Applications of the Mellin-Perron Formula in Number Theory: Addendum Number II: asymptotics of the coefficients of a Dirichlet series

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1 Introduction

This addendum to my MSc. thesis treats the following problem: given the closed form of a Dirichlet series, how can we recover the asymptotics of the coefficients of the series?

There are two parts to this document: first, we apply generalized Mellin summation to obtain an integral formula for the coefficients and second, we use this to compute the asymptotics of the sum-of-divisors function.

References to my thesis will be provided throughout.

2 Generalized Mellin summation

We will be using generalized Mellin summation (harmonic sums) as described in section 2.10, page 87 of the thesis.

The function $f(x)$ is given by

$$f(x) = \delta(x-n)$$

where $\delta(x)$ is the Dirac delta function and n is a positive integer and its Mellin transform is

$$f^*(s) = \int_0^{+\infty} f(x) x^{s-1} dx = n^{s-1}.$$

We take $F(x) = \sum_k \lambda_k f(\mu_k x)$ with $\mu_k = k$ and let $\Lambda(s) = \sum_k \frac{\lambda(k)}{k^s}$. The Mellin transform of $F(x)$ is

$$F^*(s) = \Lambda(s) n^{s-1}.$$

Applying Mellin inversion now yields

$$F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Lambda(s) n^{s-1} x^{-s} ds.$$

Setting $x = 1$, we find

$$F(1) = \sum_k \lambda_k \delta(k-n) = \lambda_n$$

and hence

$$\lambda_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Lambda(s) n^{s-1} ds.$$

3 Asymptotics of the sum-of-divisors function

Let

$$\sigma_k(n) = \sum_{d|n} d^k \quad \text{and} \quad \Lambda_k(s) = \sum_n \frac{\sigma_k(n)}{n^s}.$$

Now $\sigma_0(n) = d(n)$. This immediately implies that

$$\Lambda_0(n) = \zeta^2(s).$$

In order to compute the asymptotics of the sum-of-divisors function σ_k we need a closed form of its Dirichlet series.

3.1 Dirichlet series of σ_k, $k > 0$

As σ_k is multiplicative, we compute the Dirichlet series $\Lambda_{k,p}$ for powers of a prime p. The product over all p of those series is what we are looking for.

$$\Lambda_{k,p} = 1 + \frac{1+p^k}{p^s} + \frac{1+p^k+p^{2k}}{p^{2s}} + \frac{1+p^k+p^{2k}+p^{3k}}{p^{3s}} + \cdots$$

or

$$\Lambda_{k,p} = \frac{p^k-1}{p^k-1} + \frac{p^{2k}-1}{p^k-1}\frac{1}{p^s} + \frac{p^{3k}-1}{p^k-1}\frac{1}{p^{2s}} + \frac{p^{4k}-1}{p^k-1}\frac{1}{p^{3s}} + \cdots$$

This yields

$$\Lambda_{k,p} = p^s \frac{1}{p^k-1}\left(\frac{p^k-1}{p^s} + \frac{p^{2k}-1}{p^{2s}} + \frac{p^{3k}-1}{p^{3s}} + \frac{p^{4k}-1}{p^{4s}} + \cdots\right)$$

or

$$\Lambda_{k,p} = p^s \frac{1}{p^k-1}\left(1 - 1 + \frac{1}{p^{s-k}} - \frac{1}{p^s} + \frac{1}{p^{2(s-k)}} - \frac{1}{p^{2s}} + \frac{1}{p^{3(s-k)}} - \frac{1}{p^{3s}} + \frac{1}{p^{4(s-k)}} - \frac{1}{p^{4s}} + \cdots\right)$$

which is

$$\Lambda_{k,p} = p^s \frac{1}{p^k-1}\left(\frac{1}{1-\frac{1}{p^{s-k}}} - \frac{1}{1-\frac{1}{p^s}}\right) = p^s \frac{1}{p^k-1} \frac{1-\frac{1}{p^s}-1+\frac{1}{p^{s-k}}}{\left(1-\frac{1}{p^{s-k}}\right)\left(1-\frac{1}{p^s}\right)}$$

and finally

$$\Lambda_{k,p} = \frac{1}{p^k-1}\frac{-1+p^k}{\left(1-\frac{1}{p^{s-k}}\right)\left(1-\frac{1}{p^s}\right)} = \frac{1}{\left(1-\frac{1}{p^{s-k}}\right)\left(1-\frac{1}{p^s}\right)}.$$

This immediately implies that

$$\Lambda_k(s) = \prod_p \frac{1}{\left(1-\frac{1}{p^{s-k}}\right)\left(1-\frac{1}{p^s}\right)} = \zeta(s)\zeta(s-k).$$

3.1.1 Alternate derivation

The above formula may also be derived by straightforwardly proceeding from the definition of σ_k.

$$\zeta(s)\zeta(s-k) = \sum_{p=1}^{\infty}\frac{1}{p^s}\sum_{q=1}^{\infty}\frac{q^k}{q^s} = \sum_{n=1}^{\infty}\sum_{pq=n}\frac{1}{p^s}\frac{q^k}{q^s} = \sum_{n=1}^{\infty}\sum_{q|n}\frac{q^k}{n^s} = \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{q|n}q^k = \Lambda_k(s).$$

3.2 Asymptotics of the sum-of-divisors function

We have

$$\operatorname{Res}\left[\zeta^2(s)n^{s-1};\ s=1\right] = \ln n + 2\gamma$$

and hence

$$d(n) \sim \ln n + 2\gamma.$$

Furthermore for $k > 0$, we have

$$\operatorname{Res}\left[\zeta(s)\zeta(s-k)n^{s-1};\ s=k+1\right] = \zeta(k+1)n^k$$

and

$$\operatorname{Res}\left[\zeta(s)\zeta(s-k)n^{s-1};\ s=1\right] = \zeta(1-k)$$

and hence

$$\sigma_k(n) \sim \zeta(k+1)n^k + \zeta(1-k).$$

The first few values are

$$\sigma_1(n) \quad \sim \quad \frac{\pi^2}{6}n - \frac{1}{2}$$

$$\begin{aligned}
\sigma_2(n) &\sim \zeta(3)n^2 - \frac{1}{12} \\
\sigma_3(n) &\sim \frac{\pi^4}{90}n^3 \\
\sigma_4(n) &\sim \zeta(5)n^4 + \frac{1}{120} \\
\sigma_5(n) &\sim \frac{\pi^6}{945}n^5
\end{aligned}$$

4 External links

- Marko Riedel *http://www.geocities.com/markoriedelde/index.html Applications of the Mellin-Perron Formula in Number Theory.*