**Excercises Valtice Oct. 09**

1. Prove that $\mathbf{ians}(G) = \mathbf{iams}(G^n)$ where $G^n$ is the graph obtained from $G$ by duplicating all its edges.

2. Prove that $\mathbf{iaes}(G) = \mathbf{iams}(G^e)$ where $G^n$ is the graph obtained from $G$ by subdividing all its edges.

3. Prove that $\mathbf{ians}(G) = \mathbf{iaes}(G^t) - 1$ where $G^t$ is the graph obtained from $G$ by triplicating all its edges.

4. Given a graph $G$ and a set $X \subseteq E(G)$, define $\alpha : 2^{E(G)} \rightarrow \mathbb{N}$ such that

$$\alpha(X) = (\cup_{e \in X}) \cap (\cup_{e \in E(G) \setminus X} e).$$

   Prove that $\alpha$ is a submodular function.

5. Given a graph $G$ and a set $X \subseteq V(G)$, define $\beta : 2^{V(G)} \rightarrow \mathbb{N}$ such that

$$\beta(X) = |\{e \in E(G) \mid e \cap X \neq \emptyset \text{ and } (V(G) \setminus X) \cap e \neq \emptyset\}|.$$

   Prove that $\beta$ is a submodular function.

6. A graph $G$ has *linear-width* at most $k$ ($\mathbf{lw}(G) \leq k$) if there exists a linear ordering of its edges $e_1, \ldots, e_m$ such that $\forall_{i \in \{1,\ldots,m\}}\ \alpha(\{e_1, \ldots, e_i\}) \leq k$. Prove that if $\delta(G) \geq 2$ then $\mathbf{lw}(G) = \mathbf{iams}(G)$. What is the maximum difference between $\mathbf{lw}(G)$ and $\mathbf{iams}(G)$?

7. Find $\mathbf{vans}(Q_3)$ (or equivalently $\mathbf{ilns}(Q_3)$) where $Q_3$ is the 3-dimensional cube.

8. Consider the node search game for an invisible lazy fugitive of speed 1 (i.e. each time the fugitive is threatened, he/she moves to a neighboring vertex) and denote by $\mathbf{ilns}^1$ the corresponding search parameter. Prove that $\mathbf{ilns}_1 = \delta^*(G)$ where $\delta^*$ is the degeneracy parameter defined as follows

$$\delta^*(G) = \max\{\delta(H) \mid H \text{ is a subgraph of } G\}$$

9. Prove that if we change the game of the previous exercise so that the fugitive is visible, the resulting search parameter remains the same (i.e. $\mathbf{vlns}^1 = \mathbf{ilns}^1$).

10. Prove that $\mathbf{cmvans}$ is closed under contractions, i.e. if $H$ is taken after applying edge contractions to $G$, then $\mathbf{cmvans}(H) \leq \mathbf{cmvans}(G)$. Is $\mathbf{cmvans}$ closed under subgraphs?

11. Prove the claim of page 94 in the slides of the tutorial.

12. If $\mathbf{ilns}(G) \geq k$, then $G$ contains a path of length $k$ as a subgraph.