E-matching for Fun and Profit

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Instantiation

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How do we figure the correct instantiation out?

Triggers (an idea borrowed from Simplify)

- triggers are subterms present in the quantified formula, which can be automatically generated, or come with the input
- we only consider instances, for which the trigger after substitution is present somewhere in the model returned by the SAT solver

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- triggers are subterms present in the quantified formula, which can be automatically generated, or come with the input
- we only consider instances, for which the trigger after substitution is present somewhere in the model returned by the SAT solver
- for example if we have a formula

$$\psi \equiv \forall x, y.\, F(x, y) \Rightarrow g(x) = h(y)$$

for which $F(x, y)$ is a trigger and the current model returned by the SAT solver is:

$$\psi \wedge F(1, c) \wedge g(7) = h(c)$$

we only consider instance where $x \to 1, y \to c$, because $F(1, c)$ is present in the monome.

E–matching

- this picture with the triggers is slightly more involved, because we are interested in the trigger being present in the monome *up to equivalence*
- for example trigger $F(x, c)$ does not syntactically match in $F(1, d) \ \wedge \ c = d$, but we would like it to

E–matching: definition

Input

- a finite set $\mathcal{A}$ of *active* ground terms,
- a relation $\cong_g \subseteq \mathcal{A} \times \mathcal{A}$,
- a finite set of non-variable, non-constant *triggers* $p_1, \ldots, p_n$.

Definitions

- let $\cong \subseteq \mathcal{T} \times \mathcal{T}$ be the smallest congruence relation containing $\cong_g$,
- let $\mathtt{root}(t)$ denote a canonical representative of equivalence class containing t.

The solution to the E-matching problem is the set:

$$T = \left\{ \sigma \;\middle|\; \begin{array}{l} \exists t_1, \ldots, t_n \in \mathcal{A}.\, \sigma(p_1) \cong t_1 \wedge \cdots \wedge \sigma(p_n) \cong t_n, \\ \forall x \in \mathcal{V}.\, \sigma(x) = \mathtt{root}(\sigma(x)) \end{array} \right\}$$

E–matching: complexity

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- the problem of checking if $T \neq \emptyset$ for a fixed $\mathcal{A}$ and $\cong_g$ is NP-hard
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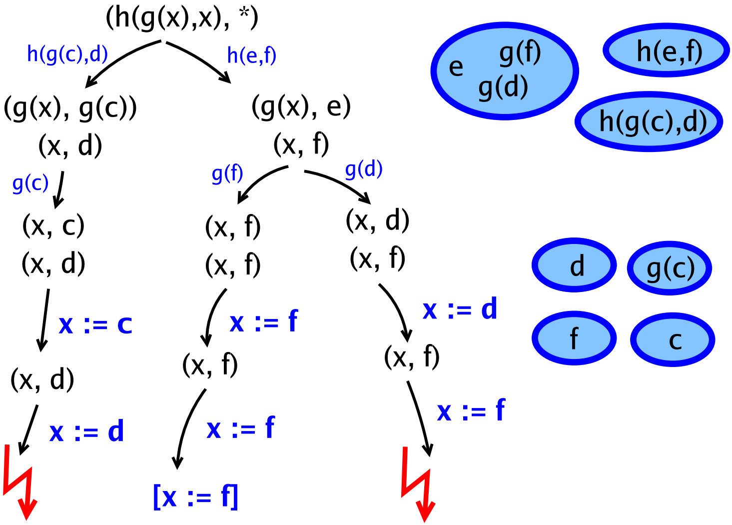
E–matching: complexity

- the problem of checking if $T \neq \emptyset$ for a fixed $\mathcal{A}$ and $\cong_g$ is NP-hard
- there can be exponential number of instances of a trigger
- the practical problem is, however, that there are often millions of matching problems to solve during solving of a single SMT query

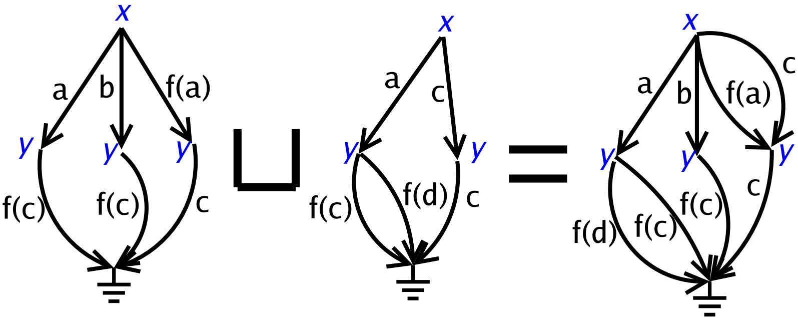
Simplify's matcher example



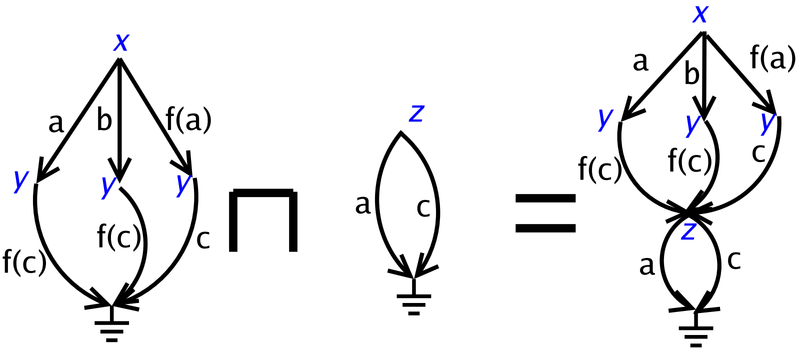
Simplify's algorithm

$$
\begin{array}{l}
\textbf{fun } \mathit{simplify\_match}([p_1, \ldots, p_n]) \\
\quad R := \emptyset \\
\quad \textbf{proc } \mathit{match}(\sigma, j) \\
\qquad \textbf{if } j = \mathit{nil} \textbf{ then } R := R \cup \{\sigma\} \\
\qquad \textbf{else case } \mathit{hd}(j) \textbf{ of} \\
\qquad\quad (c, t) \Rightarrow \qquad /\!*\ 1 *\!/ \\
\qquad\qquad \textbf{if } c \cong t \textbf{ then } \mathit{match}(\sigma, \mathit{tl}(j)) \\
\qquad\qquad \textbf{else skip} \\
\qquad\quad (x, t) \Rightarrow \qquad /\!*\ 2 *\!/ \\
\qquad\qquad \textbf{if } \sigma(x) = x \textbf{ then } \mathit{match}(\sigma[x := \mathtt{root}(t)], \mathit{tl}(j)) \\
\qquad\qquad \textbf{else if } \sigma(x) = \mathtt{root}(t) \textbf{ then } \mathit{match}(\sigma, \mathit{tl}(j)) \\
\qquad\qquad \textbf{else skip} \\
\qquad\quad (f(p_1, \ldots, p_n), t) \Rightarrow \qquad /\!*\ 3 *\!/ \\
\qquad\qquad \textbf{foreach } f(t_1, \ldots, t_n) \textbf{ in } \mathcal{A} \textbf{ do} \\
\qquad\qquad\quad \textbf{if } t = * \vee \mathtt{root}(f(t_1, \ldots, t_n)) = t \textbf{ then} \\
\qquad\qquad\qquad \mathit{match}(\sigma, (p_1, \mathtt{root}(t_1)) :: \cdots :: (p_n, \mathtt{root}(t_n)) :: \mathit{tl}(j)) \\
\quad \mathit{match}([], (p_1, *) :: \cdots :: (p_n, *) :: \mathit{nil}) \qquad /\!*\ 4 *\!/ \\
\quad \textbf{return } R
\end{array}
$$

S-tree sum

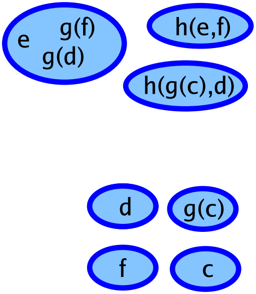


S-tree merge



Subtrigger matcher example, part I

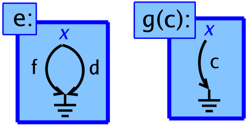
h(g(x),x) ->

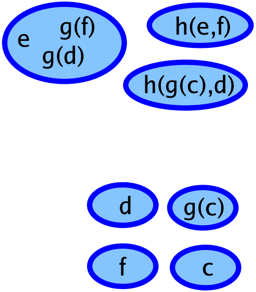


Subtrigger matcher example, part II

h(g(x),x) ->

g(x) ->

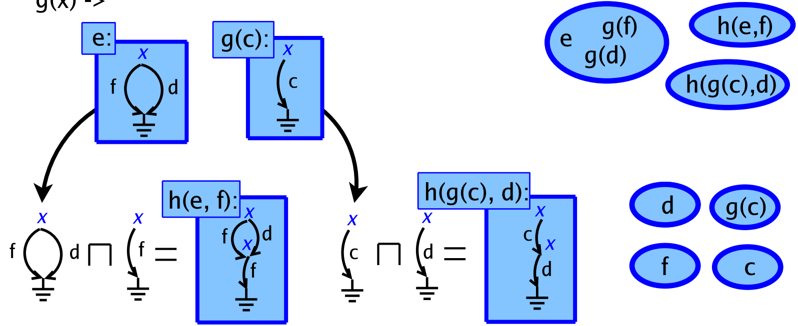




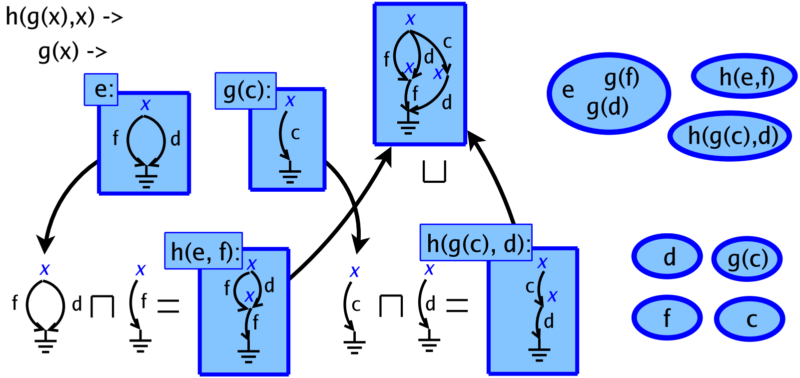
Subtrigger matcher example, part III

h(g(x),x) ->

g(x) ->



Subtrigger matcher example, part IV



Subtrigger matcher

fun *fetch*(S, t, p)
    if $S = \top$ **then return** $\{[p := \mathtt{root}(t)]\}$
    else if $S = \times \wedge t \cong p$ **then return** $\{[]\}$
    else if $S = \times$ **then return** $\emptyset$
    else return $S(\mathtt{root}(t))$
fun *match*(p)
    case p **of**
        $x \Rightarrow$ **return** $\top$
        $c \Rightarrow$ **return** $\times$
        $f(p_1, \ldots, p_n) \Rightarrow$
            foreach i **in** $1 \ldots n$ **do** $S_i = match(p_i)$ $/* \; 1 \; */$
            if $\exists i.\, S_i = \bot$ **then return** $\bot$ $/* \; 2 \; */$
            if $\forall i.\, S_i = \times$ **then return** $\times$ $/* \; 3 \; */$
            $S := \{t \mapsto \emptyset \mid t \in \mathcal{A}\}$
            foreach $f(t_1, \ldots, t_n)$ **in** $\mathcal{A}$ **do** $/* \; 4 \; */$
                $t := \mathtt{root}(f(t_1, \ldots, t_n))$
                $S := S[t \mapsto S(t) \sqcup (fetch(S_1, t_1, p_1) \sqcap \cdots \sqcap fetch(S_n, t_n, p_n))]$
          if $\forall t.\, S(t) = \bot$ **then return** $\bot$
          else return S

Subtrigger matcher, cont.

fun $\mathit{topmatch}(p)$ /* 5 */
 $S := \mathit{match}(p)$
 return $\bigsqcup_{t \in \mathcal{A}} S(t)$
fun $\mathit{subtrigger\_match}([p_1, \ldots, p_n])$
 return $\mathit{topmatch}(p_1) \sqcap \cdots \sqcap \mathit{topmatch}(p_n)$

Even simpler triggers

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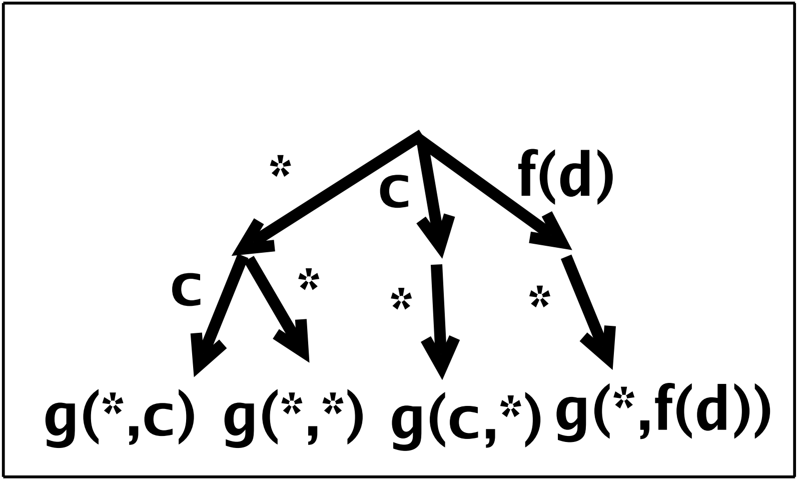
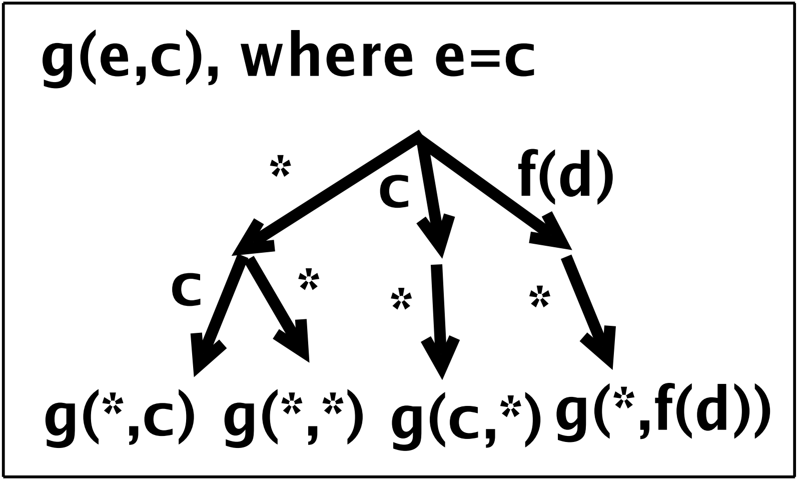
- even this algorithm was taking up a lot of time, mainly because the loop over all terms with given head is performed for each trigger
- but it seems that a lot (most?) of the triggers are even simpler – they have variables only at depth one: $f(X, Y)$, $f(c, X)$, $f(X, c, Y, g(g(d)), Z)$
- this means one can put all such *flat* triggers with head f in an indexing tree and match them all at once during one loop over terms with head f

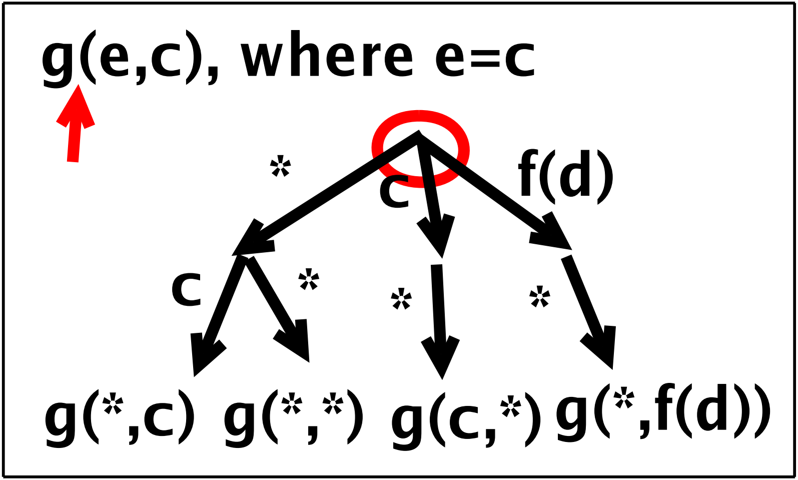
Flat trigger index



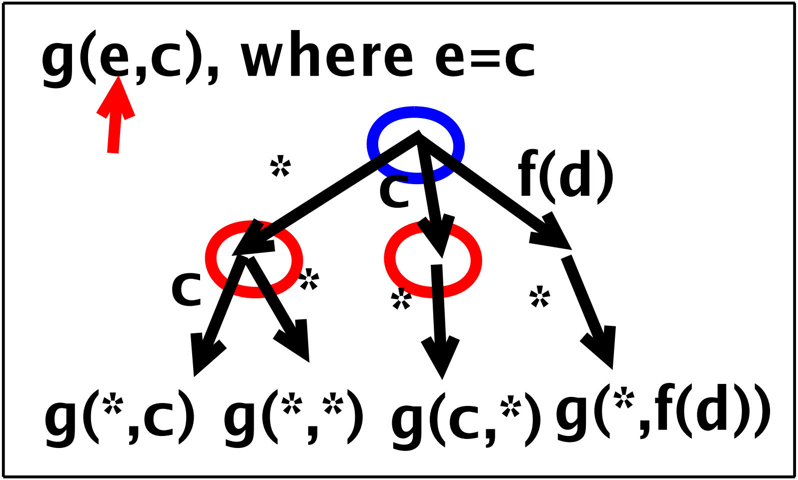
Let's match something!



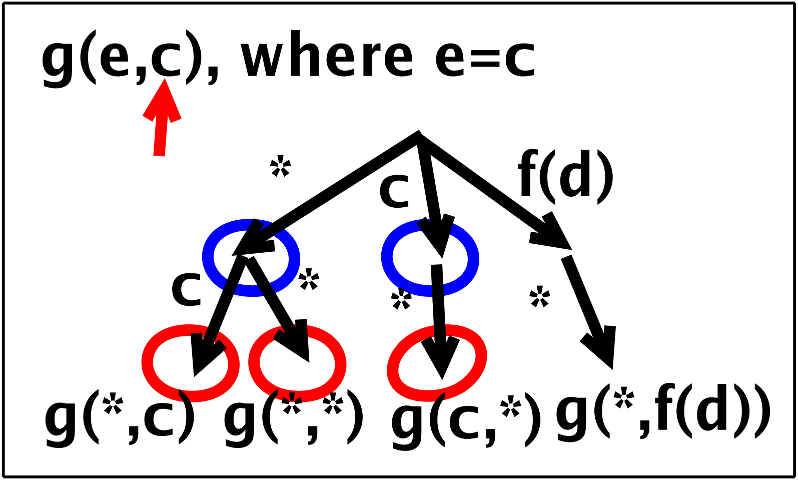
At the root



Child #1



Child #2



Memoization

- maximal sharing in terms and s-trees
- $\sqcup$, $\sqcap$ memoize results
- s-tree subtraction to remove previously returned results
- mapping of all the variables to $*$, to maximize sharing in subtrigger matcher
- mixed effects with *mod-time*

Explanation

- consider trigger: $f(g_1(x_1), \ldots, g_n(x_n))$
- $g_1(x_1)$ through $g_{n-1}(x_{n-1})$ return two matches each
- $g_n(x_n)$ does not match anything

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- $g_1(x_1)$ through $g_{n-1}(x_{n-1})$ return two matches each
- $g_n(x_n)$ does not match anything
- Simplify's matcher: $O(2^n)$ steps, subtrigger matcher: $O(n)$ steps
- even if $g_n(x_n)$ matches something, subtrigger will still do $O(n)$ steps to match, and only $O(2^n)$ much cheaper steps to walk s-tree

Thank you!

Questions?