E-matching for Fun and Profit

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Instantiation

Let’s take the formula:

\[ P(f(42)) \land \forall x. (P(f(x)) \Rightarrow x < 0) \]
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Once the solver figures out the implication:

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it can deduce the query to be unsatisfiable using only ground reasoning.
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**How do we figure the correct instantiation out?**
Triggers (an idea borrowed from Simplify)

- triggers are subterms present in the quantified formula, which can be automatically generated, or come with the input
- we only consider instances, for which the trigger after substitution is present somewhere in the model returned by the SAT solver
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- triggers are subterms present in the quantified formula, which can be automatically generated, or come with the input
- we only consider instances, for which the trigger after substitution is present somewhere in the model returned by the SAT solver
- for example if we have a formula

\[ \psi \equiv \forall x, y. \ F(x, y) \Rightarrow g(x) = h(y) \]

for which \( F(x, y) \) is a trigger and the current model returned by the SAT solver is:

\[ \psi \land F(1, c) \land g(7) = h(c) \]

we only consider instance where \( x \rightarrow 1, y \rightarrow c \), because \( F(1, c) \) is present in the monome.
E–matching

- this picture with the triggers is slightly more involved, because we are interested in the trigger being present in the monome up to equivalence
- for example trigger $F(x, c)$ does not syntactically match in $F(1, d) \land c = d$, but we would like it to
E-matching: definition

Input
- a finite set $\mathcal{A}$ of active ground terms,
- a relation $\equiv_g \subseteq \mathcal{A} \times \mathcal{A}$,
- a finite set of non-variable, non-constant triggers $p_1, \ldots, p_n$.

Definitions
- let $\equiv \subseteq \mathcal{T} \times \mathcal{T}$ be the smallest congruence relation containing $\equiv_g$,
- let $\text{root}(t)$ denote a canonical representative of equivalence class containing $t$.

The solution to the E-matching problem is the set:

$$
\mathcal{T} = \left\{ \sigma \mid \exists t_1, \ldots, t_n \in \mathcal{A}. \sigma(p_1) \equiv t_1 \land \cdots \land \sigma(p_n) \equiv t_n, \forall x \in \mathcal{V}. \sigma(x) = \text{root}(\sigma(x)) \right\}
$$
E-matching: complexity

- the problem of checking if $T \neq \emptyset$ for a fixed $A$ and $\simeq_g$ is NP-hard
- there can be exponential number of instances of a trigger
E-matching: complexity

- the problem of checking if $T \neq \emptyset$ for a fixed $\mathcal{A}$ and $\simeq_g$ is NP-hard
- there can be exponential number of instances of a trigger
- the practical problem is, however, that there are often millions of matching problems to solve during solving of a single SMT query
Simplify’s matcher example
fun simplify_match([p₁, . . . , pₙ])
  R := ∅
  proc match(σ, j)
    if j = nil then R := R ∪ {σ}
    else case hd(j) of
      (c, t) ⇒ /* 1 */
        if c ≡ t then match(σ, tl(j))
        else skip
      (x, t) ⇒ /* 2 */
        if σ(x) = x then match(σ[x := root(t)], tl(j))
        else if σ(x) = root(t) then match(σ, tl(j))
        else skip
      (f(p₁, . . . , pₙ), t) ⇒ /* 3 */
        foreach f(t₁, . . . , tₙ) in A do
          if t = * ∨ root(f(t₁, . . . , tₙ)) = t then
            match(σ, (p₁, root(t₁)) :: · · · :: (pₙ, root(tₙ)) :: tl(j))
          match([], (p₁, *) :: · · · :: (pₙ, *) :: nil) /* 4 */
    return R
S-tree sum

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Subtrigger matcher
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The matcher
Flat matcher
Implementation
Wrapping up
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S-tree merge
h(g(x),x) ->

Subtrigger matcher example, part I
Subtrigger matcher example, part II

\[ h(g(x), x) \rightarrow \]
\[ g(x) \rightarrow \]

```
\text{e:} \quad \text{g(c):}
```

```
\text{f} \quad \text{d}
```

```
\text{x}
```

```
\text{c}
```

```
\text{e} \quad \text{g(f)}
```

```
\text{h(e,f)}
```

```
\text{g(d)}
```

```
\text{h(g(c),d)}
```

```
\text{d}
```

```
\text{g(c)}
```

```
\text{f}
```

```
\text{c}
```
Subtrigger matcher example, part III

\[ h(g(x),x) \rightarrow g(x) \rightarrow \]

\[ e: \]
\[ g(c): \]
\[ h(e, f): \]
\[ h(g(c), d): \]
Subtrigger matcher example, part IV

$h(g(x), x) \Rightarrow g(x) \Rightarrow$

h(e, f):

h(g(c), d):

h(g(c), d):

h(g(c), d):
fun *fetch*(S, t, p)
    if S = ⊤ then return {[p := root(t)]}
    else if S = × ∧ t ≡ p then return {} 
    else if S = × then return ∅
    else return S(root(t))

fun *match*(p)
    case p of
    x ⇒ return ⊤
    c ⇒ return ×
    f(p₁, . . . , pₙ) ⇒
    foreach i in 1 . . . n do Sᵢ = match(pᵢ) /* 1 */
    if ∃i. Sᵢ = ⊥ then return ⊥ /* 2 */
    if ∀i. Sᵢ = × then return × /* 3 */
    S := {t ↦ ∅ | t ∈ A}
    foreach f(t₁, . . . , tₙ) in A do /* 4 */
    t := root(f(t₁, . . . , tₙ))
    S := S[t ↦ S(t) ⊔ (*fetch*(S₁, t₁, p₁) ∩ . . . ∩ *fetch*(Sₙ, tₙ, pₙ))]
    if ∀t. S(t) = ⊥ then return ⊥
    else return S
Subtrigger matcher, cont.

```
fun topmatch(p) /* 5 */
S := match(p)
return ⋃_{t ∈ A} S(t)

fun subtrigger_match([p_1, ..., p_n])
return topmatch(p_1) △ ... △ topmatch(p_n)
```
Even simpler triggers

- even this algorithm was taking up a lot of time, mainly because the loop over all terms with given head is performed for each trigger
Even simpler triggers

- even this algorithm was taking up a lot of time, mainly because the loop over all terms with given head is performed for each trigger
- but it seems that a lot (most?) of the triggers are even simpler – they have variables only at depth one: \( f(X, Y) \), \( f(c, X) \), \( f(X, c, Y, g(g(d))), Z \)
- this means one can put all such flat triggers with head \( f \) in an indexing tree and match them all at once during one loop over terms with head \( f \)
Flat trigger index

\[ g(\ast, c) \quad g(\ast, \ast) \quad g(c, \ast) \quad g(\ast, f(d)) \]
Let’s match something!

$g(e,c)$, where $e=c$

$$g(*,c) \quad g(*,*) \quad g(c,*) \quad g(*,f(d))$$
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At the root

\[ g(e, c), \text{ where } e = c \]

\[ g(*, c) \quad g(*, *) \quad g(c, *) \quad g(*, f(d)) \]
g(e, c), where e = c

\[ g(\ast, c) \quad g(\ast, \ast) \quad g(c, \ast) \quad g(\ast, f(d)) \]
g(e,c), where e=c

\[ g(\ast, c) \quad g(\ast, \ast) \quad g(c, \ast) \quad g(\ast, f(d)) \]
Memoization

- maximal sharing in terms and s-trees
- $\sqcup$, $\sqcap$ memoize results
- s-tree subtraction to remove previously returned results
- mapping of all the variables to $\ast$, to maximize sharing in subtrigger matcher
- mixed effects with $mod$-$time$
consider trigger: \( f(g_1(x_1), \ldots, g_n(x_n)) \)
- \( g_1(x_1) \) through \( g_{n-1}(x_{n-1}) \) return two matches each
- \( g_n(x_n) \) does not match anything
Explanation

- consider trigger: $f(g_1(x_1), \ldots, g_n(x_n))$
- $g_1(x_1)$ through $g_{n-1}(x_{n-1})$ return two matches each
- $g_n(x_n)$ does not match anything
- Simplify’s matcher: $O(2^n)$ steps, subtrigger matcher: $O(n)$ steps
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- \( g_n(x_n) \) does not match anything

- Simplify’s matcher: \( O(2^n) \) steps, subtrigger matcher: \( O(n) \) steps
- even if \( g_n(x_n) \) matches something, subtrigger will still do \( O(n) \) steps to match, and only \( O(2^n) \) much cheaper steps to walk s-tree