Graphs: Minimum Spanning Trees

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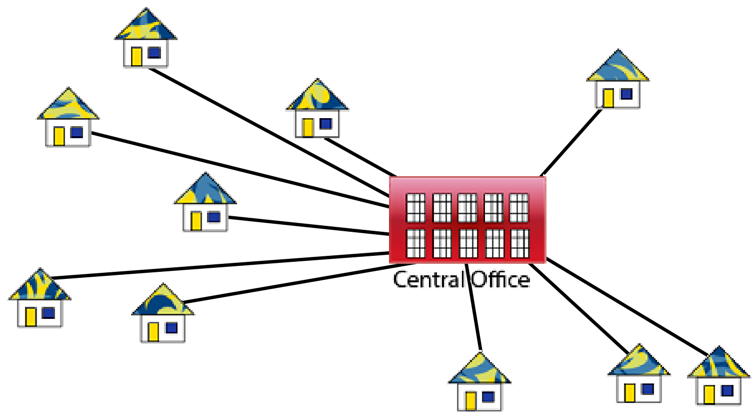
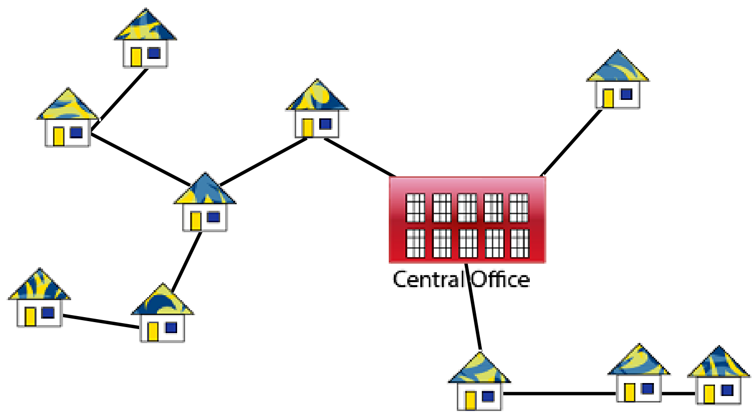
Laying a communication network



Source: https://www.javatpoint.com/applications-of-minimum-spanning-tree

Laying a communication network



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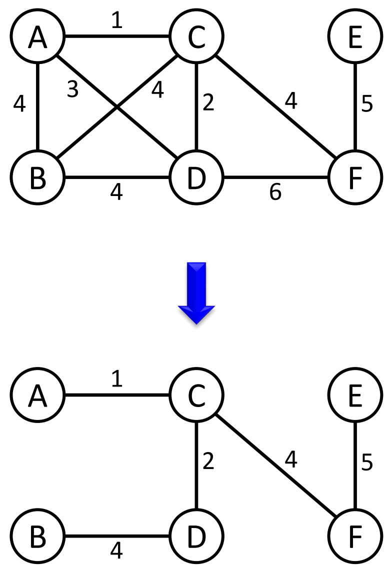
Minimum Spanning Trees



- Nodes are computers
- Edges are links
- Weights are maintenance cost
- Goal: pick a subset of edges such that
  - the nodes are connected
  - the maintenance cost is minimum

> The solution is not unique.
> Find another one !

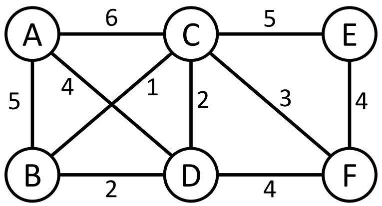
Property:
An optimal solution cannot contain a cycle.

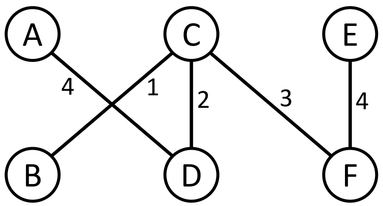
Minimum Spanning Tree

- Given un undirected graph $G = (V, E)$ with edge weights w_e, find a tree $T = (V, E')$, with $E' \subseteq E$, that minimizes

$$\text{weight}(T) = \sum_{e \in E'} w_e .$$

- Greedy algorithm: repeatedly add the next lightest edge that does not produce a cycle.



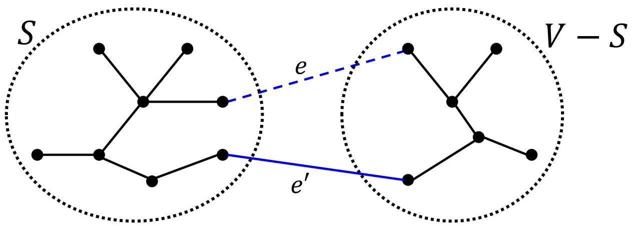


Note: We will now see that this strategy guarantees an MST.

Properties of trees

- **Definition:** A tree is an undirected graph that is connected and acyclic.
- **Property:** Any connected, undirected graph $G = (V, E)$ has $|E| \geq |V| - 1$ edges.
- **Property:** A tree on n nodes has $n - 1$ edges.
  - Start from an empty graph. Add one edge at a time making sure that it connects two disconnected components. After having added $n - 1$ edges, a tree has been formed.
- **Property:** Any connected, undirected graph $G = (V, E)$ with $|E| = |V| - 1$ is a tree.
  - It is sufficient to prove that G is acyclic. If not, we can always remove edges from cycles until the graph becomes acyclic.
- **Property:** Any undirected graph is a tree iff there is a unique path between any pair of nodes.
  - If there would be two paths between two nodes, the union of the paths would contain a cycle.

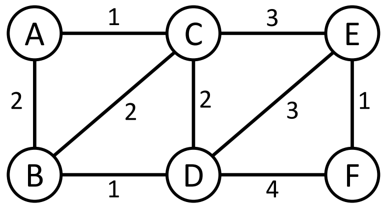
The cut property

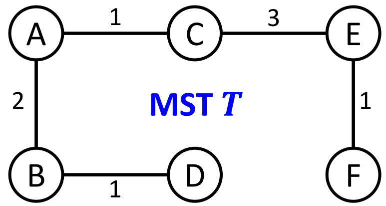


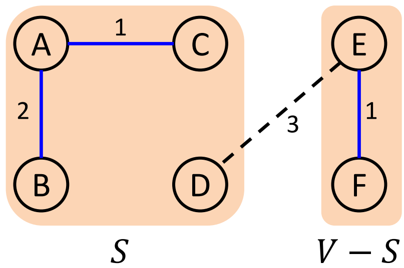
Suppose edges X are part of an MST of $G = (V, E)$. Pick any subset of nodes S for which X does not cross between S and $V - S$, and let e be the lightest edge across this partition. Then $X \cup \{e\}$ is part of some MST.

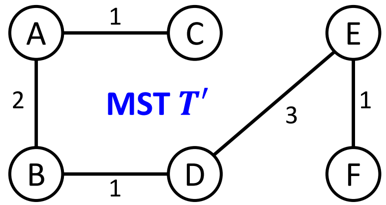
Proof (sketch): Let T be an MST and assume e is not in T. If we add e to T, a cycle will be created with another edge e' across the cut $(S, V - S)$. We can now remove e' and obtain another tree T' with $\text{weight}(T') \leq \text{weight}(T)$. Since T is an MST, then the weights must be equal.

The cut property: example





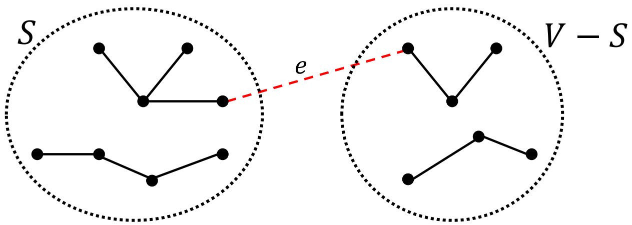




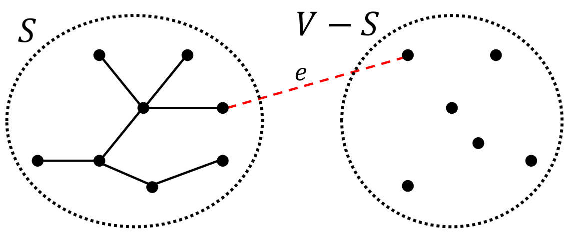
Minimum Spanning Tree

Any scheme like this works (because of the properties of trees):

```
X = {}    # The set of edges of the MST
repeat |V| − 1 times:
  pick a set S ⊂ V for which X has no edges between S and V − S
  let e ∈ E be the minimum-weight edge between S and V − S
  X = X ∪ {e}
```



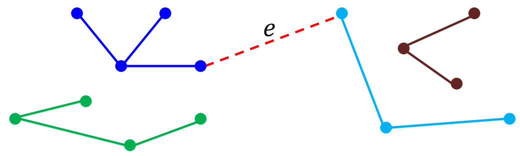
MST: two strategies



Prim's algorithm

Invariant:
- A set of nodes (S) is in the tree.

Progress:
- The lightest edge with exactly one endpoint in S is added.



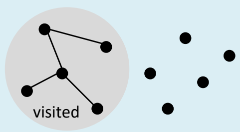
Kruskal's algorithm

Invariant:
- A set of trees (forest) has been constructed.

Progress:
- The lightest edge between two trees is added.

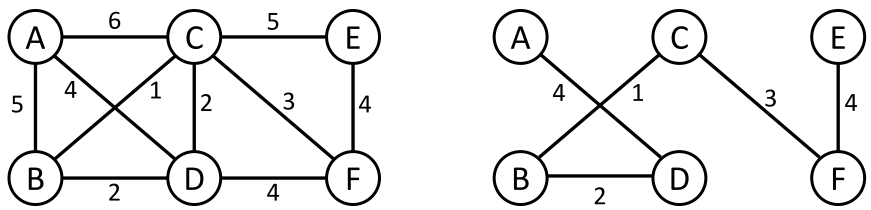
Prim's algorithm

```
def Prim(G, w) → prev:
  """Input: A connected undirected Graph G(V,E)
            with edge weights w(e).
     Output: An MST defined by the vector prev."""
  for all u ∈ V:
    visited[u] = False
    prev[u] = nil
  pick any initial node u₀
  visited[u₀] = True
  n = 1

  # Q: priority queue of edges using w(e) as priority
  Q = makequeue()
  for each (u₀,v) ∈ E: Q.insert(u₀,v)

  while n < |V|:
    (u,v) = deletemin(Q) # Edge with smallest weight
    if not visited[v]:
      visited[v] = True
      prev[v] = u
      n = n + 1
      for each (v,x) ∈ E:
        if not visited[x]: Q.insert(v,x)
```



Complexity: $O(|E| \log |V|)$

Prim's algorithm



Q:

(AD,4) (AB,5) (AC,6)
(DB,2) (DC,2) (DF,4) (AB,5) (AC,6)
(BC,1) (DC,2) (DF,4) (AB,5) (AC,6)
(DC,2) (CF,3) (DF,4) (AB,5) (CE,5) (AC,6)
(CF,3) (DF,4) (AB,5) (CE,5) (AC,6)
(DF,4) (FE,4) (AB,5) (CE,5) (AC,6)
(FE,4) (AB,5) (CE,5) (AC,6)

Kruskal's algorithm

Informal algorithm:

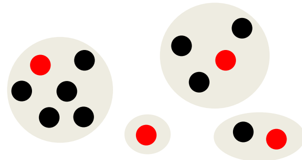
- Sort edges by weight.
- Visit edges in ascending order of weight and add them as long as they do not create a cycle.

How do we know whether a new edge will create a cycle?

```
def Kruskal(G, w) → MST:
  """Input: A connected undirected Graph G(V,E)
            with edge weights w_e.
     Output: An MST defined by the edges in MST."""

  MST = {}
  sort the edges in E by weight
  for all (u,v) ∈ E, in ascending order of weight:
      if (MST has no path connecting u and v):
          MST = MST ∪{(u,v)}
```


Disjoint sets

- A data structure to store a collection of disjoint sets.



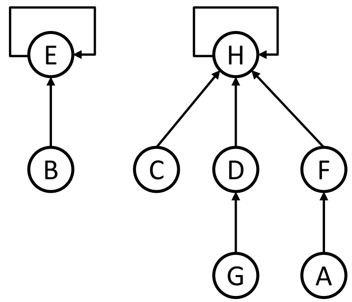
- Operations:
  - makeset(x): creates a singleton set containing just x.
  - find(x): returns the identifier of the set containing x.
  - union(x, y): merges the sets containing x and y.
- Kruskal's algorithm uses disjoint sets and calls
  - makeset: $|V|$ times
  - find: $2 \cdot |E|$ times
  - union: $|V| - 1$ times

Kruskal's algorithm

```
def Kruskal(G, w) → MST:
  """Input: A connected undirected Graph G(V,E)
            with edge weights w_e.
     Output: An MST defined by the edges in MST."""

  for all u ∈ V: makeset(u)

  MST = {}
  sort the edges in E by weight
  for all (u,v) ∈ E, in ascending order of weight:
      if (find(u) ≠ find(v)):
          MST = MST ∪{(u,v)}
          union(u,v)
```

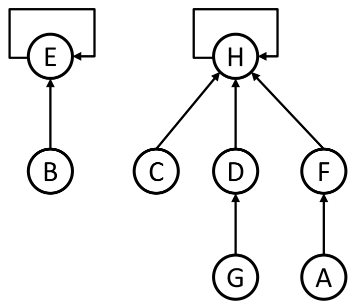

Disjoint sets

- The nodes are organized as a set of trees. Each tree represents a set.
- Each node has two attributes:
  - parent (π): ancestor in the tree
  - rank: height of the subtree
- The root element is the representative for the set: its parent pointer is itself (self-loop).
- The efficiency of the operations depends on the height of the trees.




```
def makeset(x):
  π(x) = x
  rank(x) = 0

def find(x):
  while x ≠ π(x): x = π(x)
  return x
```

Disjoint sets



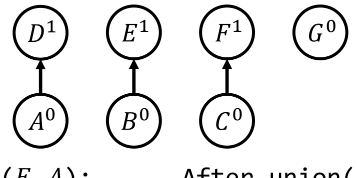

```
def union(x, y):
  r_x = find(x)
  r_y = find(y)
  if r_x = r_y: return

  if rank(r_x) > rank(r_y):
    π(r_y) = r_x
  else:
    π(r_x) = r_y
    if rank(r_x) = rank(r_y):
      rank(r_y) = rank(r_y) + 1
```

```
def makeset(x):
  π(x) = x
  rank(x) = 0

def find(x):
  while x ≠ π(x): x = π(x)
  return x
```

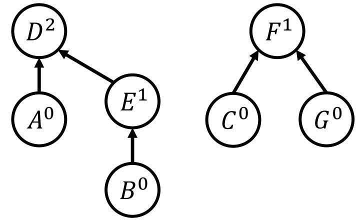

Disjoint sets

After makeset(A),…,makeset(G):

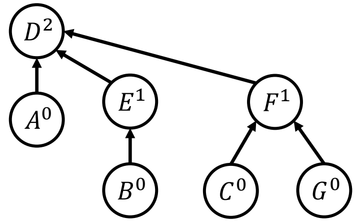
After union(A,D), union(B,E), union(C,F):



After union(C,G), union(E,A):



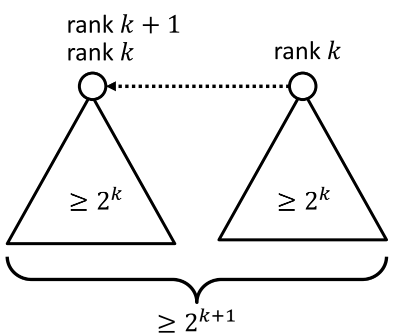
After union(B,G):



Property: Any root node of rank k has at least 2^k nodes in its tree.
Property: If there are n elements overall, there can be at most $n/2^k$ nodes of rank k.
Therefore, all trees have height $\leq \log n$.

Disjoint sets

Property 1: proof by induction



Property 2:

For n nodes, the tallest possible tree could have rank k, such that:

$$n \geq 2^k$$

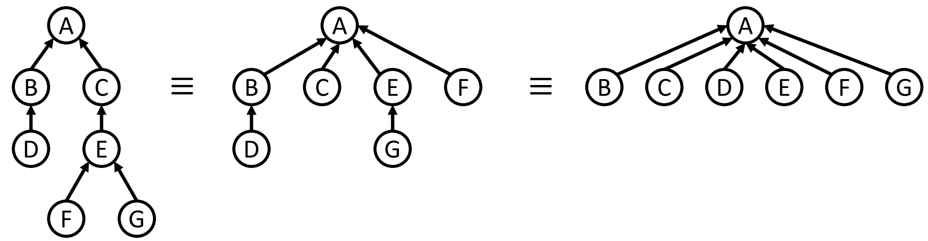
$$k \leq \log_2 n$$

Therefore, find(x) is $O(\log n)$

Property 1: Any root node of rank k has at least 2^k nodes in its tree.
Property 2: If there are n elements overall, there can be at most $n/2^k$ nodes of rank k.
Therefore, all trees have height $\leq \log n$.

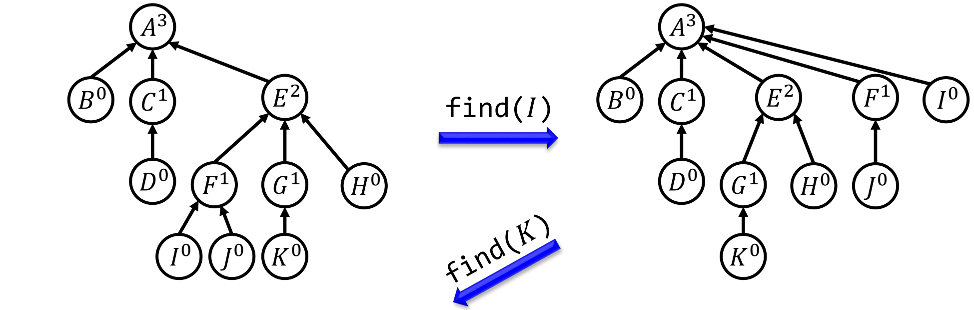
Disjoint sets: path compression

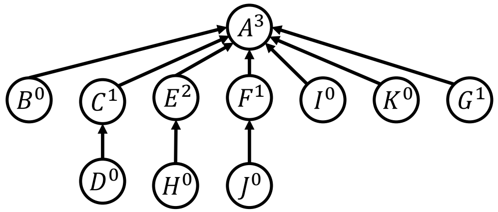
- Complexity of Kruskal's algorithm: $O(|E| \log |V|)$.
  - Sorting edges: $O(|E| \log|E|) = O(|E| \log |V|)$.
  - Find + union ($2 \cdot |E|$ times): $O(|E| \log|V|)$.
- How about if the edges are already sorted or sorting can be done in linear time (weights are integer and small)?
- Path compression:



Disjoint sets: path compression

```
def find(x):
  if x ≠ π(x): π(x)= find(π(x))
  return π(x)
```

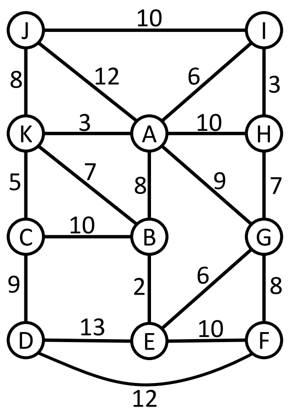




Amortized cost of find: $O(1)$
Kruskal's cost: $O(|E|)$
(if sorting has linear cost)

EXERCISES

Minimum Spanning Trees



- Calculate the shortest path tree from node A using Dijkstra's algorithm.
- Calculate the MST using Prim's algorithm. Indicate the sequence of edges added to the tree and the evolution of the priority queue.
- Calculate the MST using Kruskal's algorithm. Indicate the sequence of edges added to the tree and the evolution of the disjoint sets. In case of a tie between two edges, try to select the one that is not in Prim's tree.