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Characterizing turbulence in globally coupled maps with stochastic finite automata

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Abstract

We describe a method to discriminate between ordered and turbulent behavior in a general class of collective systems known as Globally Coupled Maps (GCM). Our method is able to discover an unknown small ordered region inside the turbulent phase of GCM parameter space. The computational nature of the method is the main novelty of our approach; it is another example of how measures based on computational notions of structure may provide new information in the study of dynamical systems. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

There are a plethora of systems in nature that compute, at least in a naive sense of computation. Ant colonies and brains are some unquestionable examples: they store, transmit and manipulate information. However, research on these systems has usually emphasized pattern formation and dynamical behaviour, leaving aside their computational properties and how dynamics and computation are mutually

involved. Of course, this bias in interests is a consequence of a lack of an adequate general theory of computation in dynamical systems, though some recent proposals, such as that of *Computational Mechanics* [1], aim to fill that gap. The purpose of the new field of computational mechanics is to uncover the *implicit manipulation of information embedded in natural systems* (also called intrinsic computation), connecting pattern discovering and pattern formation with the computational capabilities of the system [2].

The study of computation in physical/biological systems cannot be accomplished without a thorough understanding of the interplay between dynamics and computation in a more formal, and therefore more manageable, setting. Up to now, computational me-

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chanics has been applied to formal dynamical systems (but see [3]) leading to a determination of computational features of cellular automata [4], transitions in the period-doubling route to chaos [5] and one-dimensional spin systems [2], revealing some new properties of these systems not accounted for by classical measures such as entropies and algorithmic definitions of complexity [6]. Computational mechanics has also been applied to globally coupled maps (GCM), which were taken as models of collectives of dynamical complex agents in order to speculate about the possible trade-off between collective complexity and individual complexity observed in social insects [7,8].

The work we introduce here aims to go further into the study of the above mentioned trade-off, through a new method based on a computational measure of the interaction of the collective with an individual of the system. This method will allow us to characterize properly the turbulent phase of GCMs, that is, the same region of the parameter space that was found turbulent using either Kaneko’s cluster distribution function [9] or the mutual information between two randomly chosen elements of the system [7]. Besides, the method introduced here will allow us to find a small ordered region inside the turbulent phase, showing once more (see [2]) how computational measures of structure provide new information on dynamical systems under investigation.

2. Globally coupled maps

Globally coupled maps are systems of N coupled maps. In this Letter we will work with the logistic map

$$x_{n+1} = f_{\mu}(x_n) = 1 - \mu x_n^2 \tag{1}$$

This map is very well known because of its interesting dynamical behavior: depending on μ , the asymptotic behavior of the logistic map may be a fixed point, a limit cycle or a chaotic attractor. Furthermore, it is a representative element of a very large class of dynamical systems: Those with a period doubling route to chaos (see [5,6]). In GCMs we

have N maps interacting with a sort of mean state of the system (mean field)

$$h_n = \frac{1}{N} \sum_{i=1}^N f_{\mu}(x_n^i) \tag{2}$$

by means of the ‘interaction parameter’ ϵ , that is, we have $i = 1 \dots N$ elements defined in the following way:

$$x_{n+1}^i = (1 - \epsilon)f_{\mu}(x_n^i) + \epsilon h_n \tag{3}$$

This apparently simple system has indeed a very complicated dynamics with a parameter space (where $0 \leq \epsilon \leq 0.4$ and $1.4 \leq \mu \leq 2$) displaying turbulent and ordered behaviour. These diverse dynamics are due to the interplay between μ and ϵ , that is, to the interaction of the individual tendency of the system with the tendency to synchronize due to global averaging. After the GCM falls on an attractor, the N elements of the system split into k clusters (a cluster is defined by all the elements in the system with the same dynamics, i.e. all the maps that are synchronized *and* in phase). This allows one to define a cluster distribution function $Q(k)$ to characterize the different phases ($Q(k)$ is computed in the following way: ‘‘Take randomly chosen many (M) initial conditions. By counting the number of initial conditions that lead to a k -cluster attractor, and dividing it by M , we obtain $Q(k)$ ’’, see [9] for details).

Kaneko [9] reported that at the turbulent phase all attractors have many ($\approx N$) clusters, and the following condition

$$\sum_{k > \frac{N}{2}} Q(k) = 1$$

holds. Specifically, the average cluster number $R = \sum_{k=1}^N kQ(k)$ gives $R \approx N$ in the turbulent phase.

GCMs are representative of a large class of systems, since they are a mean field extension of Coupled Map Lattices (CML), collective systems where a rich variety of pattern dynamics and phase transitions are known to occur [9]. One of the implications of Kaneko’s work, as pointed out by Casti, is in fact that the coherent structures emergent from the connection of chaotic elements may be exactly what is

needed to account for things like the persistence of neural memory or, more generally, the emergence of patterns from a collection of disordered individual agents [10]. Furthermore, it has been shown that CMLs are also mathematical models of parallel deterministic machines [11].

3. Stochastic finite automata and globally coupled maps

We have shown elsewhere [7,8] that the influence of the mean field (Eq. 2, the ‘collective’) in a (randomly) chosen individual of the GCM may be measured by computing its *statistical* complexity (and its associated automaton, see below). Our method will be based on these measures to compare an individual in the GCM with an individual perturbed with a noise ξ related to the ‘collective’ h_n in a very specific way.

In order to get a computational measure of the behavior of an individual belonging to a certain GCM, we must get a long enough orbit of that individual dynamics and discretize it with the partition

$$\Pi = \{x_i^j \in [-1, 0) \Rightarrow S_i^j = 0; x_i^j \in [0, 1] \Rightarrow S_i^j = 1\}$$

where x_i^j is the state of the j -th individual at time step i and $S_i^j S_2^j \dots$ will be the discretized orbit of the j -th individual, its *symbolic dynamics* [5,12,13]. This partition is generating for the logistic map (to detail the meaning of ‘generating’ is clearly beyond the scope of this Letter, see [6,13]), but it is not known if it is generating for an individual in a GCM. The problem of constructing generating partitions in multidimensional systems (such as GCMs) has not been entirely solved yet ([6], Chap. 4). We have chosen Π as a first approximation to the problem [7].

From the bit sequence $S_1^j S_2^j \dots$ it is possible to build a (stochastic) finite automaton [14], giving us the computational counterpart of the original dynamics and its associated statistical complexity C , that is, roughly, the Shannon entropy of the stationary probability distribution of the stochastic automaton, if viewed as a Markov chain (see [1,4,5,8] for details,

though in this Letter the so called *topological* complexity – the log of the number of states of the automaton – could also be used). C will quantify the above mentioned intrinsic computation, provided that the automaton was a feasible model of the dynamical behaviour of the i -th individual [1].

Now, for a given GCM, that is, given N , ϵ , μ and a randomly generated initial condition, we can choose randomly an individual from the GCM and, after a long enough transient, get a bit sequence (using the partition Π) with which to compute its associated automaton and its complexity C_{gcm} . Besides, we can easily obtain a histogram of h_n (given a partition of the interval $[-1, 1]$ with some resolution Δx , see Fig. 1), that is, an approximation of the distribution $P(h)$ with which we build the following noisy logistic map:

$$x_{n+1} = (1 - \epsilon)f_\mu(x_n) + \epsilon\xi_n \quad (4)$$

where ξ_n are independent and identically distributed random variables (with distribution $P(\xi)$). From 4 and a randomly chosen initial condition we generate another bit sequence and compute its associated automaton and complexity C_ξ . In this case we have also used Π in order to be able to compare both automata, though some heuristic guides to find ‘adequate’ (a definition of ‘adequate’ is beyond the scope of this Letter) partitions for stochastic dynamical systems are available [15].

Once we have the automata C_{gcm} and C_ξ we can argue as follows: A (randomly) chosen individual

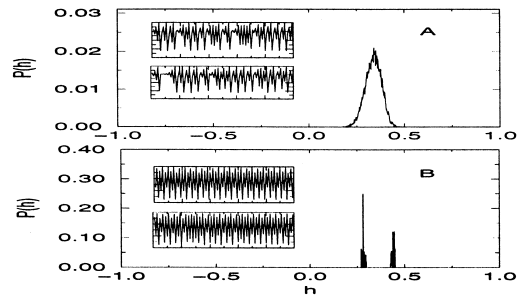


Fig. 1. Probability distributions $P(h)$ for a GCM in the turbulent phase (A, $\mu = 1.76$, $\epsilon = 0.04$) and in the ordered phase (B, $\mu = 1.58$, $\epsilon = 0.08$). Insets: in each case we show the dynamics of a single element in the GCM (above) and the corresponding single noisy logistic map (below).

inside the GCM will ‘perceive’ the rest of the system, h_n , as perturbing its own behavior (see Eq. 3); in the turbulent region of the parameter space the individual will ‘see’ the above mentioned perturbation as random, since it will be the contribution of $N - 1$ ‘almost’ independent dynamics (the $\approx N$ clusters). We just need to compare the structure of this individual behavior with the structure of another individual perturbed with a noise characterized by the same probability distribution of the perturbation h_n . Thus, we should expect $C_{\text{gcm}} = C_\xi$. However, in the ordered region, the system splits in a number of synchronized clusters where each cluster displays quite ordered behaviour [9], therefore one should observe the following inequality: $C_{\text{gcm}} < C_\xi$, since a noisy perturbation such as that of Eq. 4 is expected to cause certain disorder on the deterministic dynamics, though not enough to ‘mask’ the deterministic contribution. In fact, $P(h)$ will be, in the ordered phase, a discrete distribution with $P(h^*) > 0$ for a finite set of h^* values (Fig. 1, B). So our hypothesis would be:

- Turbulent GCM $\Rightarrow C_{\text{gcm}} = C_\xi$
- Ordered GCM $\Rightarrow C_{\text{gcm}} < C_\xi$

Thus, the automata will quantify the structure of the dynamics. If they are equal, that is, if the structure of the dynamical behavior of individuals is the same, we postulate that the given (μ, ϵ) belong to the turbulent phase. Otherwise they belong to the ordered phase. In this sense, it is the intrinsic computation of the system that is used to discriminate between different phases.

We have confirmed numerically our hypothesis using a ‘grid’ over the parameter space $0 \leq \epsilon \leq 0.2$ and $1.4 \leq \mu \leq 2$ with $\Delta\epsilon = 0.02$ and $\Delta\mu = 0.06$. We have computed C_{gcm} and C_ξ for each (μ, ϵ) (both complexities were computed with the Crutchfield–Young machine reconstruction algorithm [4,5] from a bit sequence of length 10^6) and we have classified the point as ‘ordered’ or ‘turbulent’ according to the relation between the two complexities, as stated above (we have not found any (μ, ϵ) pair in which $C_{\text{gcm}} > C_\xi$), see Figs. 2 and 3. Our resulting parameter space (see Fig. 4) is identical to that found by Kaneko (see [7,9]) using the cluster distribution func-

tion $Q(k)$, but a small domain of ordered behaviour (characterized by the spontaneous formation of few clusters, that is, identical to Kaneko’s ordered region [9]) has been found inside the previous turbulent phase.

Clearly there is a relevant difference between the GCM and the perturbed logistic map: for the single noisy map the random field ξ obtained by randomly sampling $P(\xi)$ is uncorrelated in time and this is not the case in the GCM, a completely deterministic system. So we might suspect that some relevant structure should be present in the turbulent phase. This has been conjectured by Kaneko [16,18]: It is not difficult to show that the fluctuations of the field h are given by [17]

$$\begin{aligned} \langle (h - \langle h \rangle)^2 \rangle &= \left\langle \left(\frac{1}{N} \sum_{i=1}^N [f_\mu(x^i) - \langle f_\mu(x^i) \rangle] \right)^2 \right\rangle \\ &= \frac{1}{N^2} \sum_{i=1}^N \sigma_i^2 + \frac{1}{N^2} \sum_{i \neq j} \alpha_{ij} \end{aligned} \quad (5)$$

where σ_i^2 is the variance for the single map $f_\mu(x^i)$ and α_{ij} is the covariance between $f_\mu(x^i)$ and $f_\mu(x^j)$. The non-statistical behaviour would come from the last term in the right hand side, which is zero only for some f_μ , but not for the logistic map. However, in terms of computational mechanics, no real difference arises between the GCM at the turbulent phase and a single map with a noise term with the same statistical structure. Recent work [18] on the internal structure of the turbulent phase is able to provide an explanation for this phenomenon: Hidden coherence becomes clearly manifest only at large system sizes. In the paper [18] the authors measure $\langle (h - \langle h \rangle)^2 \rangle$ and the correlation dimension for different values of N , μ and ϵ showing clearly that, for small system sizes, hidden coherence is indistinguishable from noise. Thus, this underlying structure does not contribute to the statistical complexity (approximated with the reconstructed finite automata), implying that the system lacks any information processing capability [2] beyond the ‘trivial’ one associated to the noisy map.

Finally, the small ordered region we have found inside the turbulent phase seems to be part of a much

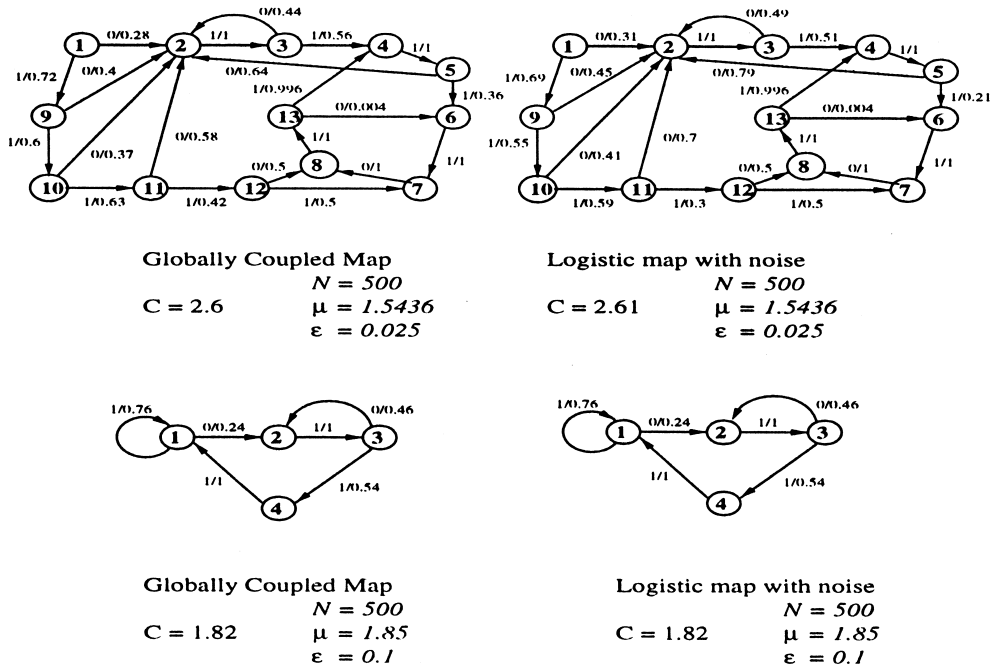


Fig. 2. C_{gcm} and C_{ξ} computed for two (μ, ϵ) pairs belonging to the turbulent phase, according to [9]. As we can see, $C_{gcm} \approx C_{\xi}$.

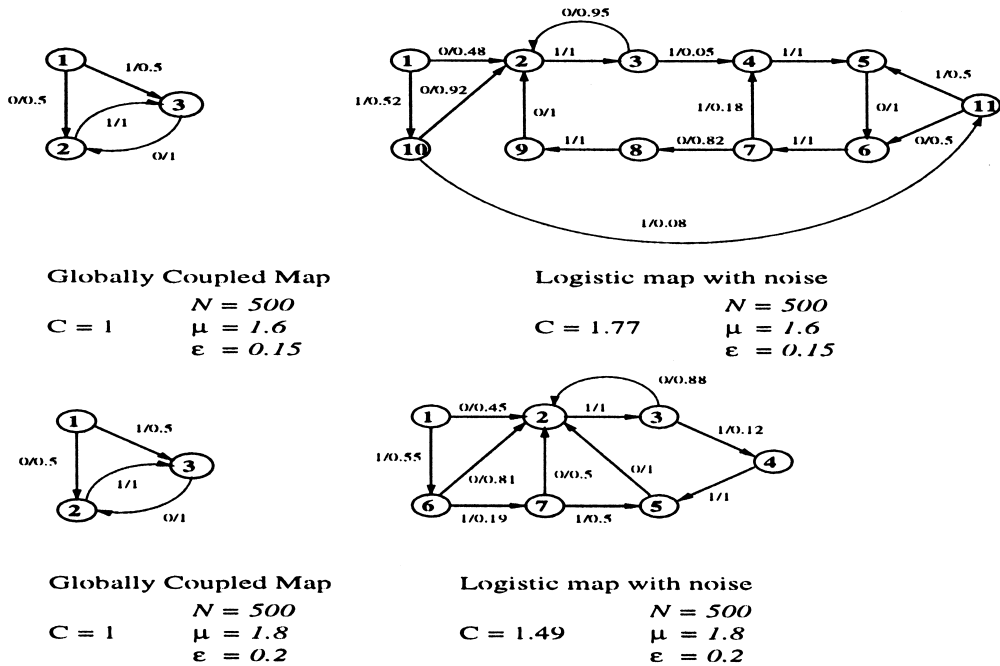


Fig. 3. C_{gcm} and C_{ξ} computed for two (μ, ϵ) pairs belonging to the ordered phase [9]. In these cases $C_{gcm} < C_{\xi}$.

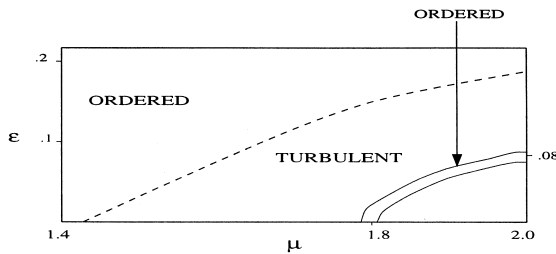


Fig. 4. Phase space of GCM ($N=500$). C_{gcm} and C_{ξ} were computed for 90 (μ, ϵ) -pairs (44 belonging to Kaneko's turbulent phase, 46 to the ordered phase) and classified according to either $C_{\text{gcm}} = C_{\xi}$ ('turbulent') or $C_{\text{gcm}} < C_{\xi}$ ('ordered'). C_{gcm} was computed choosing at random one element of the GCM (after a transient of 10^4 steps).

more complicated tongue-like bifurcation structure, as pointed out by [18]. Further analysis must be done to fully understand this behavior of the mean-field in networks of chaotic elements.

4. Conclusion

We have introduced a new method to calculate the different phases in the parameter space of GCMs, based on reconstructing the (stochastic) finite automata associated to the dynamics of individual elements in the GCM. The method consists of a comparison between the automata reconstructed from the dynamics of maps in a GCM and noisy maps (with a noise related to the GCM in a well defined way). This comparison allows one to classify (μ, ϵ) points of the parameter space as either 'turbulent' or 'ordered'. Our method has allowed us to uncover a small ordered region inside the previously known, according to Kaneko, turbulent phase. We have found that this new region has the same properties that characterized Kaneko's ordered phase. We think that our method gives support to the belief that computational analysis of dynamical systems will allow us to get new information on these systems; information that more traditional methods may be unable to provide. Further work must be done in order to detect, by computational means, the hidden coherence of the mean field h_n and to quantify, in information processing terms, its influence on individual maps in the GCMs.

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