Single-Pass List Partitioning

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MuCoCos'08

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3. The SinglePass Algorithm
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Motivation

Effectiveness of many parallel algorithms relies on partitioning the input into pieces.

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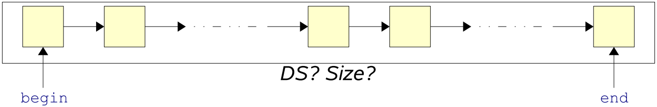
ALTHOUGH there are common settings where the input cannot be partitioned so easily.
Example: Sequences as input to algorithms in the Standard Template Library (STL), part of the C++ standard library.

Algorithms in the STL

Input given using (*forward*) *iterators*, abstract from the underlying data structure.



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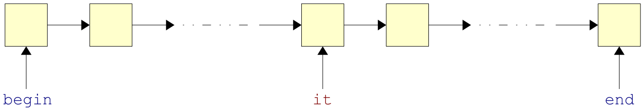

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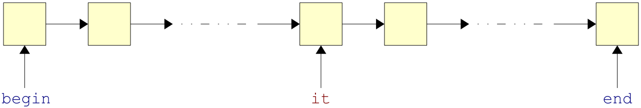

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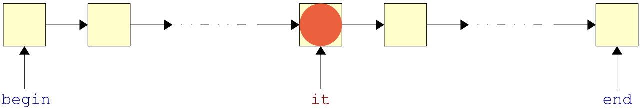

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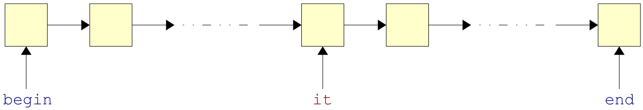

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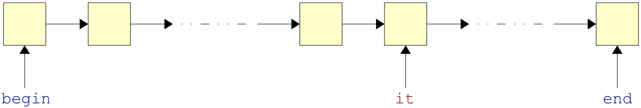

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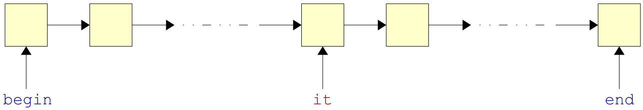

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Forward sequence

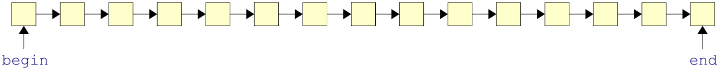
How to partition forward sequences or alike?

In compile-time:

1. The sequence is actually a random access sequence (e.g. an array)
    - More operations: `it + k`, `it - k`, `it2 - it1`, ...
    - Sequence length can be known in constant time

2. The sequence is not random access
    - Sequence length is unknown in constant time

How to partition forward sequences or alike? (2)



Naïvely:

- TRAVERSETWICE
- POINTERARRAY

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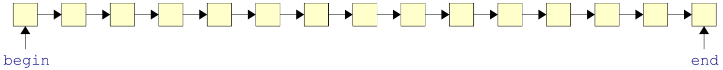

Naïvely:

- TRAVERSETWICE
  1. Determine length (1st traversal)
  2. Partition (2nd traversal)
- POINTERARRAY

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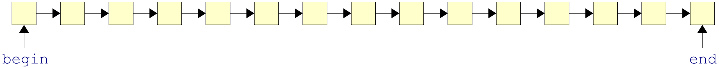
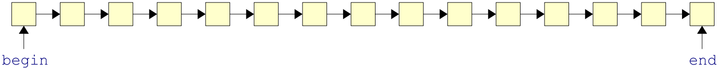


Naïvely:

- TraverseTwice
- PointerArray
  1. Store pointers in a dynamic array (linear auxiliary memory)
  2. Trivial index calculation

How to partition forward sequences or alike? (2)



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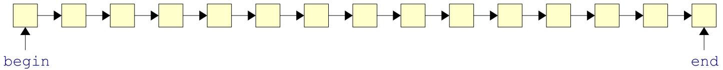

Naïvely:

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Cannot this be done more efficiently?
Amdahl's law: speedup limited by the sequential portion.

Our contribution

An efficient sequential algorithm to divide *forward sequences*.

- Only one traversal
- Sub-linear additional space

List Partitioning problem

Given a *forward sequence*, divide it into p parts of almost equal length.

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Given a *forward sequence*, divide it into p parts of almost equal length.

Quality ratio r: $1 \leq \frac{|\text{longest part}|}{|\text{shortest part}|} \leq R$

r correlates to the efficiency of processing the parts in parallel (given that processing time is proportional to parts length)

R: constant, depends only on a tuning parameter, namely the oversampling factor σ.

- $\sigma \in \mathbb{N} \setminus \{0\}$.

List Partitioning as an online problem

Only one element is given at a time, no global information.

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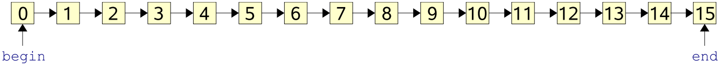
Only one element is given at a time, no global information.

Optimal offline algorithm: the difference in length between the parts is at most 1.
Quality ratio: $r_{\mathrm{OPT}} = \lceil n/p \rceil / \lfloor n/p \rfloor \overset{n \to \infty}{\longrightarrow} 1$.

Algorithm

Let $\sigma = 2$, $p = 3$

L:



0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

begin

end

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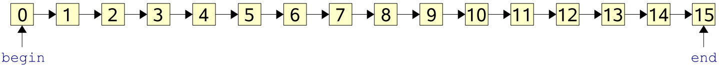

$k = 1,\ \ S = \{\}$

1. Initialization.

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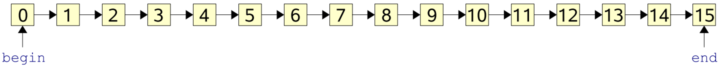

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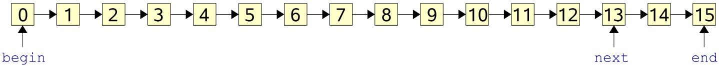

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begin next end

$k = 1, \quad S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

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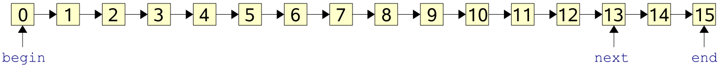

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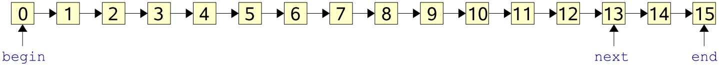

0 → 1 → 2 → 3 → 4 → 5 → 6 → 7 → 8 → 9 → 10 → 11 → 12 → 13 → 14 → 15

begin

next

end

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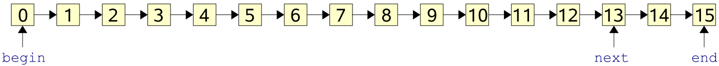

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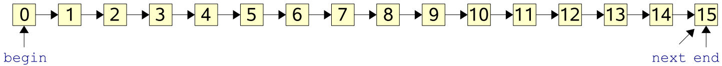

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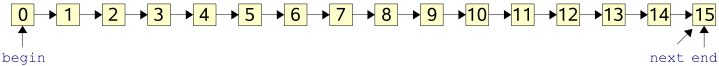

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4. Merge the subsequences in S to obtain p subsequences.

Getting p subsequences of similar length

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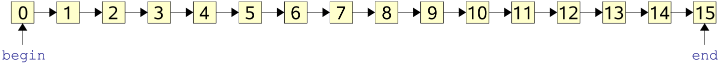

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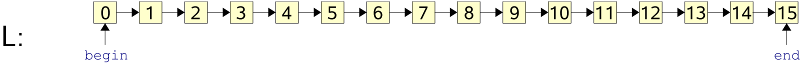

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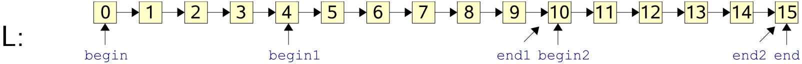

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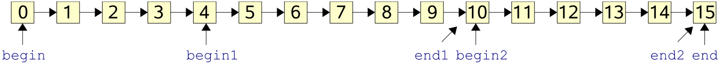

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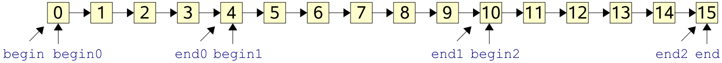

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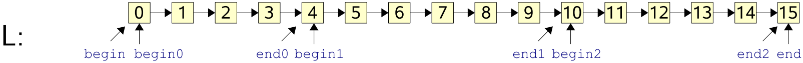

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Special care with the last subsequence in S, which may be *not* full. The algorithm guarantees that two parts differ in length in at most in k elements.

Analysis

Auxiliary space (i.e. $|S|$): $\Theta(\sigma p)$

Time: $\Theta(n + \sigma p \log n)$.

- L traversal: $\Theta(n)$
- Step 3 visits $\Theta(\sigma p)$ elements of S in $\Theta(\log n)$ iterations.

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Ratio:

- worst-case: r bounded by $\frac{\sigma+1}{\sigma}$.
- average: $\mathbb{E} r < \frac{1}{\sigma p} \sum_{\ell=\sigma p}^{2\sigma p-1} \frac{\lceil \ell/p \rceil}{\lfloor \ell/p \rfloor} \approx 1 + \frac{1}{\sigma p} \left((p-1) \ln(2) \right)$

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E. g. if $\sigma = 10$ and $p = 32$, then $r <= 1.1$ and $\mathbb{E}r < 1.07$

Generalization of the SINGLEPASS Algorithm

Performs *merge* steps only every m^{th} loop iteration.

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Equivalent to increasing the oversampling factor to σn^{γ} with $\gamma = 1 - 1/m$.

Analysis

$n^{\gamma} = \frac{n}{\sqrt[m]{n}} = \sqrt[m]{n^{m-1}}$
Auxiliary space (i.e. $|S|$): $O(\sigma p n^{\gamma})$.

Time: $\Theta(n + \sigma p(n^{\gamma} + \log n))$.

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Time: $\Theta(n + \sigma p(n^\gamma + \log n))$.

Ratio:

$|\texttt{longest}| = (\sigma n^\gamma + 1)k$ $|\texttt{shortest}| = \sigma n^\gamma k$

$\frac{|\texttt{longest}|}{|\texttt{shortest}|} = 1 + \frac{1}{\sigma n^\gamma} = 1 + \frac{\sqrt[m]{n}}{\sigma n} \overset{n\to\infty}{\longrightarrow} 1$

Choosing m

The choice of m trades off time and space versus solution quality (better r as m larger).

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Some interesting cases:

- $m = 1$: *merge* is performed each iteration → simple SINGLEPASS Algorithm
- $m = 2$: *merge* is performed once each two iterations
  - $n^{\gamma} = \sqrt{n}$
  - Auxiliary space: $O(\sigma p \sqrt{n})$
  - Time: $\Theta(n + \sigma p(\sqrt{n} + \log n))$.
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  - Time: $\Theta(n + \sigma p(\sqrt{n} + \log n))$.
  - Ratio: $1 + \frac{1}{\sqrt{n}}$

$m = 2$ appears to be a good compromise.

Implementation

C++ implementation

Algorithms

- generalized SINGLEPASS
  - included in the MCSTL [SSP]
    MCSTL = Multicore STL, parallel implementation of the STL
- TRAVERSETWICE
- POINTERARRAY

Testing

Performance and quality results for $p = 4$.
Quality evaluated according the overhead $h = r - 1$.

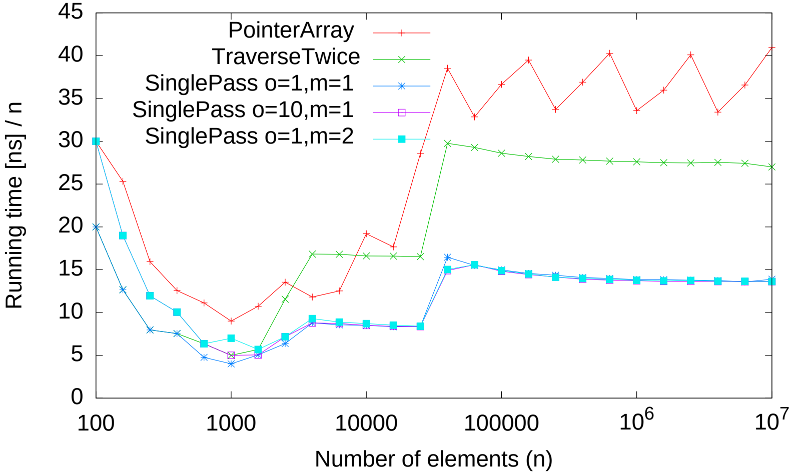
Setup

- AMD Opteron 270 (2.0 GHz, 1 MB L2 cache).
- GCC 4.2.0 + libstdc++, optimization (-O3).

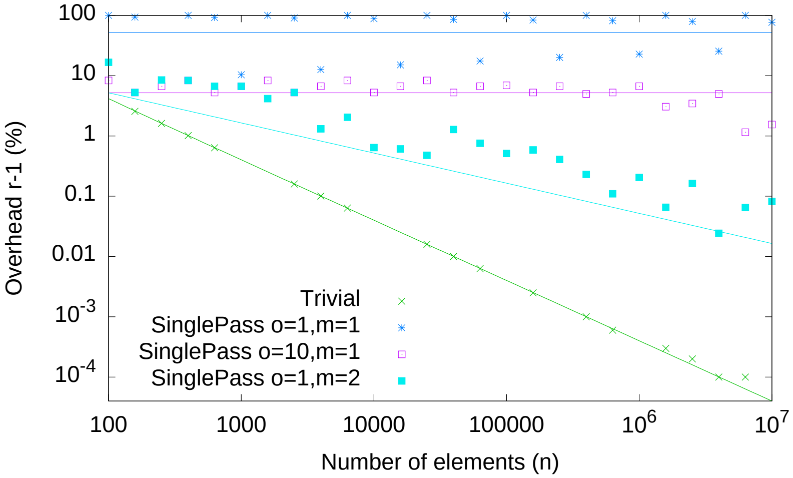
Parameters for SINGLEPASS

- $(o = 1,\ m = 1)$, $\Theta(p)$ space
- $(o = 10,\ m = 1)$, $\Theta(10p)$ space
- $(o = 1,\ m = 2)$, $\Theta(\sqrt{n}p)$

Time results



Quality results



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The larger m, the better the quality, trading off memory.
In the worst-case:

- $m = 1$: $h = 1/\sigma$
- $m > 1$: h decreases exponentially with n.

For large input instances and in most practical situations, no difference with optimally partitioned sequences.

Further reading

[SK08] describes some of the problems and challenges in parallelizing algorithms in the context of the C++ standard library.

References

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