Algorithm Analysis (II)

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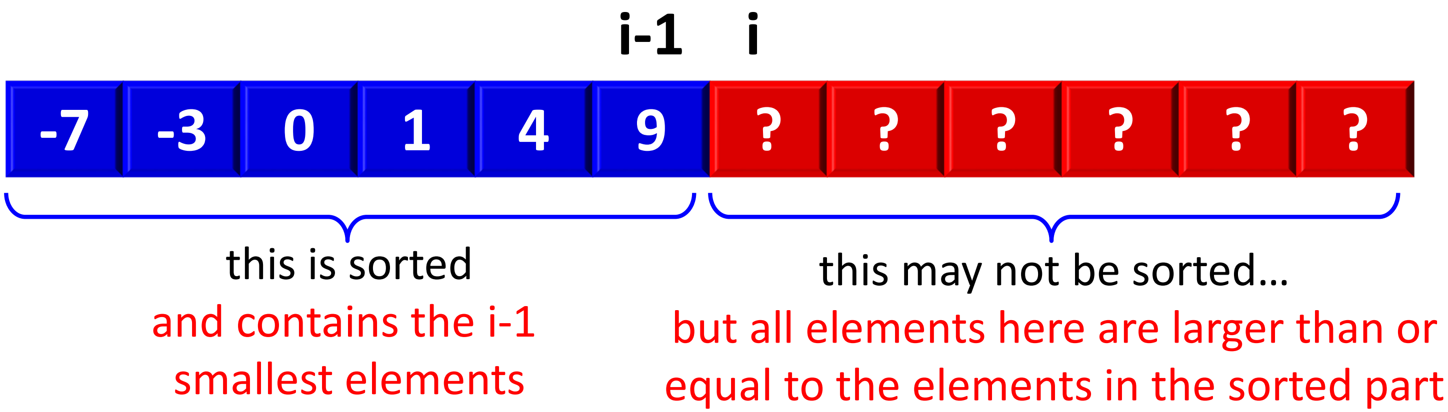
Examples

- Selection sort
- Insertion sort
- The Maximum Subsequence Sum Problem
- Convex Hull

Selection Sort

- Selection sort uses this invariant:



Selection Sort

```
def selection_sort(v: list[T]) -> None:
    """Sorts v in ascending order"""
    for i in range(len(v)-1):
        k = i
        for j in range(i+1, len(v)):
            if v[j] < v[k]:
                k = j
        v[k], v[i] = v[i], v[k]
```

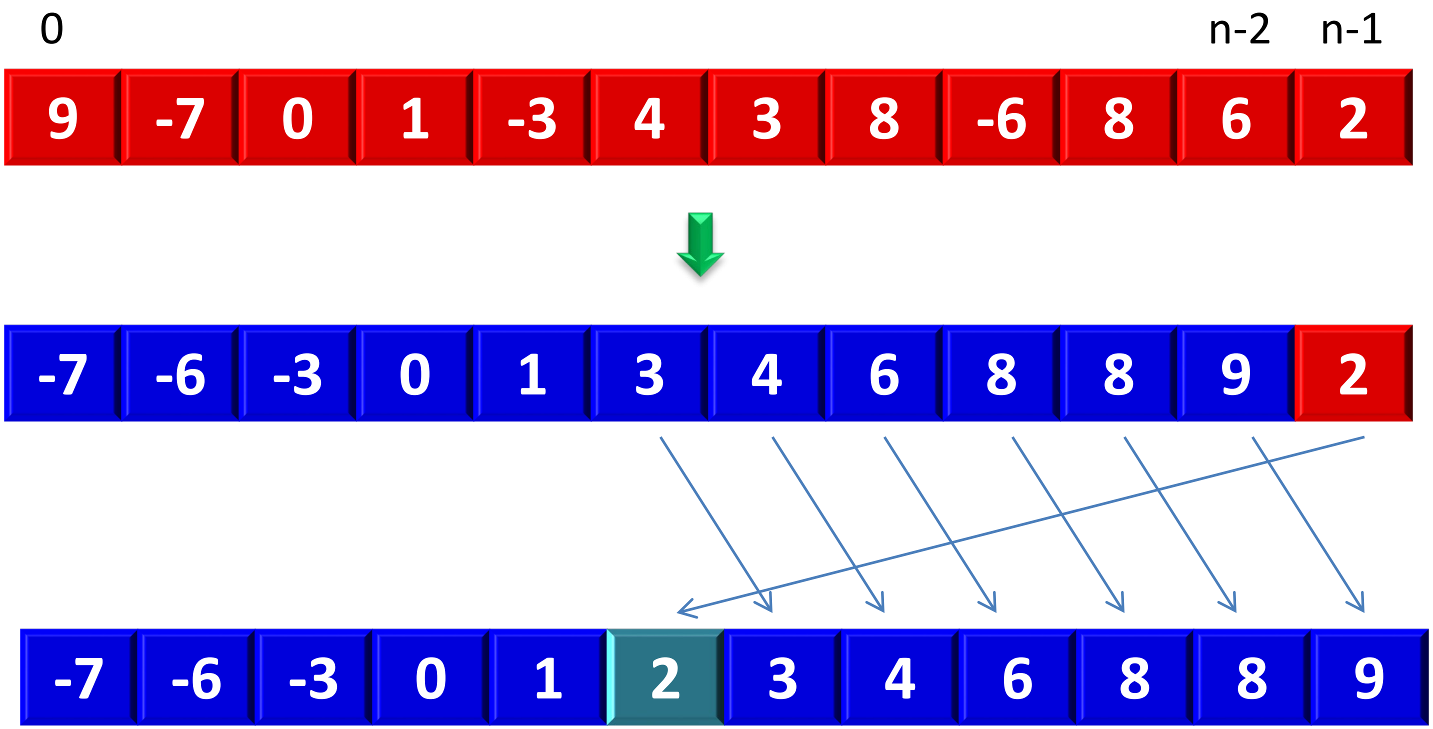
$$T(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \mathrm{O}(1) = \mathrm{O}(1) \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \mathrm{O}(1) \sum_{i=0}^{n-2} (n-i-1)$$

$$= \mathrm{O}(1) \left(\frac{n}{2}(n-1) \right) = \mathrm{O}(1) \cdot \mathrm{O}(n^2) = \mathrm{O}(n^2)$$

Observation: notice that $T(n) \in \Omega(n^2)$, also. Therefore, $T(n) \in \Theta(n^2)$.

Insertion Sort

- Let us use inductive reasoning:
  - If we know how to sort arrays of size n-1,
  - do we know how to sort arrays of size n?



Insertion Sort

```
def insertion_sort(v: list[T]) -> None:
    """Sorts v in ascending order"""
    for i in range(1, len(v)):  # n-1 times
        x = v[i]
        j = i
        while j > 0 and v[j - 1] > x:  # 0..i times
            v[j] = v[j - 1]
            j -= 1
        v[j] = x
```

$$T(n) = \Omega(n)$$
$$T(n) = \mathrm{O}(n^2)$$

$$T_{\text{worst}}(n) = \sum_{i=1}^{n-1} i \cdot \mathrm{O}(1) = \mathrm{O}(n^2)$$ $\Rightarrow$ sorted in reverse order

$$T_{\text{best}}(n) = \sum_{i=1}^{n-1} \mathrm{O}(1) = \mathrm{O}(n)$$ $\Rightarrow$ already sorted

The Maximum Subsequence Sum Problem

- Given (possibly negative) integers $A_1, A_2, \ldots, A_n$, find the maximum value of $\sum_{k=i}^{j} A_k$.
  (the max subsequence sum is 0 if all integers are negative).

- Example:
  - Input: -2, 11, -4, 13, -5, -2
  - Answer: 20 (subsequence 11, -4, 13)

(extracted from M.A. Weiss, Data Structures and Algorithms in C++, Pearson, 2014, 4th edition)

The Maximum Subsequence Sum Problem

```python
def max_sub_sum(a: list[int]) -> int:
    """Returns the sum of the maximum subsequence of a"""
    n = len(a)
    max_sum = 0
    # try all possible subsequences
    for i in range(n):
        for j in range(i, n):
            this_sum = 0
            for k in range(i, j+1):
                this_sum += a[k]
            max_sum = max(max_sum, this_sum)
    return max_sum
```

$$T(n) = \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} \sum_{k=i}^{j} 1$$

The Maximum Subsequence Sum Problem

$$T(n) = \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} \sum_{k=i}^{j} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i + 1)$$

$$= \sum_{i=0}^{n-1} \frac{(n-i+1)(n-i)}{2} = \cdots$$

$$= \frac{n^3 + 3n^2 + 2n}{6} = \Theta(n^3)$$

The Maximum Subsequence Sum Problem

```python
def max_sub_sum(a: list[int]) -> int:
    """Returns the sum of the maximum subsequence of a"""
    n = len(a)
    max_sum = 0
    # try all possible subsequences
    for i in range(n):
        this_sum = 0
        for j in range(i, n):
            this_sum += a[j]  # reuse computation
            max_sum = max(max_sum, this_sum)
    return max_sum
```

$$T(n) = \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} 1 = \Theta(n^2)$$

Max Subsequence Sum: Divide&Conquer

First half				Second half			
4	-3	5	-2	-1	2	6	-2

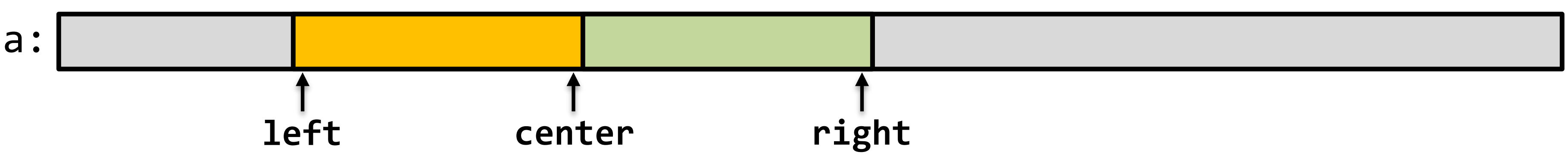
The max sum can be in one of three places:

- 1st half
- 2nd half
- Spanning both halves and crossing the middle

In the 3rd case, two max subsequences must be found starting from the center of the vector (one to the left and the other to the right)

Max Subsequence Sum: Divide&Conquer

```python
def max_sub_sum_rec(a: list[int], left: int, right: int) -> int:
    """Returns the sum of the maximum subsequence of a[left:right+1]"""
    if left == right:  # base case
        return max(a[left], 0)

    # Recursive cases: left and right halves
    center = (left + right)//2
    max_left = max_sub_sum_rec(a, left, center)
    max_right = max_sub_sum_rec(a, center+1, right)

    # Subsequence in a[center+1:right+1]
    max_rcenter, right_sum = 0, 0
    for i in range(center+1, right+1):
        right_sum += a[i]
        max_rcenter = max(max_rcenter, right_sum)

    # Subsequence in a[left:center+1]
    max_lcenter, left_sum = 0, 0
    for i in range(center, left-1, -1):
        left_sum += a[i]
        max_lcenter = max(max_lcenter, left_sum)

    return max(max_left, max_right, max_lcenter + max_rcenter)
```



Max Subsequence Sum: Divide&Conquer

$$T(1) = 1$$
$$T(n) = 2T(n/2) + \Theta(n)$$

We will see how to solve this equation formally in the next lesson (Master Theorem). Informally:

$$\begin{aligned} T(n) &= 2T(n/2) + n = 2(2T(n/4) + n/2) + n \\ &= 4T(n/4) + n + n = 8T(n/8) + n + n + n = \cdots \\ &= 2^k T\left(n/2^k\right) + \underbrace{n + n + \cdots + n}_{k} \end{aligned}$$

when $n = 2^k$, we have that $k = \log_2 n$, hence

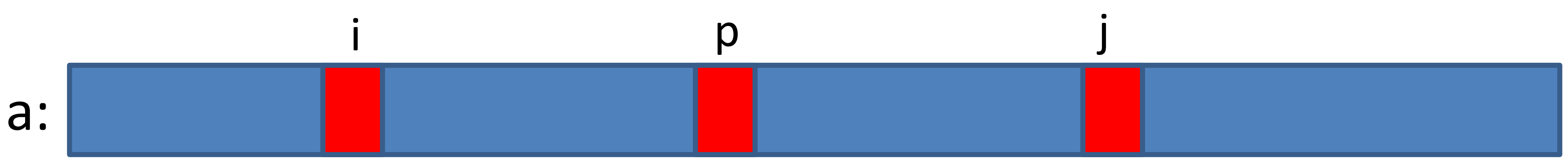
$$T(n) = 2^k T(1) + kn = n + n\log_2 n = \Theta(n \log n)$$

But, can we still do it faster?

The Maximum Subsequence Sum Problem

- Observations:
  - If a[i] is negative, it cannot be the start of the optimal subsequence.
  - Any negative subsequence cannot be the prefix of the optimal subsequence.

- Let us consider the inner loop of the $O(n^2)$ algorithm and assume that all prefixes of a[i..j-1] are positive and a[i..j] is negative:



  - If p is an index between i+1 and j, then any subsequence from a[p] is not larger than any subsequence from a[i] and including a[p-1].
  - If a[j] makes the current subsequence negative, we can advance i to j+1.

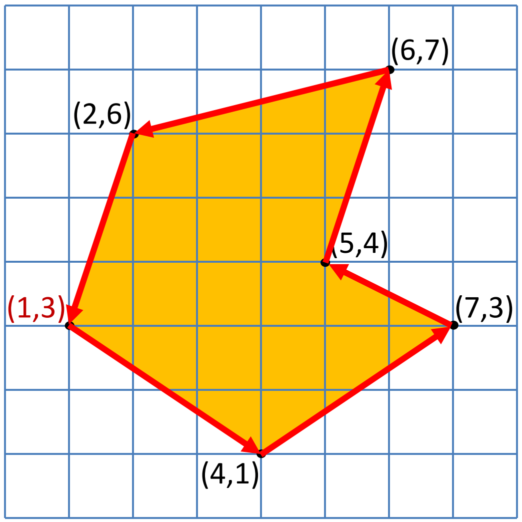
The Maximum Subsequence Sum Problem

```python
def max_sub_sum(a: list[int]) -> int:
    """Returns the sum of the maximum subsequence of a"""
    max_sum, this_sum = 0, 0
    for x in a:
        this_sum += x
        max_sum = max(max_sum, this_sum)
        this_sum = max(this_sum, 0)
    return max_sum
```

$$T(n) = \Theta(n)$$

a:	4	-3	5	-4	-3	-1	5	-2	6	-3	2
this_sum:	4	1	6	2	0	0	5	3	9	6	8
max_sum:	4	4	6	6	6	6	6	6	9	9	9

Representation of polygons

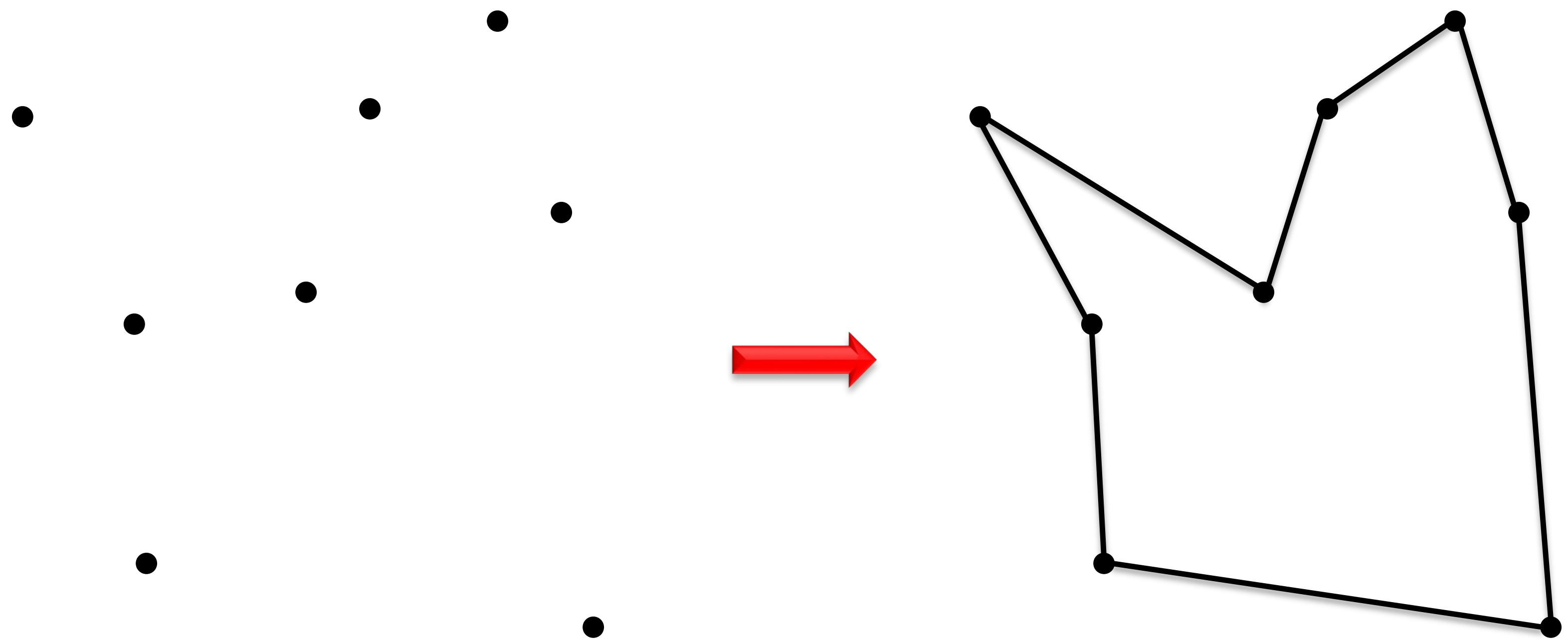


- A polygon can be represented by a sequence of vertices.
- Two consecutive vertices represent an edge of the polygon.
- The last edge is represented by the first and last vertices of the sequence.

```
Vertices:  (1,3) (4,1) (7,3) (5,4) (6,7) (2,6)

Edges: (1,3)-(4,1)-(7,3)-(5,4)-(6,7)-(2,6)-(1,3)
```

```
# A polygon (an ordered set of vertices)
Polygon = list[Point]
```

Create a polygon from a set of points

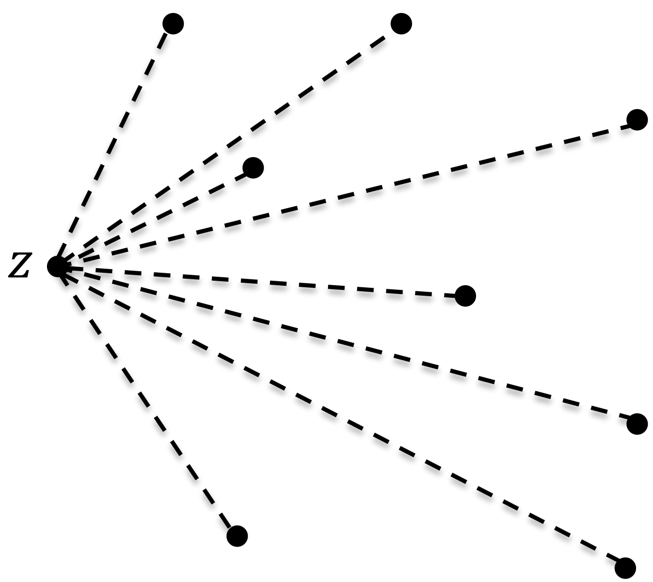


Given a set of n points in the plane, connect them in a simple closed path.

Simple polygon

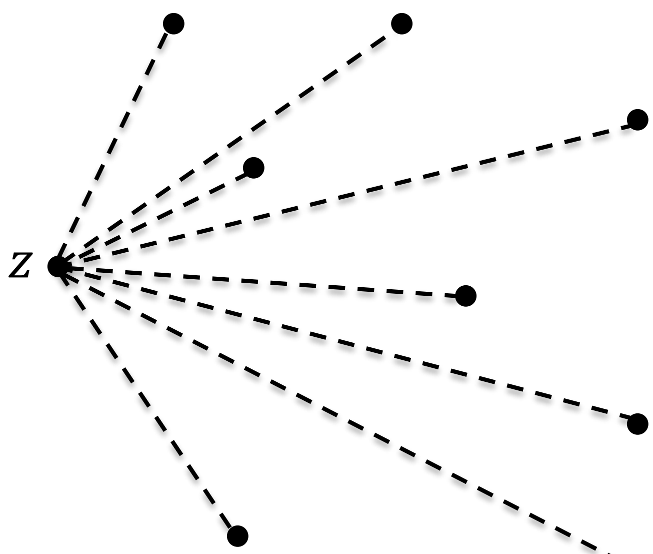
Input: $p_1, p_2, \dots, p_n$ (points in the plane).

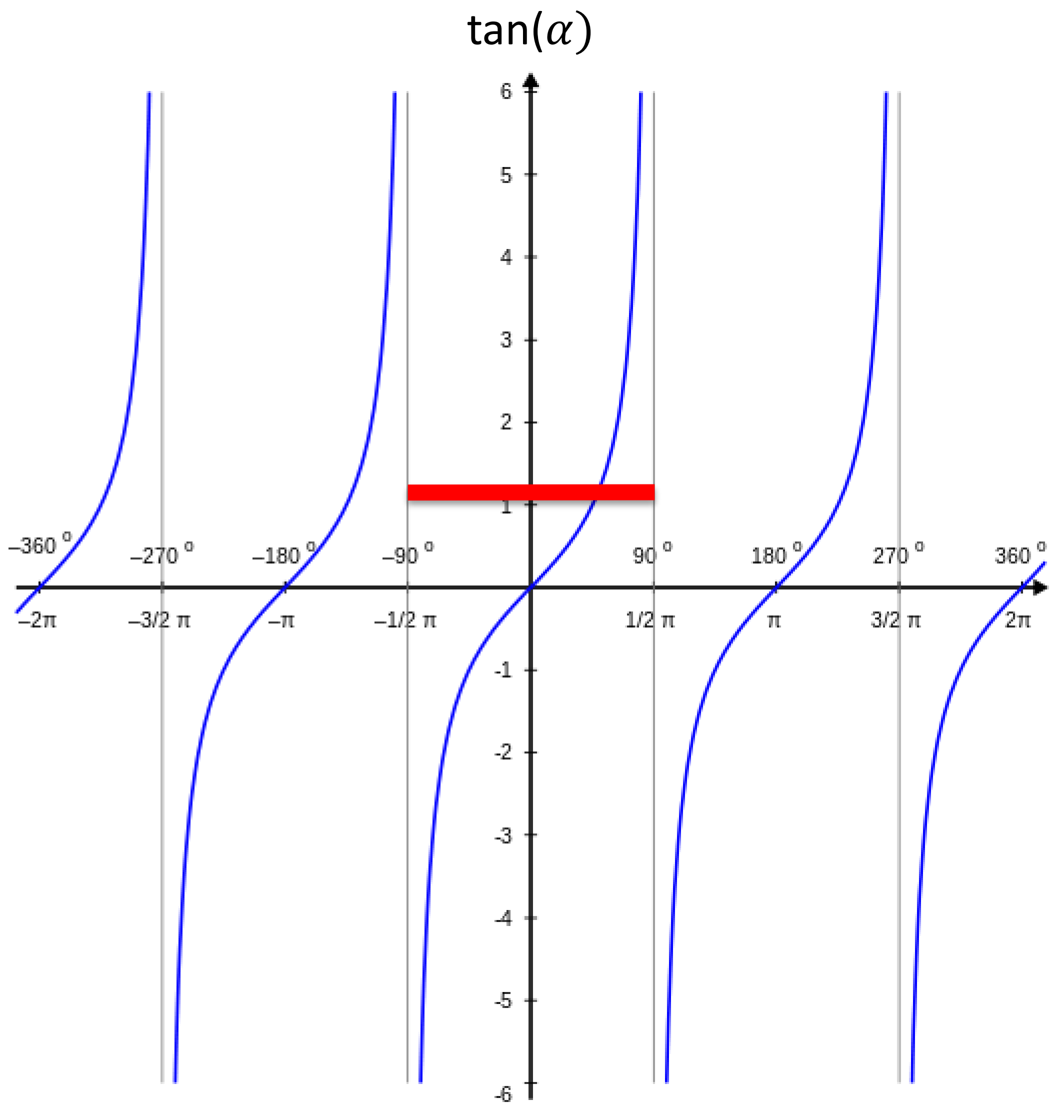
Output: P (a polygon whose vertices are $p_1, p_2, \dots, p_n$ in some order).



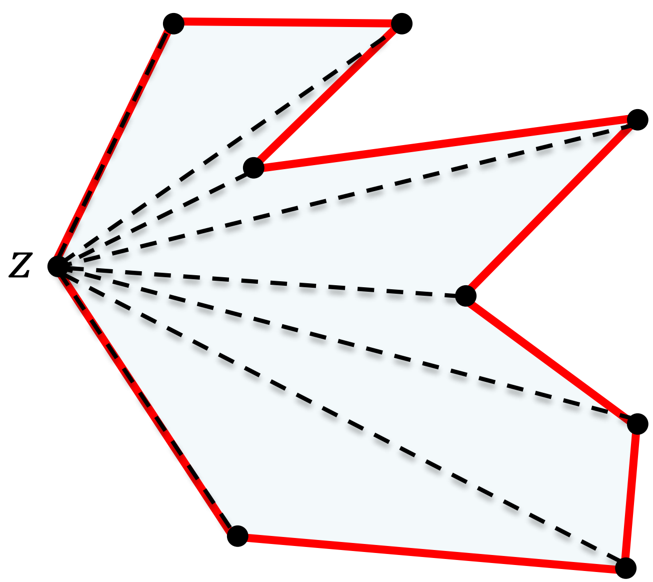
1) Select a point z with the smallest x coordinate (and smallest y in case of a tie in the x coordinate). Assume $z = p_1$.
2) For each $p_i \in \{p_2, \dots, p_n\}$, calculate the angle α_i between the lines $z - p_i$ and the x axis.
3) Sort the points $\{p_2, \dots, p_n\}$ according to their angles. In case of a tie, use distance to z.

Simple polygon





Simple polygon

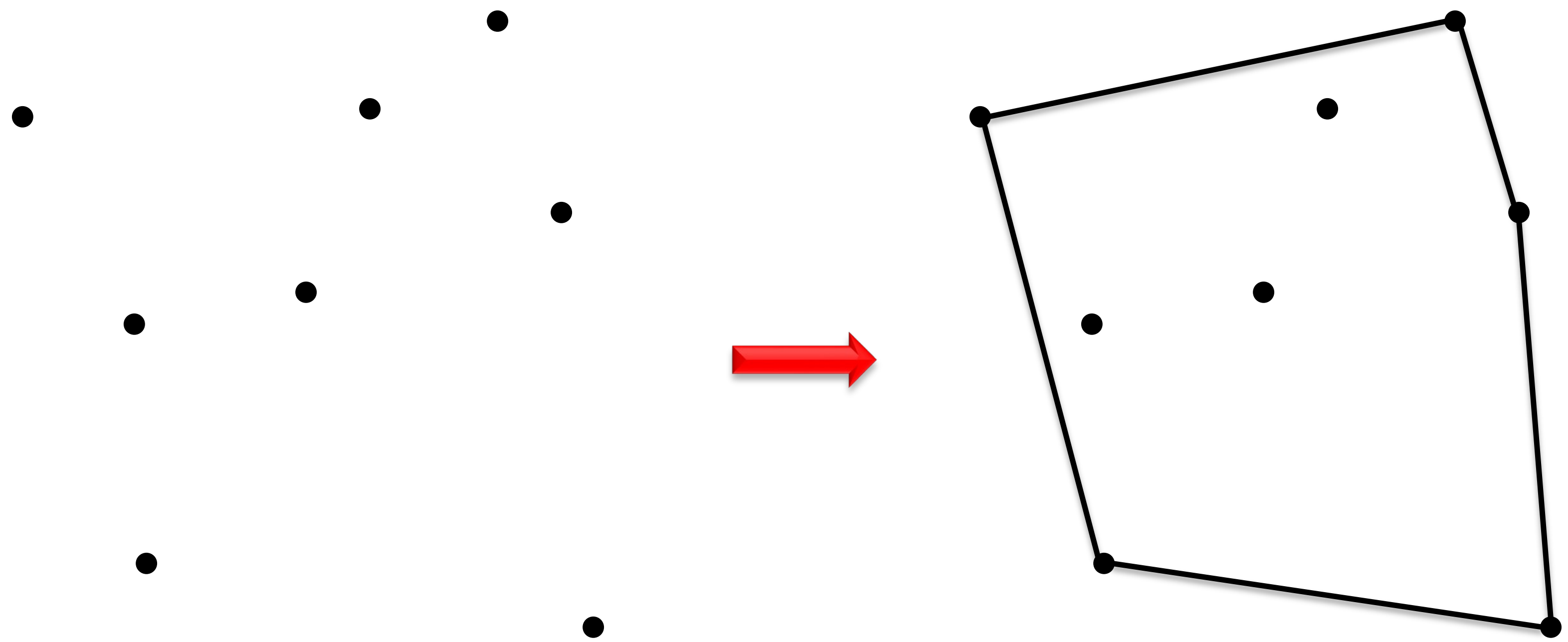


Implementation details:

- There is no need to calculate angles (requires arctan). It is enough to calculate slopes ($\Delta y/\Delta x$).
- There is not need to calculate distances. It is enough to calculate the square of distances (no sqrt required).

Complexity: $O(n \log n)$.
The runtime is dominated by the sorting algorithm.

Convex hull

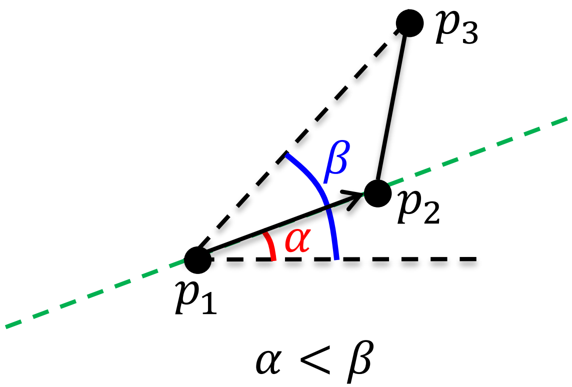


Compute the convex hull of n given points in the plane.

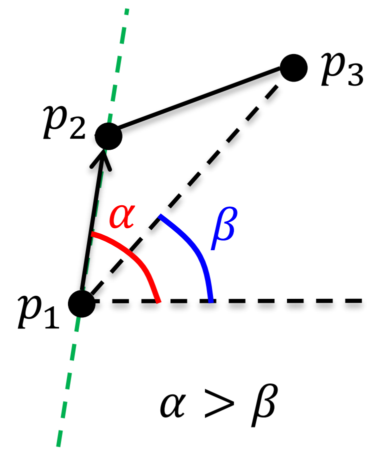
Clockwise and counter-clockwise

How to calculate whether three consecutive vertices are in a **clockwise** or ***counter-clockwise*** turn.

counter-clockwise
(p_3 at the left of $\overrightarrow{p_1 p_2}$)

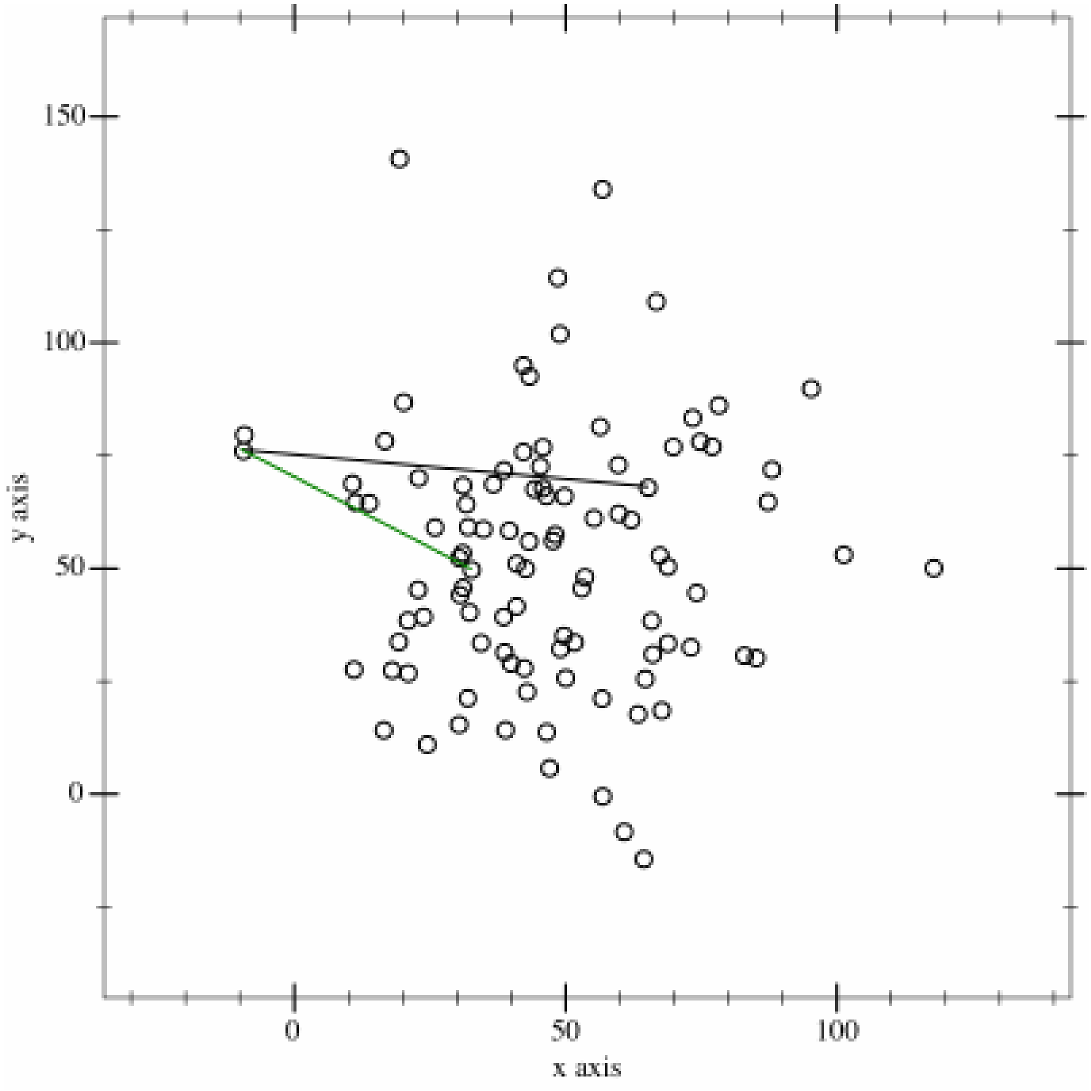


clockwise
(p_3 at the right of $\overrightarrow{p_1 p_2}$)



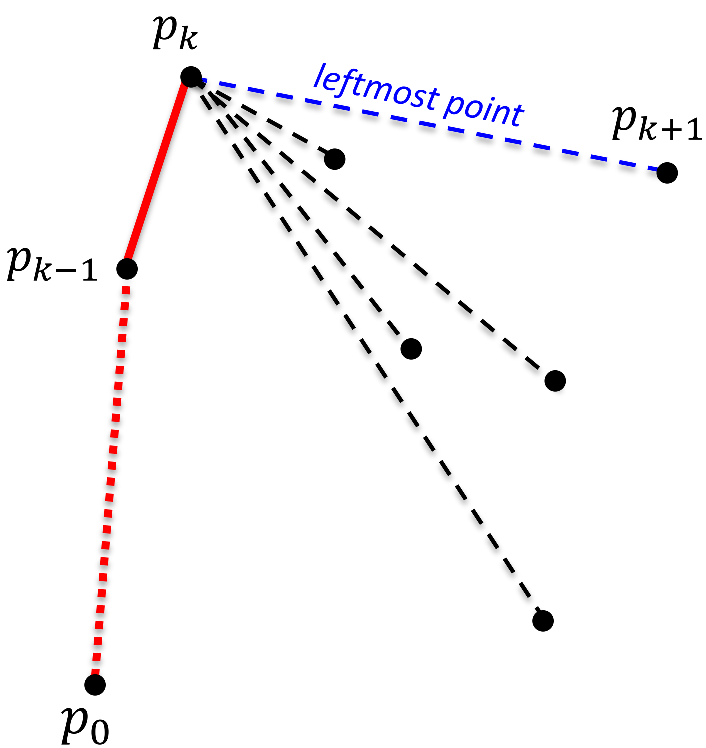

```
def left_of(p1:Point, p2:Point, p3:Point) -> bool:
    """Returns true if p3 is at the left of p1p2"""
    return (p2.x - p1.x)·(p3.y - p1.y) > (p2.y - p1.y)·(p3.x - p1.x)
```

Convex hull: gift wrapping algorithm



https://en.wikipedia.org/wiki/Gift_wrapping_algorithm

Convex hull: gift wrapping algorithm



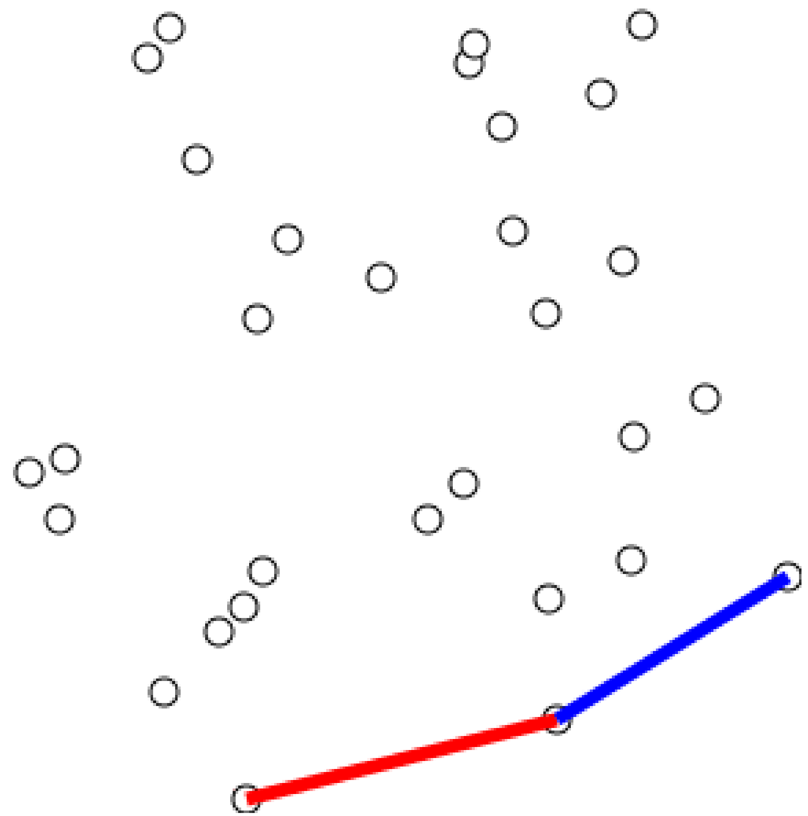
- **Input:** $p_1, p_2, \ldots, p_n$ (points in the plane).
- **Output:** P (the convex hull of $p_1, p_2, \ldots, p_n$).
- **Initial points:**
  p_0 with the smallest x coordinate.
- **Iteration:** Assume that a partial path with k points has been built (p_k is the last point). Pick some arbitrary $p_{k+1} \neq p_k$. Visit the remaining points. If some point q is at the left of $\overrightarrow{p_k p_{k+1}}$ redefine $p_{k+1} = q$.
- Stop when P is complete (back to point p_0).

Complexity: At each iteration, we calculate n angles. $T(n) = \mathrm{O}(hn)$, where h is the number of points in the convex hull. In the worst case, $T(n) = \mathrm{O}(n^2)$.

Convex hull: gift wrapping algorithm

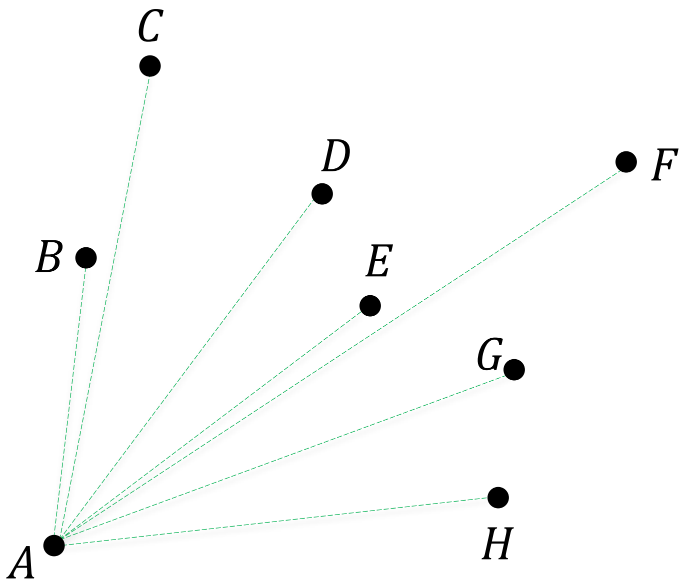
```
def gift_wrapping(pol: Polygon) -> Polygon:
    """Returns the convex-hull of a set of points"""
    hull: Polygon = []

    # Pick the leftmost point
    left = 0
    for i, p in enumerate(pol):
        if pol[i].x < pol[left].x:
            left = i

    p = left
    while True:  # Add points while the polygon is not closed
        hull.append(pol[p])  # Add point to the convex hull
        q = (p+1)%len(pol)  # Pick a point different from p

        for i, new_p in enumerate(pol): # Find leftmost point of p->q
            if left_of(pol[p], pol[q], new_p):
                q = i

        p = q  # This is the leftmost point
        if p == left:  # Stop if the point closes the polygon
            break

    return hull
```

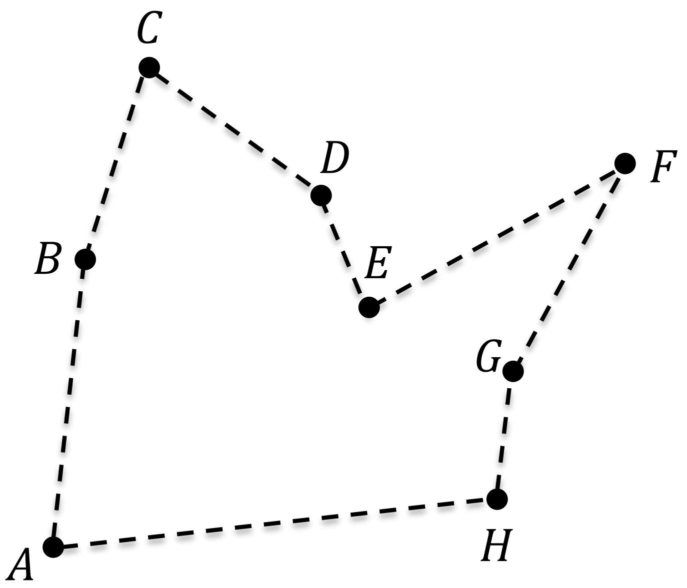
Convex hull: Graham Scan



https://en.wikipedia.org/wiki/Graham_scan

Convex hull: Graham scan





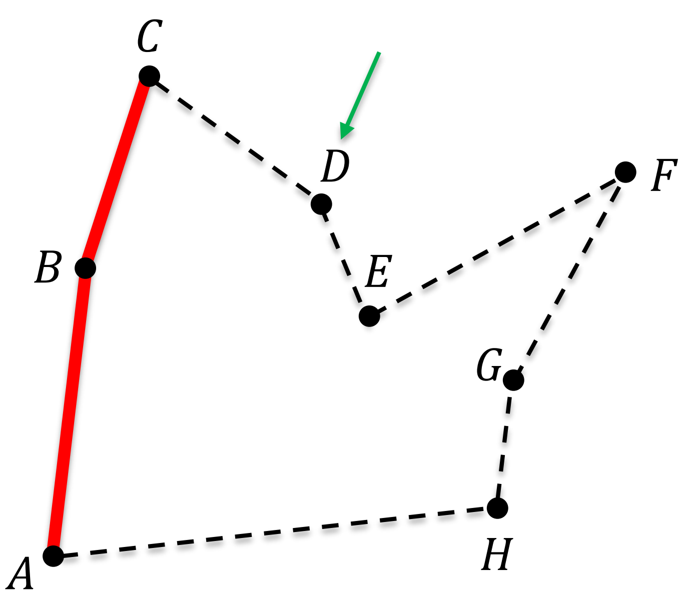
Convex hull: Graham scan

P:

A	B	C	D	E	F	G	H

Q:

A	B	C



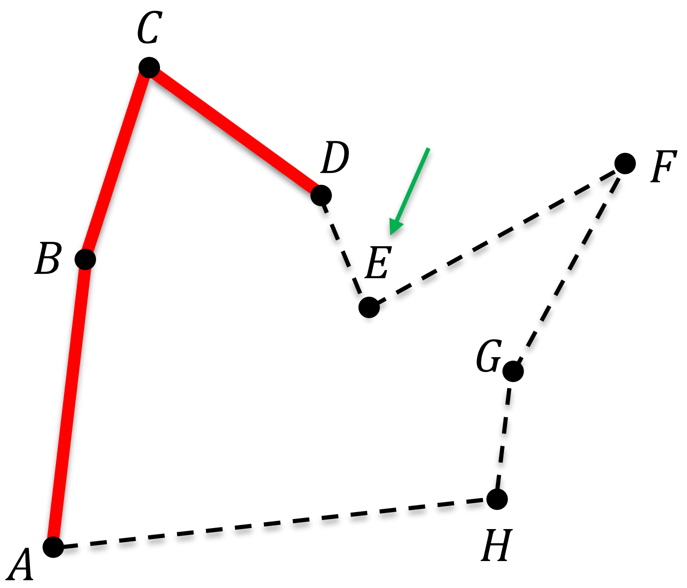
Convex hull: Graham scan

P:

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*

Q:

A	*B*	*C*	*D*



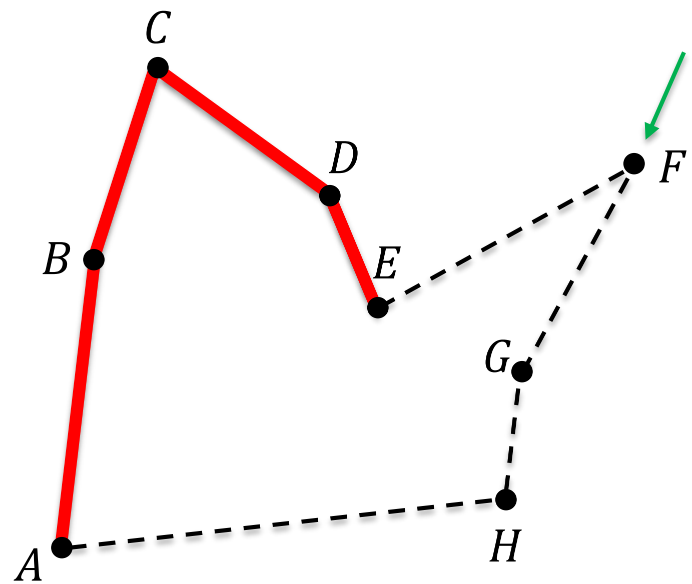
Convex hull: Graham scan

P:

A	***B***	***C***	***D***	***E***	***F***	***G***	***H***

Q:

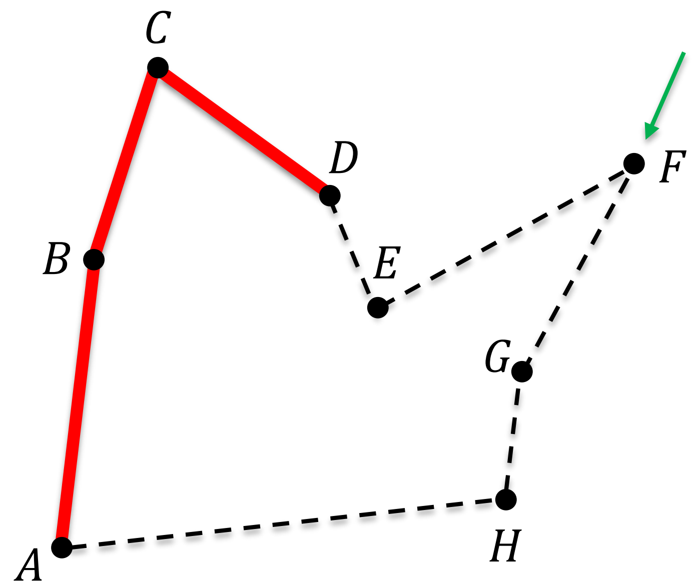
A	***B***	***C***	***D***	***E***



Convex hull: Graham scan

P:	A	B	C	D	E	F	G	H

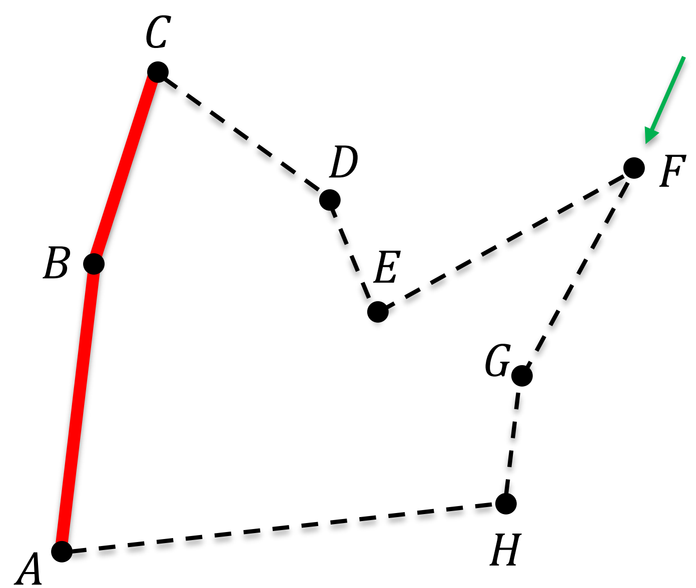
Q:	A	B	C	D



Convex hull: Graham scan

P:	A	B	C	D	E	F	G	H

Q:	A	B	C



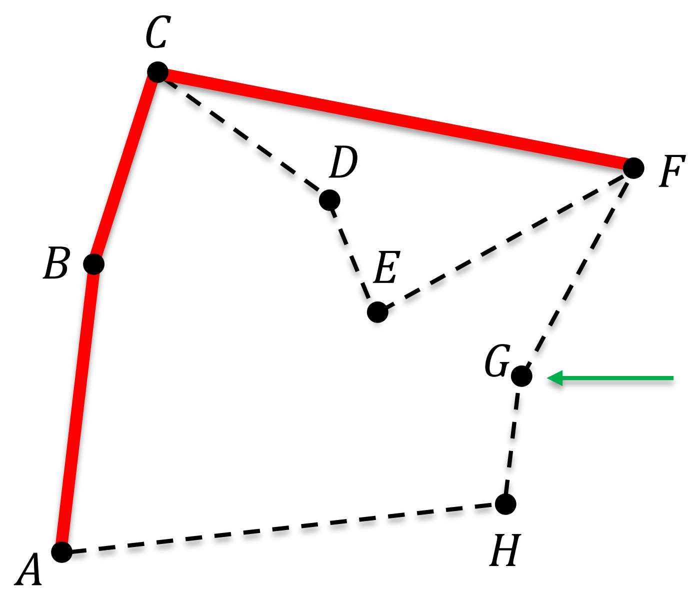
Convex hull: Graham scan

P:

A	***B***	***C***	***D***	***E***	***F***	***G***	***H***

Q:

A	***B***	***C***	***F***



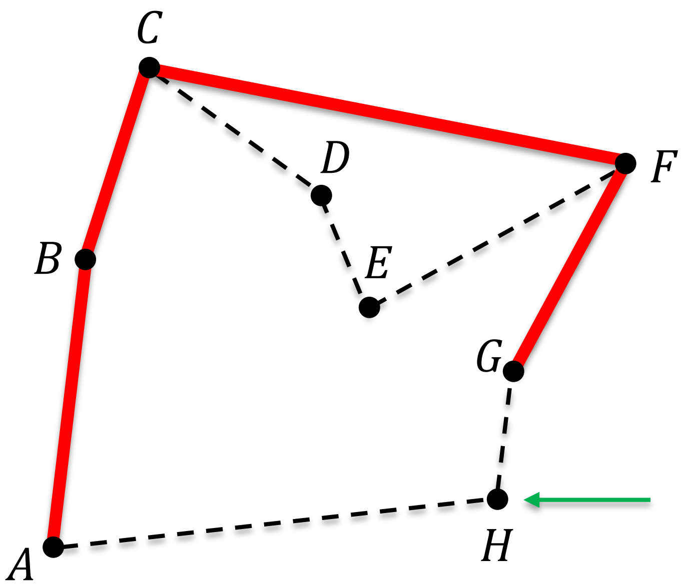
Convex hull: Graham scan

P:

A	B	C	D	E	F	G	H

Q:

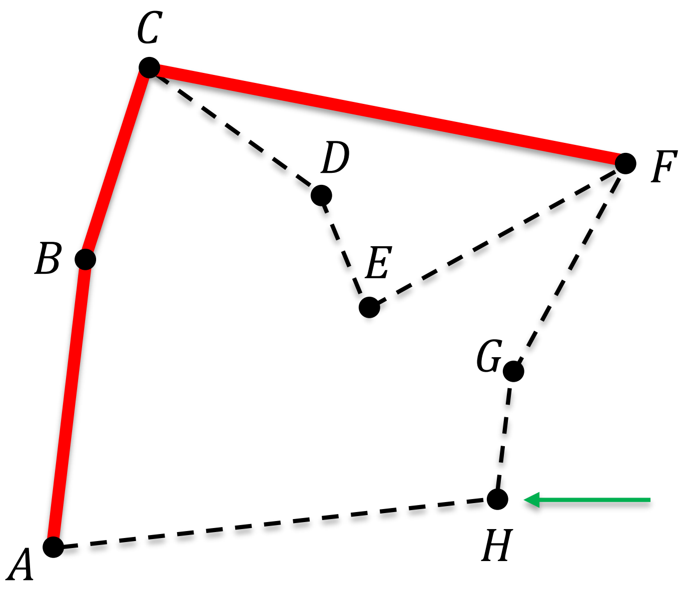
A	B	C	F	G



Convex hull: Graham scan

P: | | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H |

Q: | | | | |
|---|---|---|---|
| A | B | C | F |



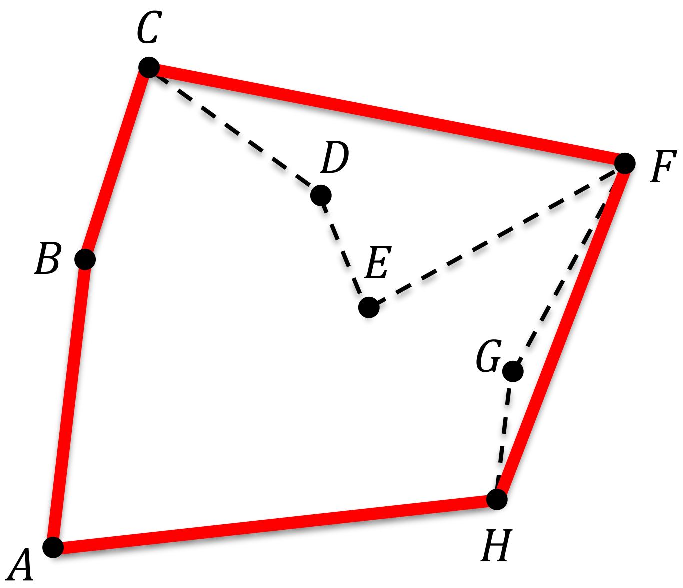
Convex hull: Graham scan

P:

A	B	C	D	E	F	G	H

Q:

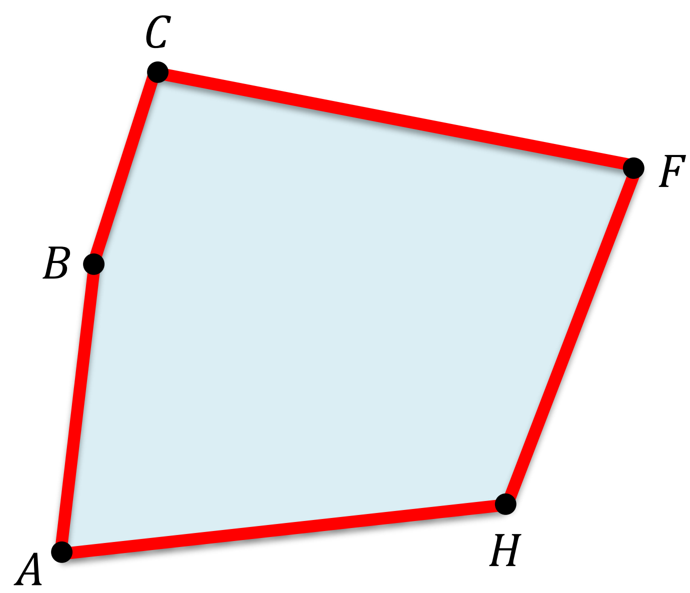
A	B	C	F	H



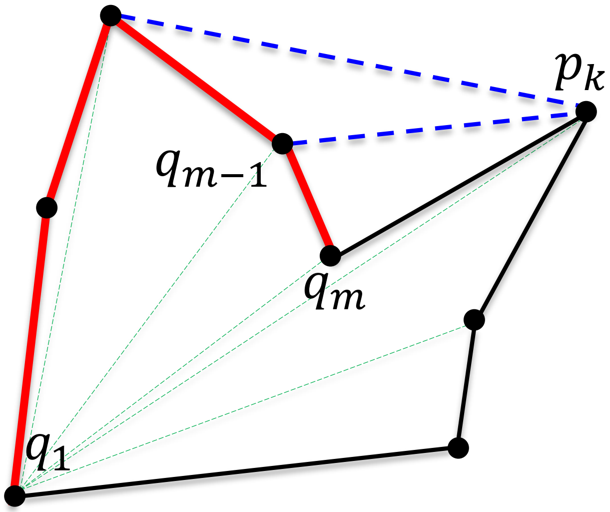
Convex hull: Graham scan

P: | A | B | C | D | E | F | G | H |

Q: | A | B | C | F | H |



Convex hull: Graham scan



Input: $p_1, p_2, \dots, p_n$ (polygon: points in the plane).

Output: $q_1, q_2, \dots, q_m$ (the convex hull).

Initially:
Create a simple polygon P (complexity $O(n \log n)$).
Assume the order of the points is $p_1, p_2, \dots, p_n$.

```python
def graham_scan(pol: Polygon) -> Polygon:
    """Returns the convex hull of a non-convex polygon"""
    hull = pol[0:3]
    for k in range(3, len(pol)):
        while left_of(hull[-2], hull[-1], pol[k]):
            hull.pop()
        hull.append(pol[k])
    return hull
```

Observation: each point p_k can be included in Q and deleted at most once.
The main loop of Graham scan has linear cost.
Complexity: dominated by the creation of the simple polygon → $O(n \log n)$.

EXERCISES

Summations

Prove the following equalities:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

For loops: analyze the cost of each code

Calculate the value of variable **s** at the end of each code

```
# Code 1
s = 0
for i in range(n):
    s += 1

# Code 2
s = 0
for i in range(0, n, 2):
    s += 1

# Code 3
s = 0
for i in range(n):
    s += 1
for j in range(n):
    s += 1
```

```
# Code 4
s = 0
for i in range(n):
    for j in range(n):
        s += 1

# Code 5
s = 0
for i in range(n):
    for j in range(i):
        s += 1

# Code 6
s = 0
for i in range(n):
    for j in range(i, n):
        s += 1
```

For loops: analyze the cost of each code

```
# Code 7
s = 0
for i in range(n):
    for j in range(n):
        for k in range(n):
            s += 1
```

```
# Code 8
s = 0
for i in range(n):
    for j in range(i):
        for k in range(j):
            s += 1
```

```
# Code 9
s = 0
i = 1
while i <= n:
    s += 1
    i *= 2
```

```
# Code 10
s = 0
for i in range(n):
    j = 1
    while j <= n:
        s += 1
        j *= 2
```

```
# Code 11
s = 0
for i in range(n):
    for j in range(i*i):
        for k in range(n):
            s += 1
```

```
# Code 12
s = 0
for i in range(n):
    for j in range(i*i):
        if j%i == 0:
            for k in range(n):
                s += 1
```

O, Ω or Θ?

The following statements refer to the ***insertion sort*** algorithm and the X's hide an occurrence of O, Ω or Θ. For each statement, find which options for $X \in \{\mathrm{O}, \Omega, \Theta\}$ make the statement true or false. Justify your answers.

1. The worst case is $X(n^2)$
2. The worst case is $X(n)$
3. The best case is $X(n^2)$
4. The best case is $X(n)$
5. For every probability distribution, the average case is $X(n^2)$
6. For every probability distribution, the average case is $X(n)$
7. For some probability distribution, the average case is $X(n \log n)$

Primality

The following algorithms try to determine whether $n \geq 0$ is prime.
Find which ones are correct and analyze their cost as a function of n.

```python
def is_prime1(n: int) -> bool:
    if n <= 1:
        return False
    for i in range(2,n):
        if n%i == 0:
            return False
    return True

def is_prime2(n: int) -> bool:
    if n <= 1:
        return False
    for i in range(2, int(math.sqrt(n))):
        if n%i == 0:
            return False
    return True

def is_prime3(n: int) -> bool:
    if n <= 1:
        return False
    for i in range(2, round(math.sqrt(n))):
        if n%i == 0:
            return False
    return True
```

```python
def is_prime4(n: int) -> bool:
    if n <= 1:
        return False
    for i in range(2, int(math.sqrt(n))+1):
        if n%i == 0:
            return False
    return True

def is_prime5(n: int) -> bool:
    if n <= 1:
        return False
    if n == 2:
        return True
    if n%2 == 0:
        return False
    for i in range(3, int(math.sqrt(n))+1, 2):
        if (n%i == 0):
            return False
    return True
```

The Sieve of Eratosthenes

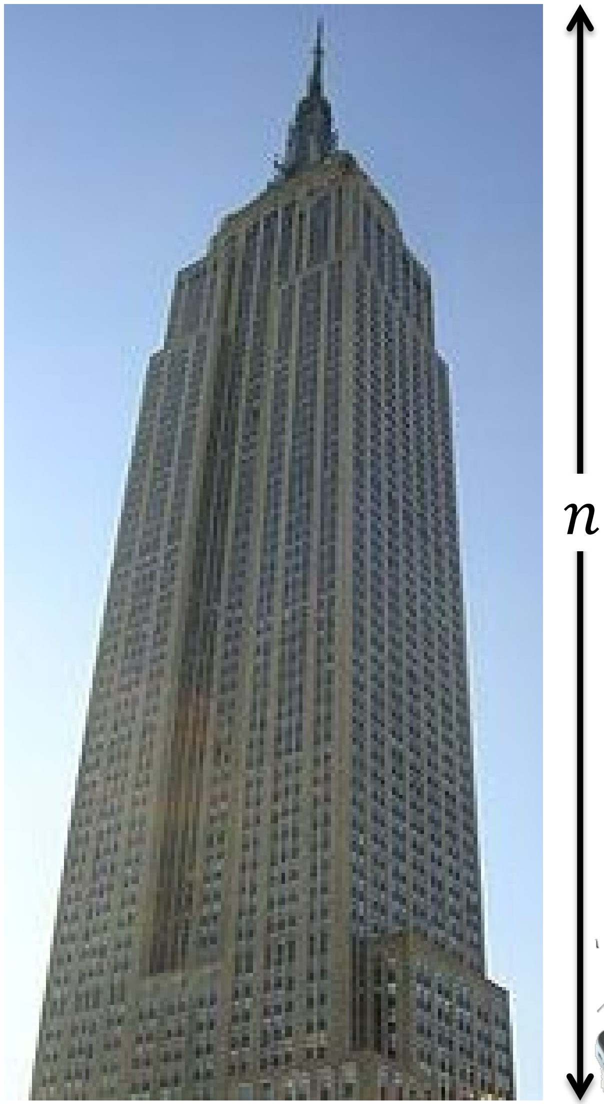
The following program is a version of the Sieve of Eratosthenes. Analyze its complexity.

```
def primes(n: int) -> list[bool]:
    p: list[bool] = [True]*(n+1)
    p[0] = p[1] = False
    for i in range(2, int(math.sqrt(n))+1):
        if p[i]:
            for j in range(i*i, n+1, i):
                p[j] = False
    return p
```

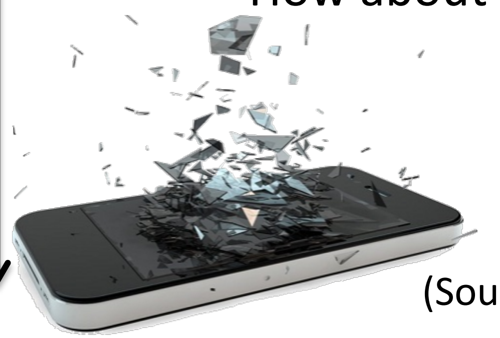
You can use the following equality, where $p \leq x$ refers to all primes $p \leq x$:

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1)$$

The Cell Phone Dropping Problem



- You work for a cell phone company which has just invented a new cell phone protector and wants to advertise that it can be dropped from the f^{th} floor without breaking.
- If you are given 1 or 2 phones and an n story building, propose an algorithm that minimizes the worst-case number of trial drops to know the highest floor it won't break.
- Assumption: a broken cell phone cannot be used for further trials.
- How about if you have p cell phones?



(Source: Wood & Yasskin, Texas A&M University)