. This inequality together with (8) implies that with probability tending to 1 as n tends to infinity,

$$\sum_{k > k_1} N_n(k) < \left\lfloor \frac{n}{2} \right\rfloor,$$

and $N_n(k)$ are non-zero for $k = 1, 2, \ldots, k_1$.

We generate a subset W of $\mathcal{X}_n$ as follows. First take the union of all clusters of size greater than k_1. Then add clusters of size k_1 until there are none left. Then add clusters of size $k_1 - 1$ until there are none left. Continue in this way. At some point, having just added a set of size i, we will have a set of size $\lfloor \frac{n}{2} \rfloor - m$ with $0 \leq m < i$. If $m = 0$, stop. If $m > 0$ then add a cluster of size m and stop. This gives a set $W \subset \mathcal{X}_n$, of size $\lfloor \frac{n}{2} \rfloor$, with no edges connecting W to $\mathcal{X}_n \backslash W$, as desired. □

Analogous results to those in Theorem 5 also hold for a percolation process in the lattice with $p < p'_c$, defined in the same way as λ'_c.

We present now our results for the CutWidth and VertSep problems. The following lemma establishes lower bounds for the cost of the optimal cutwidth and vertex separation of random geometric graphs in the subcritical regime.

Lemma 4. Suppose $\lambda < \lambda_c$ and $m_n = c/r_n$ for some constant $c > 0$. Then,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\textsc{minvs}(\mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n}; 1)) \geq \log n / \log\log n\right] = 1$$

and

$$\lim_{n\to\infty} \mathbf{Pr}\left[\textsc{mincw}(\mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n})) \geq \tfrac{1}{5}(\log n/\log\log n)^2\right] = 1.$$

Proof. Let us dissect B_{m_n} in boxes of size $\nu \times \nu$ where ν is chosen as big as possible so that all pair of points in the same box are connected by an edge in $G = \mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n})$. There are $b_n = \lfloor m_n/\nu \rfloor^2$ boxes that completely fall in B_{m_n}. For all $1 \leq i \leq b_n$, let N_i be the number of points of the process that fall in box i. By construction, we have that N_i follows a Poisson distribution with mean $\mu = \lambda\nu^2$, and that N_i is independent of N_j for all $i \neq j$. Let $M_n = \max_{i=1..b_n} N_i$ be the maximal number of points in some box. Then, $\textsc{minvs}(G) \geq M_n - 1$ and $\textsc{mincw}(G) \geq \frac{1}{2}M_n(\frac{1}{2}M_n - 1) \geq \frac{1}{5}M_n$, because G contains at least a clique with M_n points.

In the following, we show that some box contains at least $f(n) = \log n/\log\log n$ points with probability tending to one as n tends to infinity. Fix a box i. Then,

$$\mathbf{Pr}\left[N_i > f(n)\right] = \sum_{k > f(n)} \frac{\mu^k e^{-\mu}}{k!} \geq e^{-\mu}\frac{\mu^{f(n)+1}}{(f(n)+1)!}.$$

Consider now all the boxes: We have

$$\begin{aligned}
\mathbf{Pr}\left[\bigcap_{i=1..b_n} N_i \leq f(n)\right] &= \prod_{i=1..b_n} \mathbf{Pr}\left[N_i \leq f(n)\right] \\
&\leq \left(1 - e^{-\mu}\frac{\mu^{f(n)+1}}{(f(n)+1)!}\right)^{b_n} \leq \left(1 - e^{-\mu}\frac{\mu^{f(n)+1}}{(f(n)+1)!}\right)^{4c^2 n/\mu}
\end{aligned}$$

and thus the above probability goes to zero as n goes to infinity. □

In order to determine upper bounds of the same order than the lower bounds, we will need two auxiliary results, which may be useful in other situations. Let us define $\textsc{cw}(\varphi, G) = \max_{i=1..n} \theta(i, \varphi, G)$ and $\textsc{vs}(\varphi, G) = \max_{i=1..n} \delta(i, \varphi, G)$ for a graph G and a layout φ of G.

Definition 1. Given a graph $G = (V, E)$ and an integer s, let us define the *s-explosion of G* as the graph $G' = (V', E')$ where $V' = \{(u, i) \mid u \in V,\ 1 \leq i \leq s\}$ and $E' = \big\{(u, i)(u, j) \mid u \in V,\ 1 \leq i < j \leq s\big\} \cup \big\{(u, i)(v, j) \mid uv \in E,\ 1 \leq i \leq j \leq s\big\}$.

Lemma 5. Let $G' = (V', E')$ be the s-explosion of an arbitrary graph $G = (V, E)$. Then, $\textsc{mincw}(G') \leq s^2(\textsc{mincw}(G) + 1)$.

Proof. Let φ be a layout of G with minimal cutwidth. Let φ' be a layout of G' defined by $\varphi'((u,i)) = (\varphi(u)-1)s+i$ for all $u \in V$ and all $1 \le i \le s$. In plain words, the exploded nodes are placed in φ' in the same relative order that their "parents" are in φ.

Let i be an integer such that $1 \le i \le |V|$. Then, we have $\theta(si,\varphi',G') = s^2\theta(i,\varphi,G)$ because each original edge from E corresponds to s^2 edges in E'. Moreover, for all $1 \le j < s$, we have $\theta(si+j,\varphi',G') \le \theta(i+1,\varphi,G)\cdot s^2 + s^2$ because we have to count s^2 edges in E' for each edge in E plus the edges inside a clique of size s. As a consequence,

$$\textsc{mincw}(G') \le \textsc{cw}(G',\varphi') \le s^2(\textsc{cw}(G,\varphi)+1) = s^2(\textsc{mincw}(G)+1),$$

which proves the stated result. □

Definition 2. Given two integers m and l, let us define a (m,l)-*mesh* as the graph $G=(V,E)$ with vertex set $V_m = \{0,\dots,m-1\}^2$ and edge set $E = \{uv \mid u,v \in V,\ \|u-v\|_\infty \le l\}$.

Lemma 6. Given an integer $l > 0$, there exists a constant $\kappa_l > 0$ such that for any node induced subgraph L of an (m,l)-mesh with n nodes, $\textsc{minvs}(L) \le \textsc{mincw}(L) \le \kappa_l\sqrt{n}$.

Proof. First assume that $l = 1$. An argument similar to the proof of Equation (3) in Theorem 2 shows that $\textsc{mincw}(L) \le \kappa_1\sqrt{n}$ for some constant $\kappa_1 > 0$. As for any graph its minimal vertex separation can never be greater than its minimal cutwidth, Lemma 6 is proved for the case $l = 1$.

Assume now that $l > 1$. Let $m' = \lceil m/l \rceil$ and let H be a node induced subgraph of the $(m',1)$-mesh where a potential node u with coordinates x_u and y_u belongs to H if and only if some node v from L with coordinates x_v and y_v satisfies $\lfloor x_v/l \rfloor = x_u$ and $\lfloor y_v/l \rfloor = y_u$. As H is a $(m',1)$-mesh and $|H| \le |L| = n$, we have $\textsc{mincw}(H) \le \kappa_1\sqrt{n}$. Construct a graph G by s-exploding H with $s = l^2$. By Lemma 5, $\textsc{mincw}(G) \le l^4(\kappa_1\sqrt{n}+1)$. Now observe that, by construction, L is a subgraph of G! As a consequence,

$$\textsc{minvs}(L) \le \textsc{mincw}(L) \le \textsc{mincw}(G) \le \kappa_l\sqrt{n}$$

where $\kappa_l = l^4(\kappa_1+1)$. □

We are now ready to find upper bounds for the Vertex Separation and the Cutwidth problems.

Lemma 7. Suppose $\lambda < \lambda_c$ and $m_n = c/r_n$ for some constant $c > 0$ and all integer n. Then, there exist constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\lim_{n\to\infty} \mathbf{Pr}\left[\textsc{minvs}(\mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n};1)) \le \gamma_1 \log n/\log\log n\right] = 1$$

and

$$\lim_{n\to\infty} \mathbf{Pr}\left[\textsc{mincw}(\mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n};1)) \le \gamma_2 (\log n/\log\log n)^2\right] = 1.$$

Proof. We only give the proof for the Vertex Separation problem, as the proof for the Cutwidth result is similar.

Let us dissect B_{m_n} in boxes of size $\nu \times \nu$, where ν is a (small) constant that will be determined latter. There is a total number of $b_n = \lceil m_n/\nu \rceil^2$ boxes, which we will consider. Let $H = H(P_\lambda)$ be the node induced subgraph of the $(\lceil m_n/\nu \rceil), 1/\nu)$-mesh whose nodes are present if and only if their corresponding box contains at least one point in $\mathcal{P}_\lambda$. Let us denote $\mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n};1)$ by G.

Our proof consists in three steps; let us first outline them. In Step 1, we prove that, with high probability, H does not contain "too big" connected components. Assuming that H does not contain any big connected component, in Step 2, we upper bound $\textsc{minvs}(H)$ by a certain function $f(n)$, and show that if ψ is a layout such that $\textsc{vs}(\psi, H) \leq f(n)$, then $\textsc{minvs}(G)$ is bounded by M_n, a magnitude related to the number of points of $\mathcal{P}_\lambda$ in $f(n)$ boxes and to the layout ψ. Finally, in Step 3 we prove that, with high probability, $M_n \leq \nu \log n / \log\log n$. The combination of these steps shall imply the proposed result.

Step 1. For all $i = 1, \ldots, b_n$, let N_i be the number of points of $\mathcal{P}_\lambda$ that fall in box i. By construction, if box i completely falls in B_{m_n} square, we have that N_i follows a Poisson distribution with mean $\mu = \lambda\nu^2$; otherwise N_i is bounded by a Poisson distribution with mean μ^2. Moreover, for all $i \neq j$, the variables N_i and N_j are independent.

We identify each box with a node of an infinite lattice with connections between boxes whose centers lie at distance not greater than $1/\nu$. We define on this lattice a percolation process where each node is deemed open if its corresponding box contains at least one point from $\mathcal{P}_\lambda$. Observe that our lattice is a "general lattice" in the sense of Kesten [20, pp. 10–12] and Grimmett [14, p. 349]. Therefore, this percolation process must exhibit a phase transition at some critical probability, and an enough small value of ν sets us in the subcritical phase [29, Lemma 2]. As a consequence [14], there is an exponential decay in the size of a connected component C: for some constant α and all integer m,

$$\mathbf{Pr}\left[|C| > m\right] \leq \exp\left(-\alpha m\right).$$

Let $c_1 = 3/\alpha$. The probability that some connected component of H has size bigger than $c_1 \log n$ is

$$\mathbf{Pr}\left[\bigcup_{x \in V(H)} \{|C_x| > c_1 \log n\}\right] \leq b_n \cdot \exp\left(-\alpha c_1 \log n\right) \leq 4n^{-2}/\mu \to 0.$$

Step 2. We now suppose that no component of H has more than $c_1 \log n$ points. By Lemma 6, this implies that for some constant $\kappa > 0$, $\textsc{minvs}(H) \leq \kappa\sqrt{\log n}$.

Set $f(n) = \kappa\sqrt{\log n}$ and let ψ be a layout of H with $\textsc{vs}(\psi, G) \leq f(n)$. We use the layout ψ as the basis to build a layout φ of the points in $\mathcal{P}_\lambda \cap B$, taking the ordering of points within multiply occupied boxes in an arbitrary way. As $f(n)$ boxes separate any two connected boxes of H in ψ, we have that $\textsc{vs}(G, \varphi) \leq M_n$, where M_n is defined as the maximum number of points in $\mathcal{P}_\lambda$ that belong to the union of any $f(n)$ consecutive boxes according to the layout ψ.

Step 3. In the following we show that, with high probability, $f(n)$ consecutive boxes contain at most $g(n) = \nu \log n / \log\log n$ points.

The number of points in an open box follows a Poisson distribution with parameter μ, conditioned to be at least one. The reader can easily check that this is stochastically dominated by an unconditioned Poisson distribution with parameter μ plus one unit. Also, the number of points in $f(n)$ boxes follows the sum of $f(n)$ independent Poisson distributions each one with parameter μ and conditioned to be at least one. This is stochastically dominated by $f(n)$ plus the sum of $f(n)$ unconditioned Poisson distributions with parameter μ, which is the same as the variable

$$S = f(n) + \mathcal{P}(f(n)\mu) = \kappa\sqrt{\log n} + \mathcal{P}\left(\mu\kappa\sqrt{\log n}\right).$$

Since each connected component of H contains at most $c_1 \log n$ boxes, and H contains at most b_n connected components, the probability that any $f(n)$ consecutive boxes contain more than $g(n)$ points is at most

$$c_1 \log n \cdot b_n \cdot \mathbf{Pr}\left[S \geq g(n)\right].$$

In order to prove that the previous probability goes to zero as n goes to infinity, let X be some random variable following a Poisson distribution with mean μ_n to be defined latter and let α_n be a sequence to be given latter such that $\mu_n = \omega(\alpha_n)$ and that $\mu_n = \omega(1)$. Then, we claim that

$$\mathbf{Pr}\left[X \geq \alpha_n\right] \leq e^{-\mu_n} \mu_n^{\alpha_n} e^{\alpha_n} \alpha_n^{-\alpha_n}$$

and so

$$\log \mathbf{Pr}\left[X \geq \alpha_n\right] \leq -\mu_n + \alpha_n \log \mu_n + \alpha_n - \alpha_n \log \alpha_n. \tag{9}$$

Take X as a Poisson distribution with mean $\mu_n = \mu\kappa\sqrt{\log n}$ and take $\alpha_n = \gamma_1 \log n / \log\log n - \kappa\sqrt{\log n}$ for some constant γ_1. Let $\gamma_1' = \frac{3}{4}\gamma_1$ so that $\gamma_1' \log n / \log\log n < \alpha_n < \gamma_1 \log n / \log\log n$. Substituting in the right hand side of (9) and taking exponentials, we find

$$\mathbf{Pr}\left[X \geq \alpha_n\right] \geq n^{-\gamma_1/4}$$

and thus

$$c_1 \log n \cdot b_n \cdot \mathbf{Pr}\left[X \geq \alpha_n\right] \to 0$$

provided that γ_1 is chosen big enough. $\square$

We can now finally prove Theorem 6:

Proof of Theorem 6. We first couple $\mathcal{X}_n$ to a Poisson processes with a slightly lower density of points, as follows: Take $\lambda_1 \in (0, \lambda)$ and set $m_n = \lceil r_n^{-1} \rceil$ Let M_n be a Poisson variable with mean $\lambda_1 m_n^2$. Then, $\lim_{n\to\infty} \Pr[M_n > n] = 0$. Let us set $m_n\mathcal{X}_n = \{m_n X_i : 1 \leq i \leq n\}$, $\mathcal{P} = \{m_n X_i : 1 \leq i \leq M_n\}$. Notice that $\mathcal{P}$ is a Poisson process on B_{m_n} with intensity λ_1. If $M_n \leq n$ then $\mathcal{G}(\mathcal{P}; 1)$ is a subgraph of $\mathcal{G}(m_n\mathcal{X}_n; m_n r_n)$, which is isomorphic to $\mathcal{G}(\mathcal{X}_n; r_n)$. By monotonicity,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathrm{vs}(\mathcal{G}(\mathcal{X}_n; r_n)) \geq \mathrm{vs}(\mathcal{G}(\mathcal{P}; 1))\right] = 1.$$

As a consequence, by Lemma 4, we have

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathrm{vs}(\mathcal{G}(\mathcal{X}_n; r_n)) \geq \log n / \log\log n\right] = 1. \tag{10}$$

We now couple $\mathcal{X}_n$ to a Poisson processes with a slightly higher density of points: Take $\lambda_2 \in (\lambda, \lambda_c)$ and set $m_n' = \lfloor r_n^{-1} \rfloor$. Let M_n' be a Poisson variable with mean $\lambda_2 (m_n')^2$. Then, $\lim_{n\to\infty} \Pr[M_n' < n] = 0$. Let $\mathcal{P}' = \{m_n' X_i : 1 \leq i \leq M_n'\}$. Notice that $\mathcal{P}'$ is a Poisson processes on $B_{m_n'}$ with intensity λ_2. If $M_n' \leq n$ then $\mathcal{G}(\mathcal{P}; 1)$ is a super-graph of $\mathcal{G}(m_n\mathcal{X}_n; m_n r_n)$, which is isomorphic to $\mathcal{G}(\mathcal{X}_n; r_n)$. By monotonicity,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathrm{vs}(\mathcal{G}(\mathcal{X}_n; r_n)) \leq \mathrm{vs}(\mathcal{G}(\mathcal{P}'; 1))\right] = 1.$$

As a consequence, by Lemma 7, we have

$$\lim_{n\to\infty} \mathbf{Pr}\left[\textsc{vs}(\mathcal{G}(\mathcal{X}_n; r_n)) \le \gamma_1 \log n/\log\log n\right] = 1. \tag{11}$$

Combining equations (10) and (11) we obtain

$$\lim_{n\to\infty} \mathbf{Pr}\left[c_3 \le \frac{\textsc{minvs}(\mathcal{G}(\mathcal{X}_n; r_n))}{\log n/\log\log n} \le c_4\right] = 1$$

for the suitable constants c_3 and c_4.

The proof for the Cutwidth result is similar. □

We turn now to the convergence results for the MinLA and SumCut problems. We shall first prove that in the subcritical case, the expected values of MINLA and MINSC on the induced graph on C_0 are finite.

Proposition 1. Let $\lambda < \lambda_c$. Then

$$\mathbf{E}_\lambda[\textsc{minla}(C_0)] \in (0, \infty) \quad \text{and} \quad \mathbf{E}_\lambda[\textsc{minsc}(C_0)] \in (0, \infty).$$

Proof. Recall that for any graph G with n nodes, $\textsc{minla}(G) \le n^3$ and $\textsc{minsc}(G) \le n^3$. Hence to prove the statement it is enough to show that $\mathbf{E}_\lambda[|C_0|^3] < \infty$. To show this, let $B(r)$ be the ball of radius r centered at the origin and let

$$\mathcal{P}_{\lambda,0}(B(r)) = |\{x \text{ of } \mathcal{P}_{\lambda,0} \mid x \in B(r)\}|.$$

Then for any $m > 0$, the event $\{|C_0| \ge m^{1/3}\}$ is contained in the union of the events $\{\mathcal{P}_{\lambda,0}(B(m^{1/(6d)}) \ge m^{1/3}\}$ and $\{\text{diam}(C_0) \ge m^{1/(6d)}\}$, therefore using Boole's inequality we get

$$\Pr[|C_0| \ge m^{1/3}] \le \Pr[\mathcal{P}_{\lambda,0}(B(m^{1/(6d)})) \ge m^{1/3}] + \Pr[\text{diam}(C_0) \ge m^{1/(6d)}].$$

The first term in the right hand side is summable in m by standard estimates of the Poisson distribution. The second term is summable in m by Lemma 2 in [29]. Hence

$$\sum_{m\ge 1} \Pr[|C_0|^3 \ge m] < \infty$$

and the statement follows. □

Next we prove a technical lemma that will be needed later.

Lemma 8. The functions $\lambda \mapsto \mathbf{E}_\lambda\left[\frac{\textsc{minsc}(C_0)}{|C_0|}\right]$ and $\lambda \mapsto \mathbf{E}_\lambda\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right]$ are continuous in λ on $(0, \lambda_c)$.

Proof. We give the proof for the MINLA case, the proof for the MINSC is similar. Define coupled versions of the Poisson process $\mathcal{P}_\lambda, \lambda > 0$, in the following standard way. Let $\mathcal{P}$ be a Poisson process on $\mathbb{R}^2 \times [0, \infty)$ of rate 1, and let $\mathcal{P}_\lambda$ consist of the projections onto the 2 coordinates of the points of $\mathcal{P} \cap (\mathbb{R}^2 \times [0, \lambda])$. Using this coupling, write $C_0(\lambda)$ for the component including the origin of $C(\mathcal{P}_\lambda \cup \{0\}; 1)$.

Suppose (λ_n) is a sequence with $\lambda_n \to \lambda \in (0, \lambda_c)$. With this coupling, with probability one it is the case that for all large enough n the components $C_0(\lambda_n)$ and $C_0(\lambda)$ are identical. Hence by the dominated convergence theorem,

$$\mathbf{E}_{\lambda_n}[\textsc{minla}(C_0(\lambda_n))/|C_0(\lambda_n)|] \to \mathbf{E}_\lambda[\textsc{minla}(C_0(\lambda))/|C_0(\lambda)|].$$

□

We give now asymptotics for the MINLA and MINSC costs of the graphs $\mathcal{G}(\mathcal{P}_\lambda \cap B_m; 1)$.

Proposition 2. Suppose $\lambda < \lambda_c$, and let $\mathcal{G}_m = \mathcal{G}(\mathcal{P}_\lambda \cap B_m; 1)$. Then as $m \to \infty$,

$$\frac{\textsc{minla}(\mathcal{G}_m)}{m^2} \xrightarrow{\text{Pr}} \lambda \mathbf{E}_\lambda \left[\frac{\textsc{minla}(C_0)}{|C_0|} \right] \quad \text{and} \quad \frac{\textsc{minsc}(\mathcal{G}_m)}{m^2} \xrightarrow{\text{Pr}} \lambda \mathbf{E}_\lambda \left[\frac{\textsc{minsc}(C_0)}{|C_0|} \right].$$

Proof. We sketch a proof for MINLA. For each point x of $\mathcal{P}_\lambda \cap B_m$, let C_x denote the component of $\mathcal{G}(\mathcal{P}_\lambda \cap B_m; 1)$ that includes the point x, and let $\widetilde{C}_x$ denote the component of $\mathcal{G}(\mathcal{P}_\lambda; 1)$ that includes the point x. By a similar argument to the proof of Theorem 5, it suffices to prove that

$$\mathbf{E}_\lambda \left[m^{-d} \sum_{x \in \mathcal{P}_\lambda \cap B_m} \left| \frac{\textsc{minla}(\widetilde{C}_x)}{|\widetilde{C}_x|} - \frac{\textsc{minla}(C_x)}{|C_x|} \right| \right] \to 0. \tag{12}$$

For $l > 0$, let $\partial_l B_m$ be the set of points $z \in B_m$ with $\|z - y\|_\infty \le l$ for some $y \notin B_m$. The quantity inside the sum in (12) is at most $\textsc{minla}(\widetilde{C}_x) \cdot (C_x \neq \widetilde{C}_x)$, where for any statement S, (S) stands for 1 if S is true, 0 otherwise. Hence the random variable inside the expectation in (12) is at most

$$\left(m^{-d} \sum_{x \in \mathcal{P}_\lambda \cap \partial_l B_m} \textsc{minla}(\widetilde{C}_x) \right) + \left(m^{-d} \sum_{x \in B_m \setminus \partial_l B_m} \textsc{minla}(\widetilde{C}_x) \cdot (\operatorname{diam}(\widetilde{C}_x) > l) \right).$$

The expectation of the first term tends to zero, while the expectation of the second term equals $\lambda \mathbf{E}_\lambda[\textsc{minla}(C_0)(|C_0| > l)]$, which can be made arbitrarily small by the choice of l. Then (12) follows. $\square$

We can now finally prove Theorem 7:

Proof of Theorem 7. The same coupling that in the proof of Theorem 6 shows that

$$\mathbf{Pr} \left[\textsc{minla}(\mathcal{G}(\mathcal{P}_n; 1)) \le \textsc{minla}(\mathcal{G}(\mathcal{X}_n; r_n)) \le \textsc{minla}(\mathcal{G}(\mathcal{P}'_n; 1)) \right] \to 1.$$

By Proposition 2,

$$\frac{\textsc{minla}(\mathcal{G}(\mathcal{P}_n; 1))}{m_n^2} \xrightarrow{\text{Pr}} \lambda_1 \mathbf{E}_{\lambda_1} \left[\frac{\textsc{minla}(C_0)}{|C_0|} \right],$$

so that

$$\frac{\textsc{minla}(\mathcal{G}(\mathcal{P}_n; 1))}{n} \xrightarrow{\text{Pr}} \left(\frac{\lambda_1}{\lambda} \right) \mathbf{E}_{\lambda_1} \left[\frac{\textsc{minla}(C_0)}{|C_0|} \right].$$

Similarly,

$$\frac{\textsc{minla}(\mathcal{G}(\mathcal{P}'_n; 1))}{n} \xrightarrow{\text{Pr}} \left(\frac{\lambda_2}{\lambda} \right) \mathbf{E}_{\lambda_2} \left[\frac{\textsc{minla}(C_0)}{|C_0|} \right].$$

Taking $\lambda_1 \uparrow \lambda$ and $\lambda_2 \downarrow \lambda$ and using Lemma 8,

$$\frac{\textsc{minla}(\mathcal{G}(\mathcal{X}_n; r_n))}{n} \xrightarrow{\text{Pr}} \mathbf{E}_\lambda \left[\frac{\textsc{minla}(C_0)}{|C_0|} \right].$$

The proof for the convergence of MINSC is analogous. $\square$

5 Conclusions

In this paper we have considered several layout problems for specific classes of sparse graphs: lattice graphs, random lattice graphs and random geometric graphs. We have first identified the optimal solutions for the Vertex Separation, Sum Cut, and Vertex Bisection problems for the complete square lattice, and presented several worst case upper bounds for arbitrary lattices for all the addressed problems. We have then determined the order of magnitude of the Vertex Separation and Cutwidth of random lattice graphs generated by selecting at random nodes of an $m \times m$ complete lattice with probability $p < p_c$. For the Minimum Linear Arrangement and the Sum Cut problems, we have even been able to prove that their optimal cost divided by m^2 converges to a particular constant. Finally, we have considered random geometric graphs in the subcritical regime. We have shown that the Minimum Linear Arrangement and the Sum Cut measures divided by the number of nodes of the graph converge to another particular constant. For the Vertex Separation and Cutwidth of random geometric graphs, we have identified their order of magnitude. We also have shown that one may obtain an empty bisection of random geometric graphs and random lattice graphs (both in the subcritical case). Table 1 summarizes the obtained results.

For the sake of clarity, we contented ourselves in this paper with demonstrating convergence in probability; however, the convergence in our theorems actually holds in the stronger sense of *complete convergence* which implies convergence almost surely (see [38]). In all the cases, our results are given for the bidimensional space; the extension to higher dimensions remains open.

Our results shows that the optimal costs for all problems are different in the supercritical and subcritical phases: Let us recall the order of magnitudes in the supercritical case [30]: For random lattice graphs $\mathcal{L}_{m,p}$ with $p > p_c$,

$$\begin{aligned}
\mathbf{Pr}\left[\text{MINLA}(\mathcal{L}_{m,p}) = \Theta(m^3)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINCW}(\mathcal{L}_{m,p}) = \Theta(m)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINSC}(\mathcal{L}_{m,p}) = \Theta(m^3)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINVS}(\mathcal{L}_{m,p}) = \Theta(m)\right] &= 1.
\end{aligned}$$

For random geometric graphs $\mathcal{G}(\mathcal{X}_n; r_n)$ with $\lim_{n\to\infty} nr_n = \lambda$ and $\lambda > \lambda_c$, it is the case that

$$\begin{aligned}
\mathbf{Pr}\left[\text{MINLA}(\mathcal{L}_{m,p}) = \Theta(n^3 r_n^3)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINCW}(\mathcal{L}_{m,p}) = \Theta(n^2 r_n^3)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINSC}(\mathcal{L}_{m,p}) = \Theta(n^2 r_n)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINVS}(\mathcal{L}_{m,p}) = \Theta(nr_n)\right] &= 1.
\end{aligned}$$

In definitive, there also exists a transition phase for the considered layout measures on random geometric graphs and random lattice graphs.

As with many other optimization problems in the plane that exhibit convergence phenomena, an interesting problem is to find good methods for evaluating numerically the constants in Theorem 4 as functions of the open vertex density p and the analogous constants in Theorem 7 as functions of λ. Preliminary estimations for$\beta_{\text{LA}}(p)$ were given in [9]; the used method was a raw simulation of the percolation process on the lattice and computation of lower and upper bounds with several heuristics [32]. According to [37, Section 2.4], in the case of the TSP, D.S. Johnson obtained empirically that $0.70 \leq \beta_{\text{TSP}}(2) \leq 0.73$ with high confidence. Our experimental results are not so tight; we leave the seek of new empirical or analytical methods to estimate these constants as an open problem.

F	L_m	L	$\mathcal{L}_{m,p}$	$\mathcal{G}(\mathcal{X}_n; r_n)$
MINEB	$F = m + (\text{odd } m)$ [22]	$F \leq 2\sqrt{2n} + 1$	$F \xrightarrow{\text{Pr}} 0$	$F \xrightarrow{\text{Pr}} 0$
MINVB	$F = m$	$F \leq 2\sqrt{2n} + 1$	$F \xrightarrow{\text{Pr}} 0$	$F \xrightarrow{\text{Pr}} 0$
MINCW	$F = m + (\text{odd } m)$ [22]	$F \leq 14\sqrt{n}$	$c_1 \leq \dfrac{F}{\sqrt{\log m}} \leq c_2$ whp	$c_3 \leq \dfrac{F}{(\log n / \log\log n)^2} \leq c_4$ whp
MINVS	$F = m$	$F \leq 14\sqrt{n}$	$c_1 \leq \dfrac{F}{\sqrt{\log m}} \leq c_2$ whp	$c_5 \leq \dfrac{F}{\log n / \log\log n} \leq c_6$ whp
MINLA	$F = (4 - \sqrt{2})m^3/3 + o(m^3)$ [24, 25]	$F \leq 14n\sqrt{n}$	$F/m^2 \xrightarrow{\text{Pr}} \beta_{\text{LA}}(p)$	$F/n \xrightarrow{\text{Pr}} \tilde{\beta}_{\text{LA}}(\lambda)$
MINSC	$F = 2m^3/3 + o(m^3)$	$F \leq 14n\sqrt{n}$	$F/m^2 \xrightarrow{\text{Pr}} \beta_{\text{SC}}(p)$	$F/n \xrightarrow{\text{Pr}} \tilde{\beta}_{\text{SC}}(\lambda)$

Table 1: Summary of results for full $m \times m$ square lattice graphs (L_m), general lattice graphs with n nodes (L), random lattice graphs in $m \times m$ lattices with probability p ($\mathcal{L}_{m,p}$), and random geometric graphs with n nodes and radius r_n where $\lim nr_n^d = \lambda$ ($\mathcal{G}(\mathcal{X}_n; r_n)$). In the first two rows we require that $p < p_c'$ and $\lambda < \lambda_c'$. For the remaining rows $p < p_c$ and $\lambda < \lambda_c$.

References

[1] D. Adolphson. Single machine job sequencing with precedence constraints. *SIAM Journal on Computing*, 6:40–54, 1977.

[2] D. Adolphson and T. C. Hu. Optimal linear ordering. *SIAM Journal on Applied Mathematics*, 25(3):403–423, November 1973.

[3] J. Beardwood, J. Halton, and J. M. Hammersley. The shortest path through many points. *Proceedings of the Cambridge Philos. Society*, 55:299–327, 1959.

[4] J. W. Berry and M. K. Goldberg. Path optimization for graph partitioning problems. *Discrete Applied Mathematics. Combinatorial Algorithms, Optimization and Computer Science*, 90(1-3):27–50, 1999.

[5] S. N. Bhatt and F. T. Leighton. A framework for solving VLSI graph layout problems. *Journal of Computer and System Sciences*, 28:300–343, 1984.

[6] B. Bollobas and I. Leader. Compressions and isoperimetric inequalities. *Journal of Combinatorial Theory Series A*, 56:47–62, 1991.

[7] R. A. Botafogo. Cluster analysis for hypertext systems. In *16th Annual ACM SIGIR Conference on Research and Development in Information Retrieval*, pages 116–125, 1993.

[8] K. L. Chung. *A Course in Probability Theory.* Academic Press, New York, 1974.

[9] J. Díaz, M. D. Penrose, J. Petit, and M. Serna. Layout problems on lattice graphs. In T. Asano, H. Imai, D. T. Lee, S. Nakano, and T. Tokuyama, editors, *Computing and Combinatorics — 5th Annual International Conference, COCOON'99, Tokyo, Japan, July 26-28, 1999*, number 1627. Springer, Lecture Notes in Computer Science, July 1999.

[10] J. Díaz, M. D. Penrose, J. Petit, and M. Serna. Linear orderings of random geometric graphs. In ?, editor, *WG99*, number To Appear. Springer, Lecture Notes in Computer Science, ? 1999.

[11] J. Díaz, J. Petit, and M. Serna. Random geometric problems on $[0,1]^2$. In M. Luby, J. Rolim, and M. Serna, editors, *Randomization and Approximation Techniques in Computer Science*, number 1518 in Lecture Notes in Computer Science. Springer, Lecture Notes in Computer Science, October 1998.

[12] J. Díaz, J. Petit, M. Serna, and L. Trevisan. Approximating layout problems on random sparse graphs. Report de recerca LSI-98-44-R, Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya, July 1998. Accepted in the Fifth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications. Discrete Mathematics, to appear.

[13] N. Dunford and J. Schwartz. *Linear Operators. Part I: General Theory.* Interscience Publisher, New York, 1958.

[14] G. Grimmett. *Percolation. (2nd edition).* Springer-Verlag, Heidelberg, 1999.

[15] L. H. Harper. Stabilization and the edgesum problem. *Ars Combinatoria*, 4:225–270, December 1977.

[16] D. S. Johnson, C. R. Aragon, L. A. McGeoch, and C. Schevon. Optimization by simulated annealing: an experimental evaluation; part I, graph partitioning. *Operations Research*, 37(6):865–892, November 1989.

[17] M. Junguer, G. Reinelt, and G. Rinaldi. The travelling salesman problem. In C. L. Monma M. Ball, T. Magnanti and G. L. Nemhauser, editors, *Handbook on Operations Research and Management Sciences*, volume 7, pages 225–330. North–Holland, 1995.

[18] R. M. Karp. Mapping the genome: some combinatorial problems arising in molecular biology. In *24th ACM Symposium on the Theory of Computing*, pages 278–285, 1993.

[19] D. G. Kendall. Incidence matrices, interval graphs, and seriation in archeology. *Pacific Journal of mathematics*, 28:565–570, December 1969.

[20] H. Kesten. *Percolation theory for mathematicians.* Birkhäuser Boston, Mass., 1982.

[21] K. Lang and S. Rao. Finding Near-Optimal Cuts: An Empirical Evaluation. In *Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 212–221, 1993.

[22] F. T. Leighton. *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes.* Morgan Kaufmann, San Mateo, CA, 1993.

[23] F. Makedon and I. H. Sudborough. Minimizing width in linear layouts. In J. Díaz, editor, *Proc. 10th. International Colloquium on Automata, Languages and Programming*, pages 478–490. Springer-Verlag, Lectures Notes in Computer Science, 1983.

[24] G. Mitchison and R. Durbin. Optimal numberings of an $n \times n$ array. *SIAM Journal on Discrete Mathematics*, 7(4):571–582, 1986.

[25] D. O. Muradyan and T. E. Piliposjan. Minimal numberings of vertices of a rectangular lattice. *Akad. Nauk. Armjan. SRR*, 1(70):21–27, 1980. In Russian.

[26] P. Mutzel. A polyedral approach to planar augmentation and related problems. In P. Spirakis, editor, *European Symposium on Algorithms*, volume 979 of *Lecture Notes in Computer Science*, pages 497–507. Springer-Verlag, 1995.

[27] J. Pach, F. Shahrokhi, and M. Szegedy. Applications of the crossing number. *Algoritmica*, 16:111–117, 1996.

[28] C. H. Papadimitriou and M. Sideri. The bisection width of grid graphs. In *First ACM-SIAM Symposium on Discrete Algorithms*, pages 405–410, San Francisco, 1990.

[29] M. D. Penrose. Single linkage clustering and continuum percolation. *Journal of Multivariate Analysis*, 53:94–109, 1995.

[30] M. D. Penrose. Vertex ordering and partition problems for random spatial graphs. Technical report, To appear in the Annals of Applied Probability, 1999.

[31] J. Petit. Approximation Heuristics and Benchmarkings for the MinLA Problem. In R. Battiti and A. Bertossi, editors, *Alex '98 — Building bridges between theory and applications*, http://www.lsi.upc.es/~jpetit/MinLA/Benchmark, February 1998. Università di Trento.

[32] J. Petit. Approximation Heuristics and Benchmarkings for the MinLA Problem. In Roberto Battiti and Alan A Bertossi, editors, *Alex '98 — Building bridges between theory and applications* . Università di Trento, February 1998.

[33] R. Ravi, A. Agrawal, and P. Klein. Ordering problems approximated: single-processor scheduling and interval graph completition. In M. Rodriguez J. Leach, B. Monien, editor, *18th. International Colloquium on Automata, Languages and Programming*, volume 510 of *Lecture Notes in Computer Science*, pages 751–762. Springer-Verlag, 1991.

[34] Neil Robertson and P. D. Seymour. Graph minors—a survey. In *Surveys in combinatorics 1985 (Glasgow, 1985)*, pages 153–171. Cambridge Univ. Press, Cambridge, 1985.

[35] Y. Saad. *Iterative methods for sparse linear systems.* PWS Publishing Company, Boston, 1996.

[36] M. T. Shing and T. C. Hu. Computational complexity of layout problems. In T. Ohtsuki, editor, *Layout design and verification*, pages 267–294, Amsterdam, 1986. North-Holland.

[37] J. M. Steele. *Probability theory and Combinatorial Optimization.* SIAM CBMS-NSF Regional Conference Series in Applied Mathematics, 1997.

[38] J. E. Yukich. *Probability Theory of classical Euclidian Optimization Problems.* Springer–Verlag, Heidelberg, 1998.