Ferrer i Cancho, R. & Hernández Fernández, A. (2008). Power laws and the golden number. In: "Problems of general, germanic and slavic linguistics", Altmann, G., Zadorozhna, I. & Matskulyak, Y. (eds.), Chernivtsi: Books - XXI. pp. 518-523.

# Power laws and the golden number

Ramon Ferrer i Cancho [a],*

[a] Departament de Física Fonamental, Universitat de Barcelona.
Martí i Franquès 1, 08028 Barcelona, Spain.

Antoni Hernández Fernández [b]

[b]Departament de Lingüística General. Universitat de Barcelona. Gran Via de les Corts Catalanes, 585. 08007 Barcelona, Spain.

**Abstract**

The distribution of many real discrete random variables (e.g., the frequency of words, the population of cities) can be approximated by a zeta distribution, that is known popularly as Zipf's law, or power law in physics. Here we revisit the relationship between power law distribution of a magnitude and the corresponding power relationship between the magnitude of a certain element and its rank. We show that the exponents of the two power laws coincide when its value is the famous golden number, $\varphi = \left(1+\sqrt{5}\right)/2$.

**Key words**: Zipf's law, power laws, zeta distribution, Golden number and Fibonacci series.

## 1 Introduction

A discrete random variable *x* (such that $x \geq 1$) follows a zeta distribution (Wimmer and Altmann, 1999) if

$$P(x) = b(\beta) x^{-\beta}, \qquad (1)$$

where $b(\beta)$ is a normalization function and $\beta$ is the so-called exponent.

---

* Corresponding author. Phone: +34 934137870. Fax: +34 934137787.
Email address: rferrericancho@lsi.upc.edu (Ramon Ferrer i Cancho).

We have that $b(\beta) = \zeta(\beta)^{-1}$, where

$$\zeta(\beta) = \sum_{x=1}^{\infty} x^{-\beta} \tag{2}$$

is the Riemann zeta function. Therefore, the exponent $\beta$ is the only parameter of the distribution. Since $\zeta(\beta)$ converges only for $\beta>1$, Eq. 1 is only well-defined when $\beta>1$. Hereafter we assume $\beta>1$. Examples of real discrete random variables following this distribution are word frequencies and the population size of cities (Gisiger, 2001; Newman, 2005). The zeta distribution receives equivalent names: power law distribution in physics or Zipf's law in general (Newman, 2005). See Newman (2005); Ferrer i Cancho and Servedio(2005); Simkin and Roychowdhury (2006); Mitzenmacher (2003) for a review of the most popular explanations of the zeta distribution.

If the discrete random variable $x$ is a frequency of occurrence (e.g., the frequency of a word), then its probability distribution is called a frequency spectrum (Tuldava, 1996). Historically, besides considering the probability of a discrete magnitude $x$, researchers have also been concerned about the relationship between $x$ or the normalized value of $x$ versus rank (Zipf, 1972; Wimmer et al., 1999). Imagine that we have a set of elements (e.g., words) that are characterized by a certain magnitude $x$ (e.g., their frequency of occurrence in a text). Imagine that we sort these elements decreasingly by their selected magnitude (e.g., decreasingly by their frequency). The element with the largest value of $x$ is assigned rank $i$=1, the element with the second largest value of $x$ is assigned rank $i$=2 and so on. We define $x(i)$ as the value of $x$ of the element of rank $i$ (e.g., $x(i)$ is the frequency of the $i$-th most frequent word of a text). It is known that if $x$ is zeta distributed then $x(i)$ obeys (Chitashvili and Baayen, 1993; Adamic, 2000; Pietronero et al., 2001; Adamic and Huberman, 2002)

$$x(i) \sim i^{-\alpha}. \tag{3}$$

The organization of the remainder of this article is as follows. Firstly, we will review how $b(\beta)$ can be calculated (Section 2). This will provide us with some basic strategies for the next step. Secondly, we will review the relationship between the exponents $\alpha$ and $\beta$ (Section

3). Thirdly, we will show that both exponents equate when their value is the golden number (Section 4). Finally, we will discuss the finding (Section 5).

## 2 Calculation of *b*(*β*)

For certain natural values of *β*, exact values of *b*(*β*) can be obtained. It is known that (Spiegel and Liu, 1999)

$$\zeta(2k) = \frac{2^{k-1}\pi^{2k}B_k}{(2k)!}, \tag{4}$$

where $k$=1,2,3,... and $B_k$ is the $k$-th Bernoulli number. Thus, trivially

$$b(2k) = \frac{(2k)!}{2^{k-1}\pi^{2k}B_k}, \tag{5}$$

with $k$=1,2,3,.... Recalling $B_1$=1/6, $B_2$=1/30 and $B_3$=1/42 and Eq. 2, we obtain for example $b(2) = 6/p^2$, $b(4) = 90/p^4$ and $b(6) = 945/p^6$.

As for other values of *β*, we can find tight bounds using integrals. Knowing that any (integrable) monotonically decreasing function $f(k)$ satisfies (Cormen et al., 1990)

$$\int_{m}^{n+1} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)dx, \tag{6}$$

we obtain

$$b(\beta) \le \beta\text{-}1 \tag{7}$$

for $\beta$>1. Notice that we cannot obtain a lower bound for $b(\beta)$ using Eq. 6 because $x \ge 1$. As for a lower bound for $b(\beta)$, it is easy to see that if $f(x)$ is an integrable monotonically decreasing function we have that

$$\sum_{k=m}^{n} f(k) \le f(m) + \int_{m}^{n} f(x)dx \tag{8}$$

and thus

$$b(\beta) \ge 1 - \frac{1}{\beta}. \tag{9}$$

In sum,

$$1 - \frac{1}{\beta} \le b(\beta) \le \beta - 1. \tag{10}$$

# 3 The equivalence between $\alpha$ and $\beta$

The relationship between the exponents $\beta$ and $\alpha$ is well-known (Chitashvili and Baayen, 1993; Mandelbrot, 1997; Adamic, 2000; Pietronero et al., 2001; Adamic and Huberman, 2002)

$$\beta = \frac{1}{\alpha} + 1 \tag{11}$$

for $\beta$>1.

Now we provide a simple but not too simplified derivation of the previous equation (something in-between the sophisticated maths of (Chitashvili and Baayen, 1993; Mandelbrot, 1997) and the informal approaches of (Adamic, 2000; Pietronero et al., 2001; Adamic and Huberman, 2002)). We focus on the estimation of $x(i)$ from a sample of $T$ occurrences of elements knowing that these elements are distributed according to $P(x)$ (Eq. 1) with $\beta$>1. $N(x)=TP(x)$ is the expected number of different elements whose magnitude is $x$ in a sample of $T$ occurrences distributed according to $P(x)$ (Eq. 1). The expected smallest rank of an element of magnitude $x$ is

$$i_{\min}(x) = 1 + \sum_{x'=x+1}^{\infty} N(x') \tag{12}$$

and the expected largest rank is

$$i_{\max}(x) = \sum_{x'=x}^{\infty} N(x'). \tag{13}$$

Imagine that we assign an arbitrary rank within the interval [$i_{min}$(x), $i_{max}$($x$)], to each of the $N$($x$) elements of magnitude $x$. Then, the mean rank of elements of frequency $x$ is

$$\bar{i}(x) = \frac{1}{N(x')} \sum_{j=1}^{N(x)} \left( j + \sum_{x'=x+1}^{\infty} N(x') \right). \tag{14}$$

Knowing that (Spiegel and Liu, 1999)

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}, \tag{15}$$

we obtain

$$\bar{i}(x) = \frac{N(x)+1}{2} + \sum_{x'=x+1}^{\infty} N(x') \tag{16}$$

after some algebra.

Applying the integral bounds of summations given in Eq. 6 to Eq. 16 with $\beta$>1 we obtain, after some work,

$$\bar{i}(x) \geq \frac{1}{2}\left[c(\beta)\left((x+1)^{1-\beta} + x^{1-\beta}\right) + 1\right] \tag{17}$$

and

$$\bar{i}(x) \leq \frac{1}{2}\left[c(\beta)\left(x^{1-\beta} + (x-1)^{1-\beta}\right) + 1\right], \tag{18}$$

where

$$c(\beta) = \frac{Tb(\beta)}{\beta - 1} \tag{19}$$

Knowing that $x^{1-\beta} + (x-1)^{1-\beta} \le 2(x-1)^{1-\beta}$ and $(x+1)^{1-\beta} + x^{1-\beta} \ge 2(x+1)^{1-\beta}$ we can rewrite Eq. 18 as

$$\bar{i}(x) \ge c(\beta)(x+1)^{1-\beta} + \frac{1}{2} \tag{20}$$

$$\bar{i}(x) \le c(\beta)(x-1)^{1-\beta} + \frac{1}{2}. \tag{21}$$

Writing $x$ as a function of $\bar{i}$ in the two previous Eqs. we obtain

$$x(\bar{i}) = \left[\frac{1}{c(\beta)}\left(\bar{i} - \frac{1}{2}\right)\right]^{-\frac{1}{\beta-1}} \pm 1. \tag{22}$$

Put differently, in the previous Eq. we have derived the relationship between the magnitude $x$ and the mean rank of elements whose magnitude is zeta distributed. Interestingly, our error in the value of such magnitude is of only ±1, which can be neglected for large $\bar{i}$.
Applying the method above, we also obtain

$$\left[\frac{1}{c(\beta)}(i_{\min} - 1)\right]^{-\frac{1}{\beta-1}} - 1 \le x(i_{\min}) \le \left[\frac{1}{c(\beta)}(i_{\min} - 1)\right]^{-\frac{1}{\beta-1}} \tag{23}$$

and

$$\left(\frac{1}{c(\beta)} i_{\max}\right)^{-\frac{1}{\beta-1}} \le x(i_{\max}) \le \left(\frac{1}{c(\beta)} i_{\max}\right)^{-\frac{1}{\beta-1}} + 1. \tag{24}$$

# 4 The golden number

Equating the l.h.s. of Eq. 24 and the r.h.s. of Eq. 3 we obtain

$$\alpha = \frac{1}{\beta - 1} \tag{25}$$

for $\beta > 1$. Similarly, comparing Eq. 22 with Eq. 3, we obtain that Eq. 25 also holds approximately.

Knowing Eq. 25, $\beta$ equates $\alpha$ when $\beta = 1/(\beta - 1)$, which leads to the quadratic equation

$$\beta^2 - \beta - 1 = 0. \tag{26}$$

The previous Eq. has two solutions, i.e.

$$\beta = \frac{1 \pm \sqrt{5}}{2} \tag{27}$$

of which the negative must be discarded for two reasons. First, Eq. 25 is obtained assuming $\beta > 1$. Second, $x(i)$ is, by definition, monotonically decreasing. The positive solution is $\varphi = \left(1 + \sqrt{5}\right)/2$, the famous golden number (Ghyka, 1977; Walser, 2001).

# 5 Discussion

We have seen that the exponent $\varphi$ is the value where the exponents of the probability distribution of a discrete magnitude and that of the value of the magnitude versus its rank coincide. This is one among many contexts in which the golden number appears (Ghyka, 1977; Walser, 2001). Probably, one of the most famous places where this number appears is the Fibonacci series, that is defined by the recurrence relation

$$F_{n+1} = F_{n-1} + F_n \tag{28}$$

for $n \geq 2$ with $F_0=0$ and $F_1=1$. The begining of the series is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811,... It is well-known that the golden number is the limit ratio between two consecutive numbers of the series, i.e.

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \varphi \qquad (29)$$

At this time, we believe that it cannot yet be said if our discovery of $\varphi$ is just a mathematical curiosity a the beginning of a series of discoveries that may change the way in which scientists study power laws in mathematical (Mitzenmacher, 2003; Mandelbrot, 1997) and natural sciences (Gisiger, 2001; Stanley, 1999; Newman, 2005). We hope that our finding stimulates further research.

## Acknowledgements

RFC's work was funded by the projects FIS2006-13321-C02 and BFM2003-08258-C02-02 of the Spanish Ministry of Education and Science under a Juan de la Cierva contract from the Spanish Ministry of Education and Science.

It is strictly prohibited to use, to investigate or to develop, in a direct or indirect way, any of the scientific contributions of the author contained in this work by any army or armed group in the world, for military purposes and for any other use which is against human rights or the environment, unless a written consent of all the persons in the world is obtained.

## References

Adamic, L., (2000). Zipf, power-law, Pareto - a ranking tutorial. Available from: http://www.hpl.hp.com/research/idl/papers/ranking/.

Adamic, L. A., Huberman, B. A., (2002). Zipf's law and the internet. Glottometrics 3, 143–150.

Chitashvili, R. J., Baayen, R. H., (1993). Word frequency distributions. In: Altmann, G., Hřebíček, L. (Eds.), Quantitative Text Analysis. Wissenschaftlicher Verlag Trier,

Trier, pp. 54–135.

Cormen, T. H., Leiserson, C. E., Rivest, R. L., (1990). Introduction to algorithms. The MIT Press, Cambridge, MA.

Ferrer i Cancho, R., Servedio, V., (2005). Can simple models explain Zipf's law for all exponents? Glottometrics 11, 1–8.

Ghyka, M., (1977). The geometry of art and life. Dover, New York, 1st appeared as Ghyka, C. M. (1927). Esthétique des proportions dans la nature et dans les arts. Paris: Gallimard.

Gisiger, T., (2001). Scale invariance in biology: coincidence of footprint of a universal mechanism. Biol. Rev. 76, 161–209.

Mandelbrot, B., (1997). Fractals and scaling in finance: discontinuity, concentration, risk. Springer, New York, Ch. Rank-size plots, Zipf's law, and scaling.

Mitzenmacher, M., (2003). A brief history of generative models for power law and lognormal distributions. Internet Mathematics 1, 226–251.

Newman, M. E. J., (2005). Power laws, Pareto distributions and Zipf's law. Contemporary Physics 46, 323–351.

Pietronero, L., Tosatti, E., Tosatti, V., Vespignani, A., (2001). Explaining the uneven distribution of numbers in nature: the laws of Benford and Zipf. Physica A 293, 297–304.

Simkin, M. V., Roychowdhury, V. P., (2006). Re-inventing Willis. physics/0601192.

Spiegel, M., Liu, J., (1999). Mathematical handbook of formulas and tables. McGraw-Hill, New York, 2nd edition.

Stanley, E., (1999). Scaling, universality and renormalization: three pillars of modern critical phenomena. Reviews of Modern Physics 71 (2), S358–S366.

Tuldava, J., (1996). The frequency spectrum of text and vocabulary. J. Quantitative Linguistics 3 (1), 38–50.

Walser, F., (2001). The golden section. The Mathematical Association of America, Washington.

Wimmer, G., Altmann, G., 1(999). Thesaurus of univariate discrete probability distributions. Stamm, Germany.

Wimmer, G., Sidlik, P., Altmann, G., (1999). A new model of rank-frequency distribution. Journal of quantitative linguistics 6, 188–193.

Zipf, G. K., 1972. Human behaviour and the principle of least effort. An introduction to human ecology. Hafner reprint, New York, 1st edition: Cambridge, MA: Addison-Wesley, 1949.