Applications of the Mellin-Perron Formula in Number Theory: Addendum Number III: Two Asymptotic Expansions from the Handbook of Algorithms and Data Structures

Marko R. Riedel

May 27, 2009

1 Introduction

This addendum to my MSc. thesis treats the following problem: find the complete asymptotic expansion of the following two finite sums:

$$\sum_{k=1}^{n-1} \sqrt{k(n-k)} \quad \text{and} \quad \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}}.$$

There are two parts to this document: first, we apply generalized Mellin summation to compute the asymptotics of these sums and second, we show how to compute the Mellin transforms involved.

References to my thesis will be provided throughout.

2 Generalized Mellin summation

We will be using generalized Mellin summation (harmonic sums) as described in section 2.10, page 87 of the thesis.

The functions $f_1(x)$ and $f_2(x)$ are given by

$$f_1(x) = H_0\left(\frac{x}{n}\right)\sqrt{x(n-x)} \quad \text{and} \quad f_2(x) = H_0\left(\frac{x}{n}\right)\frac{1}{\sqrt{x(n-x)}}.$$

where n is a positive integer and H_0 is the Heaviside-like step function that is 1 on the interval $[0,1)$ and zero elsewhere. Its Mellin transforms are

$$f_1^*(s) = \int_0^{+\infty} f_1(x)x^{s-1}dx = \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma(s+1/2)}{\Gamma(s+2)}n^{s+1}$$

and

$$f_2^*(s) = \int_0^{+\infty} f_2(x)x^{s-1}dx = \sqrt{\pi}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}n^{s-1}.$$

We take $F_{1,2}(x) = \sum_{k\geq 1} \lambda_k f_{1,2}(\mu_k x)$ with $\mu_k = k$ and $\lambda_k = 1$ so that

$$F_1(1) = \sum_{k\geq 1} f_1(k) = \sum_{k\geq 1} H_0\left(\frac{k}{n}\right)\sqrt{k(n-k)} = \sum_{k=1}^{n-1}\sqrt{k(n-k)}$$

and

$$F_2(1) = \sum_{k\geq 1} f_2(k) = \sum_{k\geq 1} H_0\left(\frac{k}{n}\right)\frac{1}{\sqrt{k(n-k)}} = \sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}}$$

and the Mellin transforms of $F_{1,2}(x)$ are

$$F_1^*(s) = \int_0^{+\infty} F_1(x)x^{s-1}dx = \frac{1}{2}\sqrt{\pi}\,\zeta(s)\frac{\Gamma(s+1/2)}{\Gamma(s+2)}n^{s+1}$$

and

$$F_2^*(s) = \int_0^{+\infty} F_2(x) x^{s-1} dx = \sqrt{\pi}\,\zeta(s) \frac{\Gamma(s-1/2)}{\Gamma(s)} n^{s-1}.$$

Applying Mellin inversion now yields

$$F_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{2}\sqrt{\pi}\,\zeta(s) \frac{\Gamma(s+1/2)}{\Gamma(s+2)} n^{s+1} x^{-s} ds$$

and

$$F_2(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sqrt{\pi}\,\zeta(s) \frac{\Gamma(s-1/2)}{\Gamma(s)} n^{s-1} x^{-s} ds.$$

Setting $x = 1$, we find

$$F_1(1) = \sum_{k=1}^{n-1} \sqrt{k(n-k)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{2}\sqrt{\pi}\,\zeta(s) \frac{\Gamma(s+1/2)}{\Gamma(s+2)} n^{s+1} ds$$

and

$$F_2(1) = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sqrt{\pi}\,\zeta(s) \frac{\Gamma(s-1/2)}{\Gamma(s)} n^{s-1} ds.$$

2.1 Computing the asymptotic expansions

To keep things simple, we introduce $G_{1,2}(s)$:

$$G_1(s) = \frac{1}{2}\sqrt{\pi}\,\zeta(s) \frac{\Gamma(s+1/2)}{\Gamma(s+2)} n^{s+1} \quad \text{and} \quad G_2(s) = \sqrt{\pi}\,\zeta(s) \frac{\Gamma(s-1/2)}{\Gamma(s)} n^{s-1}.$$

The poles are contributed by $\zeta(s)$ at $s = 1$ with residue 1 and $\Gamma(s)$ at $s = -m, m \geq 0$ with residue $\frac{(-1)^m}{m!}$. It follows that

$$\text{Res}\,[G_1(s);\ s = 1] = \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma(3/2)}{\Gamma(3)} n^2 = \frac{\pi}{8} n^2$$

and

$$\text{Res}\,[G_2(s);\ s = 1] = \sqrt{\pi}\,\frac{\Gamma(1/2)}{\Gamma(1)} = \pi.$$

Furthermore, with $m \geq 0$, we have

$$\text{Res}\,[G_1(s);\ s = -m-1/2] = \frac{1}{2}\sqrt{\pi}\,\zeta(-m-1/2) \frac{(-1)^m}{m!} \frac{1}{\Gamma(-m+3/2)} n^{1/2-m}.$$

This gives the following table of values for $G_1(s)$:

$$\begin{aligned}
\text{Res}\,[G_1(s);\ s = -1/2] &= \sqrt{n}\,\zeta(-1/2) \\
\text{Res}\,[G_1(s);\ s = -3/2] &= -\sqrt{n}\frac{1}{2}\frac{\zeta(-3/2)}{n} \\
\text{Res}\,[G_1(s);\ s = -5/2] &= -\sqrt{n}\frac{1}{8}\frac{\zeta(-5/2)}{n^2} \\
\text{Res}\,[G_1(s);\ s = -7/2] &= -\sqrt{n}\frac{1}{16}\frac{\zeta(-7/2)}{n^3} \\
\text{Res}\,[G_1(s);\ s = -9/2] &= -\sqrt{n}\frac{5}{128}\frac{\zeta(-9/2)}{n^4} \\
\text{Res}\,[G_1(s);\ s = -11/2] &= -\sqrt{n}\frac{7}{256}\frac{\zeta(-11/2)}{n^5} \\
\text{Res}\,[G_1(s);\ s = -13/2] &= -\sqrt{n}\frac{21}{1024}\frac{\zeta(-13/2)}{n^6}
\end{aligned}$$

$$\text{Res}\,[G_1(s);\ s=-15/2] \quad = \quad -\sqrt{n}\frac{33}{2048}\frac{\zeta(-15/2)}{n^7}.$$

Again with with $m \geq 0$, we have

$$\text{Res}\,[G_2(s);\ s=-m+1/2] = \sqrt{\pi}\,\zeta(-m+1/2)\frac{(-1)^m}{m!}\frac{1}{\Gamma(-m+1/2)}n^{-m-1/2}.$$

This gives the following table of values for $G_2(s)$:

$$\begin{aligned}
\text{Res}\,[G_2(s);\ s=1/2] &= \frac{1}{\sqrt{n}}\zeta(1/2)\\
\text{Res}\,[G_2(s);\ s=-1/2] &= \frac{1}{\sqrt{n}}\frac{1}{2}\frac{\zeta(-1/2)}{n}\\
\text{Res}\,[G_2(s);\ s=-3/2] &= \frac{1}{\sqrt{n}}\frac{3}{8}\frac{\zeta(-3/2)}{n^2}\\
\text{Res}\,[G_2(s);\ s=-5/2] &= \frac{1}{\sqrt{n}}\frac{5}{16}\frac{\zeta(-5/2)}{n^3}\\
\text{Res}\,[G_2(s);\ s=-7/2] &= \frac{1}{\sqrt{n}}\frac{35}{128}\frac{\zeta(-7/2)}{n^4}\\
\text{Res}\,[G_2(s);\ s=-9/2] &= \frac{1}{\sqrt{n}}\frac{63}{256}\frac{\zeta(-9/2)}{n^5}\\
\text{Res}\,[G_2(s);\ s=-11/2] &= \frac{1}{\sqrt{n}}\frac{231}{1024}\frac{\zeta(-11/2)}{n^6}\\
\text{Res}\,[G_2(s);\ s=-13/2] &= \frac{1}{\sqrt{n}}\frac{429}{2048}\frac{\zeta(-13/2)}{n^7}
\end{aligned}$$

Collecting all these values together, we get the following two asymptotic expansions:

$$\begin{aligned}
\sum_{k=1}^{n-1}\sqrt{k(n-k)} &= \frac{\pi}{8}n^2+\sqrt{n}\left(\zeta(-1/2)-\frac{1}{2}\frac{\zeta(-3/2)}{n}-\frac{1}{8}\frac{\zeta(-5/2)}{n^2}-\frac{1}{16}\frac{\zeta(-7/2)}{n^3}\right.\\
&\quad \left. -\frac{5}{128}\frac{\zeta(-9/2)}{n^4}-\frac{7}{256}\frac{\zeta(-11/2)}{n^5}-\frac{21}{1024}\frac{\zeta(-13/2)}{n^6}-\frac{33}{2048}\frac{\zeta(-15/2)}{n^7}-\cdots\right).
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} &= \pi+\frac{1}{\sqrt{n}}\left(\zeta(1/2)+\frac{1}{2}\frac{\zeta(-1/2)}{n}+\frac{3}{8}\frac{\zeta(-3/2)}{n^2}+\frac{5}{16}\frac{\zeta(-5/2)}{n^3}\right.\\
&\quad \left. +\frac{35}{128}\frac{\zeta(-7/2)}{n^4}+\frac{63}{256}\frac{\zeta(-9/2)}{n^5}+\frac{231}{1024}\frac{\zeta(-11/2)}{n^6}+\frac{429}{2048}\frac{\zeta(-13/2)}{n^7}+\cdots\right).
\end{aligned}$$

3 Details of the Mellin transform computation

Recall that the two Mellin transforms were

$$f_1^*(s) = \int_0^{+\infty} f_1(x)x^{s-1}dx = \int_0^n \sqrt{x(n-x)}x^{s-1}dx \quad\text{and}\quad f_2^*(s) = \int_0^{+\infty} f_2(x)x^{s-1}dx = \int_0^n \frac{1}{\sqrt{x(n-x)}}x^{s-1}dx.$$

Let $x = nq$ so that $dx = n\,dq$:

$$f_1^*(s) = n^{s+1}\int_0^1 \sqrt{q(1-q)}q^{s-1}dq \quad\text{and}\quad f_2^*(s) = n^{s-1}\int_0^1 \frac{1}{\sqrt{q(1-q)}}q^{s-1}dq.$$

Collecting terms, we find

$$f_1^*(s) = n^{s+1} \int_0^1 q^{s-1/2}(1-q)^{1/2} dq = n^{s+1} B(s+1/2, 3/2) \quad \text{and} \quad f_2^*(s) = n^{s-1} \int_0^1 q^{s-3/2}(1-q)^{-1/2} dq = n^{s-1} B(s-1/2, 1/2).$$

This finally yields

$$f_1^*(s) = n^{s+1} \frac{\Gamma(s+1/2)\Gamma(3/2)}{\Gamma(s+2)} = n^{s+1} \frac{1}{2} \sqrt{\pi} \frac{\Gamma(s+1/2)}{\Gamma(s+2)}$$

and

$$f_2^*(s) = n^{s-1} \frac{\Gamma(s-1/2)\Gamma(1/2)}{\Gamma(s)} = n^{s-1} \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)}.$$

4 Generalization to (almost) arbitrary exponents

Now suppose we want to find the asymptotic expansion of

$$\sum_{k=1}^{n-1} (k(n-k))^{-z} \quad \text{where} \quad z > 2, z \neq 2, 3, 4, \ldots$$

We use the function $f(x)$, which is given by

$$f(x) = H_0 \left(\frac{x}{n} \right) (x(n-x))^{-z}.$$

4.1 Computation of the Mellin transform

Recall that the Mellin transforms of $f(x)$ is

$$f^*(s) = \int_0^{+\infty} f(x) x^{s-1} dx = \int_0^n (x(n-x))^{-z} x^{s-1} dx.$$

Let $x = nq$ so that $dx = n\, dq$:

$$f^*(s) = n^{s-2z} \int_0^1 (q(1-q))^{-z} q^{s-1} dq.$$

Collecting terms, we find

$$f^*(s) = n^{s-2z} \int_0^1 q^{s-z-1}(1-q)^{-z} dq = n^{s-2z} B(s-z, 1-z).$$

This finally yields

$$f^*(s) = n^{s-2z} \frac{\Gamma(s-z)\Gamma(1-z)}{\Gamma(s+1-2z)}.$$

4.2 Computing the asymptotic expansion

We take $F(x) = \sum_{k \geq 1} \lambda_k f(\mu_k x)$ with $\mu_k = k$ and $\lambda_k = 1$ so that

$$F(1) = \sum_{k \geq 1} f(k) = \sum_{k \geq 1} H_0 \left(\frac{k}{n} \right) (k(n-k))^{-z} = \sum_{k=1}^{n-1} (k(n-k))^{-z}$$

and the Mellin transforms of $F(x)$ is

$$F^*(s) = \int_0^{+\infty} F(x) x^{s-1} dx = \zeta(s) n^{s-2z} \frac{\Gamma(s-z)\Gamma(1-z)}{\Gamma(s+1-2z)}.$$

Applying Mellin inversion now yields

$$F(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)\frac{\Gamma(s-z)\Gamma(1-z)}{\Gamma(s+1-2z)} n^{s-2z} x^{-s} ds.$$

Setting $x = 1$, we find

$$F(1) = \sum_{k=1}^{n-1} (k(n-k))^{-z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)\frac{\Gamma(s-z)\Gamma(1-z)}{\Gamma(s+1-2z)} n^{s-2z} ds.$$

To keep things simple, we introduce $G(s)$:

$$G(s) = \zeta(s)\frac{\Gamma(s-z)\Gamma(1-z)}{\Gamma(s+1-2z)} n^{s-2z}.$$

The poles are contributed by $\zeta(s)$ at $s = 1$ with residue 1 and $\Gamma(s)$ at $s = -m, m \geq 0$ with residue $\frac{(-1)^m}{m!}$. It follows that

$$\text{Res}\,[G(s);\; s = 1] = n^{1-2z}\frac{(\Gamma(1-z))^2}{\Gamma(2-2z)}.$$

Furthermore, with $m \geq 0$, we have

$$\text{Res}\,[G(s);\; s = z-m] = \zeta(z-m)\frac{(-1)^m}{m!}\frac{\Gamma(1-z)}{\Gamma(z-m+1-2z)} n^{z-m-2z} = \zeta(z-m)\frac{(-1)^m}{m!}\frac{\Gamma(1-z)}{\Gamma(1-z-m)} n^{-z-m}.$$

Now observe that

$$(-1)^m\frac{\Gamma(1-z)}{\Gamma(1-z-m)} = (-1)^m(-z)(-z-1)(-z-2)\cdots(-z-m+1) = z(z+1)(z+2)\cdots(z+m-1) = \frac{\Gamma(z+m)}{\Gamma(z)}$$

so that finally

$$\text{Res}\,[G(s);\; s = z-m] = n^{-z}\zeta(z-m)\frac{\Gamma(z+m)}{\Gamma(z)\, m!\, n^m}.$$

The conclusion is that

$$\begin{aligned}\sum_{k=1}^{n-1}(k(n-k))^{-z} &= n^{1-2z}\frac{(\Gamma(1-z))^2}{\Gamma(2-2z)} + n^{-z}\left(\zeta(z) + \zeta(z-1)\frac{z}{n} + \zeta(z-2)\frac{z(z+1)}{2\,n^2}\right.\\ &+ \left.\zeta(z-3)\frac{z(z+1)(z+2)}{6\,n^3} + \cdots + \zeta(z-m)\frac{\Gamma(z+m)}{\Gamma(z)\, m!\, n^m} + \cdots\right).\end{aligned}$$

Note that we can use the following identity to simplify the leading term:

$$\Gamma(2z) = (2\pi)^{-1/2}\, 2^{2z-1/2}\,\Gamma(z)\Gamma(z+1/2).$$

This gives

$$n^{1-2z}\frac{(\Gamma(1-z))^2}{\Gamma(2-2z)} = n^{1-2z}\frac{(\Gamma(1-z))^2}{(2\pi)^{-1/2}\, 2^{2(1-z)-1/2}\,\Gamma(1-z)\Gamma(3/2-z)} = n^{1-2z}\sqrt{\pi}\,\frac{\Gamma(1-z)}{2^{-1/2+2-2z-1/2}\,\Gamma(3/2-z)}$$

or

$$\left(\frac{n}{2}\right)^{1-2z}\sqrt{\pi}\,\frac{\Gamma(1-z)}{\Gamma(3/2-z)}.$$

5 External links

- Marko Riedel *http://www.geocities.com/markoriedelde/index.html Applications of the Mellin-Perron Formula in Number Theory.*
- Baeza-Yates *Handbook of Algorithms and Data Structures*