What about Internet?

Christos Papadimitriou (STOC 2001)

"The internet is unique among all the computer systems in that it is build, operated and used by multitude of diverse economic interests, in varing relationships of collaboration and competition with each other. This suggest that the mathematical tools and insights most appropriate for understanding the Internet may come from the fusion of algorithmic ideas with concepts and techniques from Mathematical Economics and Game Theory."

http://www.cs.berkeley.edu/~christos/games/cs294.html

What is Game Theory?

Game theory is often described as a branch of applied mathematics and economics that studies situations where players choose different actions in an attempt to maximize their returns.

The essential feature, however, is that it provides a formal modelling approach to social situations in which decision makers interact with other minds.

Game theory extends the simpler optimization approach developed in neoclassical economics.

Where to use game theory?

Game theory studies decisions made in an environment in which players interact. In other words, game theory studies choice of optimal behavior when costs and benefits of each option depend upon the choices of other individuals.

What for?

Game theory looks for states of equilibrium sometimes calles solutions

Basic Reference

- Osborne. An Introduction to Game Theory, Oxford University Press, 2004
- Ad-hoc references.

Game Theory for CS?

- Framework to analyze equilibrium states of protocols used by rational agents.
  Price of anarchy/stability.
- Tool to design protocols for internet with guarantees.
  Mechanism design.
- New concepts to analyze/justify behavior of on-line algorithms
  Give guarantees of stability to dynamic network algorithms.
- Source of new computational problems to study.
  Algorithmic game theory

Non-cooperative games

Strategic game

A strategic game Γ (with ordinal preferences) consists of:

- A finite set $N = \{1, \ldots, n\}$ of players.
- For each player $i \in N$, a nonempty set of actions A_i.
- Each player chooses his action once. Players choose actions simultaneously. No player is informed, when he chooses his action, of the actions chosen by others.
- For each player $i \in N$, a preference relation (a complete, transitive, reflexive binary relation) $\preceq_i$ over the set $A = A_1 \times \cdots \times A_n$.

It is frequent to specify the players' preferences by giving utility functions $u_i(a_1, \ldots a_n)$. Also called pay-off functions.

Example: Prisoner's Dilemma

The story

- Two suspects in a major crime are held in separate cells.
- Evidence to convict each of them of a minor crime.
- No evidence to convict either of them of a major crime unless one of them finks.

The penalties

- If both stay quiet, be convicted for a minor offense (one year prison).
- If only one finks, he will be will be freed (and used as a witness) and the other will be convicted for a major offense (four years in prison).
- If both fink, each one will be convicted for a major offense with a reward for coperation (three years each).

The Prisoner's Dilemma models a situation in which

- there is a gain from cooperation,
- but each player has an incentive to free ride.

Game representation

- Players $N = \{\text{Suspect 1}, \text{Suspect 2}\}$.
- Actions $A_1 = A_2 = \{\text{Quiet}, \text{Fink}\}$.
- Action profiles
  $A = A_1 \times A_2 = \{(\text{Quiet}, \text{Quiet}), (\text{Quiet}, \text{Fink}), (\text{Fink}, \text{Quiet}), (\text{Fink}, \text{Fink})\}$
- Preferences
  - Player 1

$$(\text{Fink}, \text{Quiet}) \preceq_1 (\text{Quiet}, \text{Quiet}) \preceq_1 (\text{Fink}, \text{Fink}) \preceq_1 (\text{Quiet}, \text{Fink})$$

  - Player 2

$$(\text{Quiet}, \text{Fink}), \preceq_2 (\text{Quiet}, \text{Quiet}) \preceq_2 (\text{Fink}, \text{Fink}) \preceq_2 (\text{Fink}, \text{Quiet})$$

- Utilities
  $u_1(\text{Fink}, \text{Quiet}) = 3, u_1(\text{Quiet}, \text{Quiet}) = 2, u_1(\text{Fink}, \text{Fink}) = 1, u_1(\text{Quiet}, \text{Fink}) = 0$
  $u_2(\text{Quiet}, \text{Fink}) = 3, u_2(\text{Quiet}, \text{Quiet}) = 2, u_2(\text{Fink}, \text{Fink}) = 1, u_2(\text{Fink}, \text{Quiet}) = 0$

We can represent pay-offs in a compact way on a bi-matrix.

		Suspect 2	
		Quiet	Fink
Suspect 1	Quiet	2,2	0,3
	Fink	3,0	1,1

Example: Matching Pennies

- Two people choose, simultaneously, whether to show the head or tail of a coin.
- If they show same side, person 2 pays person 1, otherwise person 1 pays person 2.
- Payoff are equal to the amounts of money involved.

Person 2

Person 1		Head	Tail
	Head	1,-1	-1,1
	Tail	-1,1	1,-1

This is an example of a zero-sum game

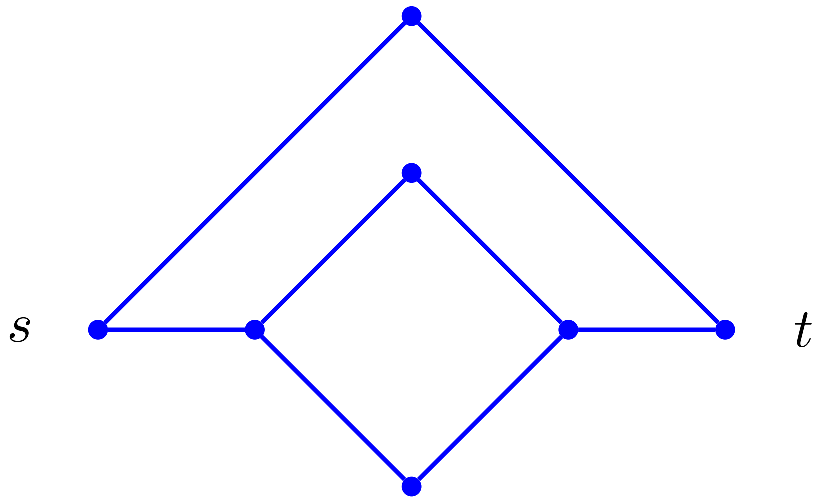
Example: Sending from s to t

- We have a graph $G = (V, E)$ and two vertices $s, t \in V$.
- There is one player for each vertex $v \in V$, $v \neq t$.
- The set of actions for player u is $N_G(u)$.
- A strategy profile is a set of vertices $(v_1, \ldots, v_{n-1})$.
- Pay-offs are defined as follows:

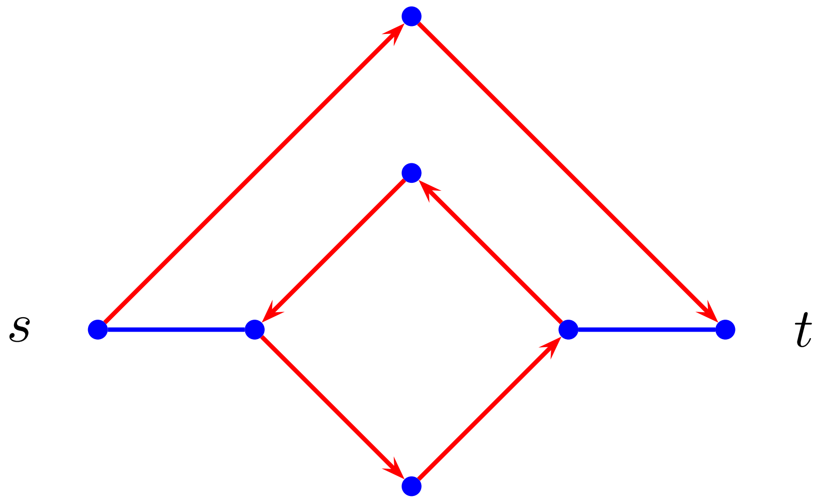
  player u gets 1 if the shortest path joining s to t in the digraph induced by $v_1, \ldots, v_{n-1}$ contains (u, v_u), otherwise gets 0.

Players are selfish but the system can get a different reward/cost. For example the cost of the shortest path.

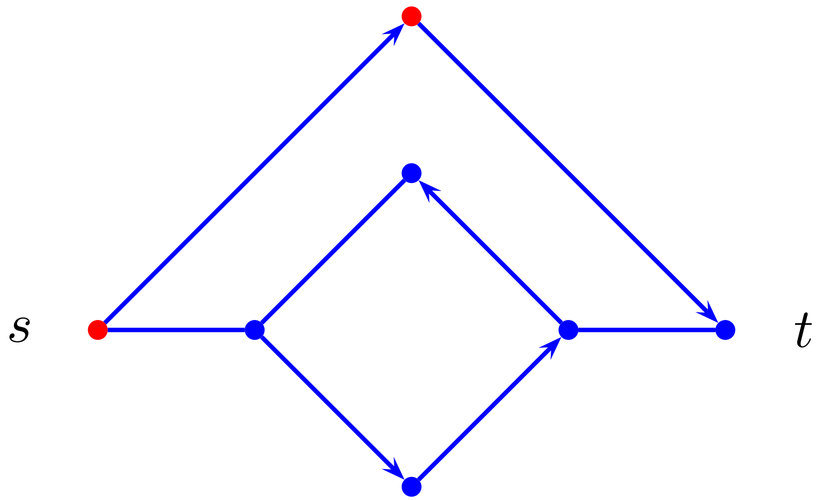
Sending from s to t: example



Sending from s to t: strategies



Sending from s to t: pay-offs



Red nodes get pay-off 1, blue nodes get pay-off 0.

Strategies

A strategy of player $i \in N$ in a strategic game is an action $a_i \in A_i$ to each non-terminal history $h \in H \setminus Z$ for which $P(h) = i$.

A *strategy profile* $s = (s_1, \ldots, s_n)$ consists of a strategy for each player.

For each $s = (s_1, \ldots s_n)$ we denote by

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

$$(s_{-i}, s'_i) = (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$$

Best response

Let Γ be an strategic game defined trhough pay-off functions

The set of best responses for player i to strategy s_i is

$$BR(s_{-i}) = \{a_i \in A_i \mid u_i(s_{-i}, a_i) = \max_{a'_i \in A_i} u_i(s_{-i}, a'_i)\}$$

Those are the actions that give maximum pay-off provided the other players do not change their strategies.

Pure Nash equilibrium

A pure Nash equilibrium is an strategy profile $a^* = (a_1^*, \ldots, a_n^*)$ such that no player i can do better choosing an action different from a_i^*, given that every other player j adheres to a_j^*:

> for every player i and for every action $a_i \in A_i$ it holds
> $u_i(a_{-i}^*, a_i^*) \geqslant u_i(a_{-i}^*, a_i)$.
>
> Equivalently, for every player i and for every action $a_i \in A_i$ it holds
> $a_i^* \in BR(a_{-i}^*)$.

Pure Nash Equilibrium

- Is a strategy profile in which all players are happy.
- Identified with a fixed point of an iterative process of computing a best response.
- However, the game is played only once!
- GT deals with the existence and analysis of equilibria assuming rational behavior.

  players try to maximize their benefit
- GT does not provide algorithmic tools for computing such equilibrium if one exists.

Pure Nash equilibria, examples

	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

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- Prisoner's Dilemma, (Fink, Fink).
- Bach or Stravinsky, (Bach, Bach), (Stravinsky, Stravinsky).
- Stag Hunt, (Stag, Stag), (Hunt, Hunt).
- Matching Pennies, none.

Example: Sending from s to t

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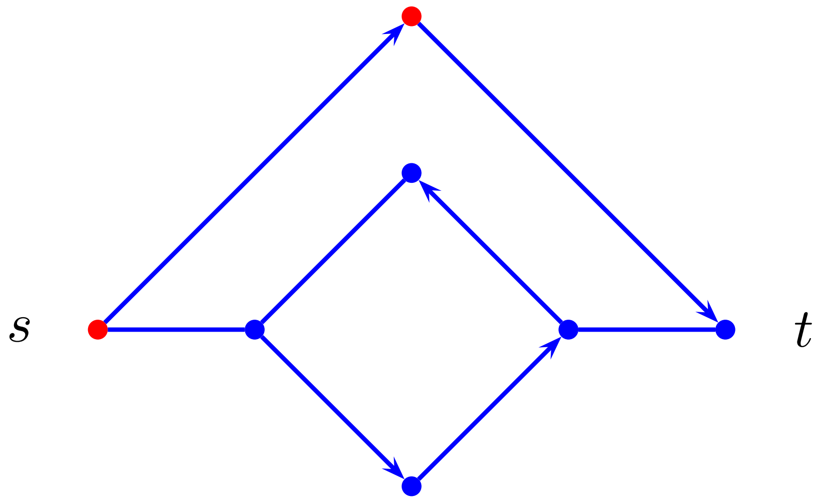
Sending from s to t: pay-offs



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Is a Nash equilibrium?

Pure Nash equilibrium

- First notion of equilibria for non-cooperative games.
- There are strategic games with no pure Nash equilibrium.
- There are games with more than one pure Nash equilibrium.
- How to compute a Nash equilibrium if there is one?

Mixed Nash equilibrium

Until now players are selecting as strategy an action.

When each player i is allowed to select as strategy a distribution σ_i on the set of actions A_i we have a mixed strategy mixed strategy.

The utility function for player i is the expected utility under the distribution σ_i.

A mixed Nash equilibrium is a profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ such that no player i can get better utility choosing a distribution different from σ_i^*, given that every other player j adheres to σ_j^*.

Theorem (Nash): Every strategic game has a mixed Nash equilibrium.