On the Connectivity of Dynamic Random Geometric Graphs*

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Abstract

We provide the first analytical results for the connectivity of dynamic random geometric graphs — a model of mobile wireless networks in which vertices move in random directions, and an edge exists between two vertices if their Euclidean distance is below a given value. We provide precise asymptotic results for the expected length of the connectivity and disconnectivity periods of the network. We believe the formal tools developed in this work could be of use in more concrete settings, in the same manner as the development of connectivity threshold for static random geometric graphs has affected a lot of research done on ad hoc networks. In the process of proving results for the dynamic case we also obtain new asymptotically precise bounds for the probability of the existence of a component of fixed size ℓ, $\ell \geq 2$, for the static case.

1 Introduction

Random Geometric Graphs (RGG) have been a very influential and well-studied model of large networks, such as sensor networks, where the network agents are represented by the vertices of the RGG, and the direct connectivity between agents is represented by the edges. Informally, given a radius r, a random geometric graph results from placing a set of n vertices uniformly and independently at random on the unit torus $[0, 1)^2$ and connecting two vertices if and only if their *distance* is at most r, where the distance depends on the chosen metric.

In the late 90s, Penrose Gupta-Kumar and Apple and Russo gave accurate estimations for the value of r at which with high probability, a RGG becomes connected (see [9], for the historical references). This happens at $r_c = \sqrt{\frac{\ln n \pm O(1)}{\pi n}}$, for the Euclidean distance in $[0, 1)^2$. Thereafter, hundreds of researchers have used those basic

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results on connectivity to design algorithms for more efficient coverage and communication in ad hoc networks, and in particular for sensor networks . On the other hand, much work has been done on the graph theoretical properties of *static* RGG, which is comprehensively summarized in [9].

Recently, there has been an increasing interest for MANETs (mobile ad hoc networks). Several "practical" models of mobility have been proposed in the literature — for a survey of these models we refer to [6]. In all these models, the connections in the network are created and destroyed as the vertices move closer together or further apart. In all previous work, the authors performed **empirical studies** on connectivity issues and routing performance.

The particular mobility model we study in the present paper was introduced by Guerin [4], and it is often called in the literature the *Random Walk* model. This model can be seen as the foundation for most of the mobility models developed afterwards (see [6]). In the Random Walk model [4], each vertex chooses uniformly at random a direction (angle) in which to travel and also a velocity from a given distribution of velocities. Then, each vertex moves at its selected velocity towards its selected direction. After some randomly chosen period of time, each vertex halts, chooses a new direction and velocity, and the process repeats. An **experimental study** of the connectivity of the resulting ad hoc network for different values of n and r for this particular model is presented in [10]. As it is stated in the same paper, in many applications which are not life-critical, *"temporary network disconnections can be tolerated, especially if this goes along with a significant decrease of energy consumption."* This means, that the communication distance r should be kept as small as possible, but still large enough to guarantee a connected graph, *so it is desirable to set* r *to be around* r_c.

In the present paper, we perform the first **analytical study** of connectivity in the Random Walk model, presented in [4]. The setting of the model that we study, is the following: Given an initial RGG with n vertices and a radius r set to be at the known connectivity threshold r_c, each vertex chooses independently and uniformly at random an angle $\alpha \in [0, 2\pi)$, and moves a distance s in that direction for a period of m steps. Therefore the total distance before changing direction is $d = sm$. Then, a new angle is selected independently for each vertex, and the process repeats. We denote this graph model the *Dynamic Random Geometric Graph.*

Our main result (Theorem 1 in Section 2) provides precise asymptotic results for the expected number of steps that the dynamic graph remains connected once it becomes connected, and the expected number of steps the graph remains disconnected once it becomes disconnected. We remark that we only consider the case $r = r_c$ here, since we are mainly interested in situations in which the network is neither highly connected nor highly disconnected most of the time. By such choice of r, connectivity can be guaranteed with (arbitrarily large) constant probability and at the same time the energy consumption is as small as possible. Our results are expressed in terms of n, s and m. Surprisingly, the final expression on the length of connectivity periods does not depend on the number m of the steps (and therefore on the distance d covered by the vertices) between each change of angles, as long as the angles do eventually change, no matter how large the value of m is. It is worth to note here that the evolution of connectivity of this model **is not Markovian**, in the sense that staying connected for a large number of steps does have an impact on the probability of being connected at the next step. However, one key (and rather counterintuitive) fact is that, despite this absence of the Markovian property, the argument to prove our result is mainly based on the analysis of the connectivity changes in two consecutive steps (see Lemma 8).

Throughout the paper, we consider the usual Euclidean distance on the unit torus $[0, 1)^2$, but similar results can be obtained for any ℓ_p-normed distance, $1 \leq p \leq \infty$. Our results can also be extended to the k-cube $[0, 1]^k$, for any fixed k. Moreover, our argument can be easily adapted to cover the more general model in which each vertex i covers a different distance $d_i = m_i s$ in one direction before changing the angle.

To the best of our knowledge, the present work is the first one in which the dynamic connectivity of RGG is studied theoretically. In [3] the loosely related problem of the connectivity of the ad hoc graph produced by w vertices moving randomly along the edges of a $n \times n$ grid is studied. The authors of [7] use a similar model to the one used in the present paper to prove that if the vertices are initially distributed uniformly at random, the distribution remains uniform at any time.

As a side product we also derive an interesting new result for the *static case*: At the threshold of connectivity r_c and for any fixed integer $\ell > 1$, the probability of having some component of size at least ℓ other than the giant component is asymptotically $\Theta(1/\log^{\ell-1} n)$. Moreover, the most common of such components are cliques with exact size ℓ.

Notation and Organisation. Unless otherwise stated, all our stated results are asymptotic as $n \to \infty$. As usual, the abbreviation a.a.s. stands for *asymptotically almost surely*, i.e. with probability $1 - o(1)$. In Section 2 we state our results and give an outline of the proof. Section 3 contains technical definitions and statements of auxiliar results needed in our argument. Due to lack of space, most of the actual proofs are deferred to the long version.

2 Main Result and Idea of the Proof

2.1 The Model and our Result

The formal definition of a random geometric graph is the following (see [9]): Given a set of n vertices and a positive real $r = r(n)$, each vertex is placed at some random position in the unit torus $[0, 1)^2$ selected independently and uniformly at random (u.a.r.). We denote by $X_i = (x_i, y_i)$ the random position of vertex i for $i \in \{1, \ldots, n\}$, and let $\mathcal{X} = \mathcal{X}(n) = \bigcup_{i=1}^{n} X_i$. Note that with probability 1 no two vertices choose the same position and thus we restrict the attention to the case that $|\mathcal{X}| = n$. We define $G(\mathcal{X}; r)$ as the random graph having $\mathcal{X}$ as the vertex set, and with an edge connecting each pair of vertices X_i and X_j in $\mathcal{X}$ at distance $\mathsf{d}(X_i, X_j) \leq r$, where $\mathsf{d}(\cdot, \cdot)$ denotes the Euclidean distance in the torus. We refer to $G(\mathcal{X}; r)$ as the *static model*.

Let us denote by $\mathcal{C}$ and $\mathcal{D}$ the events that $G(\mathcal{X};r)$ is connected and disconnected, respectively. We also define the parameter $\mu = ne^{-\pi r^2 n}$. In our analysis we restrict to the case $\mu = \Theta(1)$, which is equivalent to $r = r_c = \sqrt{\frac{\log n \pm O(1)}{\pi n}}$.

Formal definition of the dynamic model. Given $s = s(n)$, $s \in \mathbb{R}^+$ and $m = m(n)$, $m \in \mathbb{Z}^+$, consider the random process $(\mathcal{X}_t)_{t\in\mathbb{Z}} = (\mathcal{X}_t(n,s,m))_{t\in\mathbb{Z}}$: At step $t = 0$, n vertices are scattered independently and u.a.r. over $[0,1)^2$, as in the static model. Moreover, for each vertex i and for each interval of steps $[t, t+m]$ with $t \in \mathbb{Z}$ divisible by m, an angle $\alpha \in [0, 2\pi)$ is chosen independently and u.a.r., and this angle determines the direction of i between steps t and $t+m$. At every step, each one of the vertices jumps a distance s in the corresponding direction. Since the dynamic process is time-reversible, it also makes sense to consider negative steps. The dynamic random geometric graph is then defined as a sequence $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$, where for each particular value of t, $G(\mathcal{X}_t;r)$ is the random geometric graph with vertex set $\mathcal{X}_t$.

In order to get a better picture of the model, it is natural to consider the underlying continuous-time model, in which the vertices move continuously at *constant speed* around the torus rather than performing jumps at discrete steps. In this model, which we denote by $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{R}}$, the vertices travel a distance $d = sm$ between each change of direction. Observe that our model $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ can be regarded a discrete approximation to $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{R}}$, in which we take m snapshots of the process between each change of direction. Hence, for any given $d = d(n)$, we can infer the approximate behaviour of $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{R}}$ from the study of $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$, by choosing a large m and thus a small s.

To state our main theorem precisely, we need a few definitions. We denote by $\mathcal{C}_t$ ($\mathcal{D}_t$) the event that $\mathcal{C}$ ($\mathcal{D}$) holds at step t. In $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$, define $L_t(\mathcal{C})$ to be the number of consecutive steps that $\mathcal{C}$ holds starting at step t (possibly ∞ and also 0 if C_t does not hold). The distribution of $L_t(\mathcal{C})$ does not depend on t (see Lemma 3), and we often omit the t when it is understood. $L_t(\mathcal{D})$ is defined analogously by interchanging $\mathcal{C}$ and $\mathcal{D}$.

We are interested in the length of the periods in which $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ stays connected (disconnected). More precisely, we consider the expected number of steps that $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ stays connected (disconnected) starting at step t conditional upon the fact that it becomes connected (disconnected):

$$P_{\mathcal{C}} = \mathbf{E}\left(L_t(\mathcal{C}) \mid \mathcal{D}_{t-1} \wedge \mathcal{C}_t\right), \qquad P_{\mathcal{D}} = \mathbf{E}\left(L_t(\mathcal{D}) \mid \mathcal{C}_{t-1} \wedge \mathcal{D}_t\right).$$

Our main theorem then reads as follows:

Theorem 1. *Let $r = r_c$. The expected length of the connectivity (disconnectivity) periods in $\big(G(\mathcal{X}_t;r)\big)_{t\in\mathbb{Z}}$ is*

$$P_{\mathcal{C}} \sim \begin{cases} \frac{\pi}{4\mu srn} & \text{if } srn = o(1), \\ \frac{1}{1-e^{-\mu(1-e^{-4srn/\pi})}} & \text{if } srn = \Theta(1), \\ \frac{1}{1-e^{-\mu}} & \text{if } srn = \omega(1), \end{cases} \qquad P_{\mathcal{D}} \sim \begin{cases} \frac{\pi(e^{\mu}-1)}{4\mu srn} & \text{if } srn = o(1), \\ \frac{e^{\mu}-1}{1-e^{-\mu(1-e^{-4srn/\pi})}} & \text{if } srn = \Theta(1), \\ e^{\mu} & \text{if } srn = \omega(1), \end{cases}$$

Notice that surprisingly these results are independent of the number of steps m (and thus of the travelled distance d) between the changes of direction of the vertices. In fact, the proof of Theorem 1 only requires that the vertices change their direction eventually.

Intuitively speaking, the consequences of the result are the following: First observe that, for $s = o(1/(rn))$, the expected number of steps in a period of connectivity (disconnectivity) has a factor inversely proportional to s. This means that for any $s = o(1/(rn))$ the expected total distance $P_{\mathcal{C}} \cdot s$ covered by each vertex during a connectivity period (disconnectivity period, respectively) is

$$P_{\mathcal{C}} \cdot s \sim \frac{\pi}{4\mu rn} \sim \frac{\pi\sqrt{\pi}}{4\mu\sqrt{n \ln n}} \qquad \left(\text{respectively,} \quad P_{\mathcal{D}} \cdot s \sim \frac{\pi(e^{\mu}-1)}{4\mu rn} \sim \frac{\pi\sqrt{\pi}(e^{\mu}-1)}{4\mu\sqrt{n \ln n}}\right).$$

Moreover, we can choose s small enough, such that we do not expect $\left(G(\mathcal{X}_t; r)\right)_{t\in\mathbb{R}}$ to become temporarily disconnected between two consecutive steps of $\left(G(\mathcal{X}_t; r)\right)_{t\in\mathbb{Z}}$ in which the graph is connected, and we can approximate the continuous model by the discrete model.

2.2 Outline of the Proof

The main ingredient of the proof is the fact that $P_{\mathcal{C}}$ and $P_{\mathcal{D}}$ can be expressed in terms of the probabilities of events involving only two consecutive steps. We stress this fact because the sequence of connected/disconnected states of $\left(G(\mathcal{X}_t; r)\right)_{t\in\mathbb{Z}}$ is not Markovian, since staying connected for a long period of time makes it more likely to remain connected for one more step. More precisely, in Lemma 8 we show that it suffices to compute the probabilities of the events:

$$(\mathcal{C}_t \wedge \mathcal{D}_{t+1}), \quad (\mathcal{D}_t \wedge \mathcal{C}_{t+1}), \quad \mathcal{C} \quad \text{and} \quad \mathcal{D}. \tag{1}$$

However, the lemma requires that the expectations $\mathbf{E}(L(\mathcal{C}))$ and $\mathbf{E}(L(\mathcal{C}))$ are finite, which is proven in Lemma 10. From Equation (2) in Section 3 and Corollary 7 we obtain the probabilities of the events in (1). It turns out that the existence/non-existence of isolated vertices is asymptotically equivalent to the disconnectivity/connectivity of the graph, both in the static case $G(\mathcal{X}; r)$ and for two consecutive steps of $\left(G(\mathcal{X}_t; r)\right)_{t\in\mathbb{Z}}$. Proposition 5 characterizes the changes of the number of isolated vertices between two consecutive steps. The proof is based on the computation of the joint factorial moments of the variables accounting for these changes. At first sight, it is not obvious that the probability of existence of components of larger sizes is negligible compared to the probability of sudden apperance of isolated vertices, but this is indeed shown in Lemma 6. The proof is quite lengthy, since the arguments for components of different sizes and/or different diameters are very different.

As a side product of the techniques applied to the dynamic case we also obtain a new result for the static case: For a fixed integer $\ell \geq 1$, let K_ℓ be the number of components in $G(\mathcal{X}; r)$ of size exactly ℓ. For any fixed $\epsilon > 0$, let $K'_{\epsilon,\ell}$ be the number of components of size exactly ℓ which have all their vertices at distance at most ϵr from their leftmost one. Let $\widetilde{K}_\ell$ denote the number of components of size $\geq \ell$ and which are not *solitary*. We will define *solitary* components in Section 3, but intuitively a solitary component can be thought of as a unique and very large component (usually of size $\Theta(n)$). Notice that $K'_{\epsilon,\ell} \leq K_\ell \leq \widetilde{K}_\ell$. We can prove the following theorem:

Theorem 2. *Let $\ell \geq 2$ be a fixed integer. Let $0 < \epsilon < 1/2$ be fixed. Assume that $\mu = \Theta(1)$. Then*

$$\mathbf{Pr}\left[\widetilde{K}_\ell > 0\right] \sim \mathbf{Pr}\left[K_\ell > 0\right] \sim \mathbf{Pr}\left[K'_{\epsilon,\ell} > 0\right] = \Theta\left(\frac{1}{\log^{\ell-1} n}\right).$$

The theorem states that asymptotically all the weight in the probability that $\widetilde{K}_\ell > 0$ comes from components which also contribute to $K'_{\epsilon,\ell}$ for ϵ arbitrarily small. This implies that at r_c, the more common components of size $\geq \ell$ are cliques of size exactly ℓ, with all their vertices close together.

3 Sketch of Technical Details

In order to prove our main theorem, we need some background about the static case which can be regarded as a snapshot of the dynamic case. Recall that K_1 is the random variable counting the number of isolated vertices in $G(\mathcal{X}; r)$. It is well known (see [9]) that for $r = r_c$ a.a.s. there is only one giant component and a Poisson number of isolated vertices with parameter $\mu = ne^{-\pi r^2 n} = \Theta(1)$. Hence,

$$\mathbf{Pr}\left[\mathcal{C}\right] \sim \mathbf{Pr}\left[K_1 = 0\right] \sim e^{-\mu} \quad \text{and} \quad \mathbf{Pr}\left[\mathcal{D}\right] \sim \mathbf{Pr}\left[K_1 > 0\right] \sim 1 - e^{-\mu}. \tag{2}$$

Therefore, the probability that $G(\mathcal{X}; r)$ has some component of size greater than 1 other than the giant component is $o(1)$.

For the analysis of the dynamic model we need additional definitions. We denote by $X_{i,t} = (x_{i,t}, y_{i,t})$ the position of i at time t. Let $\mathcal{X}_t = \bigcup_{i=1}^n X_{i,t}$ be the set of positions of the vertices at time t. The following lemma (see [7]) indicates that the dynamic model at any fixed time t can be seen as a copy of the static model.

Lemma 3. *At any fixed step $t \in \mathbb{Z}$, the vertices are distributed over the torus $[0,1)^2$ independently and u.a.r. Consequently for any $t \in \mathbb{Z}$, $G(\mathcal{X}_t; r)$ has the same distribution as $G(\mathcal{X}; r)$.*

Let us consider two arbitrary consecutive steps t and $t+1$ of $(\mathcal{X}_t)_{t \in \mathbb{Z}}$, t an arbitrary fixed integer (omitted from notation whenever it is understood). For each $i \in \{1, \ldots, n\}$, the random positions $X_{i,t}$ and $X_{i,t+1}$ of vertex i at t and $t+1$ are denoted by $X_i = (x_i, y_i)$ and $X'_i = (x'_i, y'_i)$. Let also $\mathcal{X} = \mathcal{X}_t$ and $\mathcal{X}' = \mathcal{X}_{t+1}$. Note that X_i and X'_i are not independent. In fact if $2\pi z_i$ ($z_i \in [0,1)$) is the angle in which i moves between t and $t+1$, then $x'_i = x_i + s\cos(2\pi z_i)$ and $y'_i = y_i + s\sin(2\pi z_i)$. That motivates a description of the model at t and $t+1$ in terms of a three dimensional placement of the vertices, in which the third dimension is interpreted as a normalized angle. For each $i \in \{1, \ldots, n\}$, define the random point $\widehat{X}_i = (x_i, y_i, z_i) \in [0,1)^3$, and let $\widehat{\mathcal{X}} = \bigcup_{i=1}^{n} \widehat{X}_i$. By Lemma 3 all random points $\widehat{X}_i$ are chosen independently and u.a.r. from the 3-torus $[0,1)^3$. Moreover, $\widehat{\mathcal{X}}$ encodes all the information of the model at t and $t+1$. In fact, if we map $[0,1)^3$ onto $[0,1)^2$ by $\pi_1 : (x,y,z) \to (x,y)$ and $\pi_2 : (x,y,z) \to (x+s\cos(2\pi z), y+s\sin(2\pi z))$, we can recover the positions of i at t and $t+1$ from $\widehat{X}_i$ and write $X_i = \pi_1(\widehat{X}_i)$ and $X'_i = \pi_2(\widehat{X}_i)$. By Lemma 3, for any measurable sets $\mathcal{A} \subseteq [0,1)^2$ and $\mathcal{B} \subseteq [0,1)^3$, $\mathbf{Pr}\left[X_i \in \mathcal{A}\right] = \mathsf{Area}(\mathcal{A})$, $\mathbf{Pr}\left[X'_i \in \mathcal{B}\right] = \mathsf{Area}(\mathcal{A})$, and $\mathbf{Pr}\left[\widehat{X}_i \in \mathcal{B}\right] = \mathsf{Vol}(\mathcal{B})$.

For each $i \in \{1,\dots,n\}$, consider $\mathcal{R}_i = \{X \in [0,1)^2 : \mathsf{d}(X, X_i) \le r\}$ and $\mathcal{R}'_i = \{X \in [0,1)^2 : \mathsf{d}(X, X'_i) \le r\}$. Let $\widehat{\mathcal{R}}_i = \pi_1^{-1}(\mathcal{R}_i)$ and $\widehat{\mathcal{R}}'_i = \pi_2^{-1}(\mathcal{R}'_i)$ be their counterparts in $[0,1)^3$. Observe that X_i is isolated in $G(\mathcal{X};r)$ iff $(\widehat{\mathcal{X}} \setminus \{\widehat{X}_i\}) \cap \widehat{\mathcal{R}}_i = \emptyset$, and analogously X'_i is isolated in $G(\mathcal{X}';r)$ iff $(\widehat{\mathcal{X}} \setminus \{\widehat{X}_i\}) \cap \widehat{\mathcal{R}}'_i = \emptyset$.

For each $i \in \{1,\dots,n\}$, we define $\widehat{\mathcal{Q}}_i = \widehat{\mathcal{R}}'_i \setminus \widehat{\mathcal{R}}_i$ and $\widehat{\mathcal{Q}}'_i = \widehat{\mathcal{R}}_i \setminus \widehat{\mathcal{R}}'_i$. Given any two vertices i and j, observe that $\widehat{X}_i \in \widehat{\mathcal{Q}}'_j$ iff $\widehat{X}_j \in \widehat{\mathcal{Q}}'_i$ iff $\mathsf{d}(X_i, X_j) \le r$ and $\mathsf{d}(X'_i, X'_j) > r$ (i.e. the vertices are joined by an edge at time t but not at time $t+1$). This holds with probability $\mathsf{Vol}(\widehat{\mathcal{Q}}_i) = \mathsf{Vol}(\widehat{\mathcal{Q}}'_i)$, which neither depends on the particular vertices nor on t and will be denoted by q hereinafter. The value of this parameter depends on the asymptotic relation between r and s and is given in the following lemma:

Lemma 4. *The probability that two different vertices $i,j \in \{1,\dots,n\}$ are at distance $\le r$ at t but $> r$ at $t+1$ is $q \le \pi r^2$, which also satisfies: $q \sim \frac{4}{\pi} sr$ if $s = o(r)$, $\Theta(r^2)$ if $s = \Theta(r)$, and πr^2 if $s = \omega(r)$.*

Next, we study the changes of the isolated vertices between two consecutive steps t and $t+1$. Extending the notation in Section 2, let $K_{1,t}$ the number of isolated vertices of $G(\mathcal{X}_t;r)$. For any two consecutive steps t and $t+1$, define the following random variables: B_t is the number of vertices i such that X_i is not isolated in $G(\mathcal{X}_t;r)$ but X'_i is isolated in $G(\mathcal{X}_{t+1};r)$; D_t is the number of vertices i such that X_i is isolated in $G(\mathcal{X}_t;r)$ but X'_i is not isolated in $G(\mathcal{X}_{t+1};r)$; S_t is the number of vertices i such that X_i and X'_i are both isolated in $G(\mathcal{X}_t;r)$ and $G(\mathcal{X}_{t+1};r)$. Denote them by B, D and S whenever t and $t+1$ are understood. Note that B and D have the same distribution.

Recall that given a collection of events $\mathcal{E}_1(n),\dots,\mathcal{E}_k(n)$ and of random variables $W_1(n),\dots,W_l(n)$ taking values in $\mathbb{N}$, with k and l fixed, they are *mutually asymptotically independent* if for any $k', l', i_1,\dots,i_{k'}, j_1,\dots,j_{l'}, w_1,\dots,w_{l'} \in \mathbb{N}$ such that $k' \le k$, $l' \le l$, $1 \le i_1 < \cdots < i_{k'} \le k$, $1 \le j_1 < \cdots < j_{l'} \le l$ we have

$$\mathbf{Pr}\left[\bigwedge_{a=1}^{k'} \mathcal{E}_{i_a} \wedge \bigwedge_{b=1}^{l'} (W_{j_b} = w_b)\right] \sim \prod_{a=1}^{k'} \mathbf{Pr}\left[\mathcal{E}_{i_a}\right] \prod_{b=1}^{l'} \mathbf{Pr}\left[W_{j_b} = w_b\right]. \tag{3}$$

By computing the joint factorial moments $\mathbf{E}\left([B]_{\ell_1}[D]_{\ell_2}[S]_{\ell_3}\right)$, we can show the following proposition:

Proposition 5. *Assume $\mu = \Theta(1)$. Then for any two consecutive steps,*

$$\mathbf{E}(B) = \mathbf{E}(D) \sim \mu(1 - e^{-qn}) \quad \textit{and} \quad \mathbf{E}(S) \sim \mu e^{-qn}.$$

Moreover we have that

1. *If $s = o(1/rn)$, then* $\mathbf{Pr}[B > 0] \sim \mathbf{E}(B)$*;* $\mathbf{Pr}[D > 0] \sim \mathbf{E}(D)$*; S is asymptotically Poisson; and $(B > 0)$, $(D > 0)$ and S are asymptotically mutually independent.*

2. *If $s = \Theta(1/rn)$, then B, D and S are asymptotically mutually independent Poisson.*

3. *If $s = \omega(1/rn)$, then B and D are asymptotically Poisson;* $\mathbf{Pr}[S > 0] \sim \mathbf{E}(S)$*; and B, D and $(S > 0)$ are asymptotically mutually independent.*

Taking into account that $K_{1,t} = D_t + S_t$ and $K_{1,t+1} = S_t + B_t$, the number of isolated vertices at two consecutive steps can in the case $s = \Theta\big(1/(rn)\big)$ be completely characterized by Proposition 5. For the other ranges of s, the result is weaker but still sufficient for our further purposes. We remark that if $s = o\big(1/(rn)\big)$ then creations and destructions of isolated vertices are rare, but a Poisson number of isolated vertices is present at both consecutive steps. If $s = \omega\big(1/(rn)\big)$ then the isolated vertices which are present at both consecutive steps are rare since, but a Poisson number of them is created and also a Poisson number destroyed.

Given a component Γ of $G(\mathcal{X}; r)$, Γ is *embeddable* if it can be mapped into the square $[r, 1-r]^2$ by a translation in the torus. Embeddable components do not wrap around the torus. Components which are not embeddable must have a size of at least $\Omega(1/r)$.

Sometimes several non-embeddable components can coexist together. However, there are some non-embeddable components which are so spread around the torus that do not allow any room for other non-embeddable ones. Call these components *solitary*. By definition we can have at most one solitary component. We cannot disprove the existence of a solitary component, since with probability $1 - o(1)$ there exists a giant component of this nature. For not solitary components, we give asymptotic bounds on the probability of their existence according to their size.

In order to characterize the connectivity of $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$, we need to bound the probability that components other than isolated vertices and the giant one appear at some step. We know by the comments in Section 2 that a.a.s. this does not occur at one single step. However during long periods of time this event could affect the connectivity and must be considered. Extending the notation in Section 2, given a step t, let $\widetilde{K}_{2,t}$ be the number of non-solitary components other than isolated vertices occurring at step t. In the next lemma, we show that such components have a negligible effect compared to isolated vertices in the dynamic evolution of connectivity.

Lemma 6. *Assume that $\mu = \Theta(1)$ and $s = o\big(1/(rn)\big)$. Then,*
$\mathbf{Pr}\left[\widetilde{K}_{2,t} > 0 \wedge \widetilde{K}_{2,t+1} = 0\right] = \mathbf{Pr}\left[\widetilde{K}_{2,t} > 0 \wedge B_t > 0\right] = o(srn)$*.*

From Proposition 5 and Lemma 6 the following corollary is straightforward to prove.

Corollary 7. *Assume that $\mu = \Theta(1)$. Then,*

$$\mathbf{Pr}[\mathcal{C}_t \wedge \mathcal{D}_{t+1}] \sim e^{-\mu}(1 - e^{-\mathbf{E}(B)}),\ \mathbf{Pr}[\mathcal{D}_t \wedge \mathcal{C}_{t+1}] \sim e^{-\mu}(1 - e^{-\mathbf{E}(B)})$$

$$\mathbf{Pr}\left[\mathcal{C}_t \wedge \mathcal{C}_{t+1}\right] \sim e^{-\mu} e^{-\mathbf{E}(B)},\ \mathbf{Pr}\left[\mathcal{D}_t \wedge \mathcal{D}_{t+1}\right] \sim 1 - 2e^{-\mu} + e^{-\mu} e^{-\mathbf{E}(B)}.$$

For the next lemma, recall the definition of $L_t(\mathcal{C})$ and $L_t(\mathcal{D})$ from Section 2.

Lemma 8. *If $\mathbf{E}\left(L(\mathcal{C})\right) < +\infty$ (but possibly $\mathbf{E}\left(L(\mathcal{C})\right) \to +\infty$ as $n \to +\infty$), then conditional upon $\mathcal{C}_t$ but not C_{t-1} we have*

$$\mathbf{E}\left(L_t(\mathcal{C}) \mid \mathcal{D}_{t-1} \wedge \mathcal{C}_t\right) = \frac{\mathbf{Pr}\left[\mathcal{C}\right]}{\mathbf{Pr}\left[\mathcal{D}_{t-1} \wedge \mathcal{C}_t\right]},$$

which does not depend on t. The same statement holds if we interchange $\mathcal{C}$ and $\mathcal{D}$.

Proof. We have that $L_{t-1}(\mathcal{C}) + 1[\mathcal{D}_{t-1}]L_t(\mathcal{C}) = 1[\mathcal{C}_{t-1}] + L_t(\mathcal{C})$, and by taking expectations and using the hypothesis that $\mathbf{E}\left(L(\mathcal{C})\right) < +\infty$ we get

$$\mathbf{E}\left(1[\mathcal{D}_{t-1}]L_t(\mathcal{C})\right) = \mathbf{Pr}\left[\mathcal{C}\right], \quad \forall t.$$

The statement follows from the fact that

$$\mathbf{E}\left(L_t(\mathcal{C}) \mid \mathcal{D}_{t-1} \wedge \mathcal{C}_t\right) = \frac{\mathbf{E}\left(1[\mathcal{D}_{t-1} \wedge \mathcal{C}_t]L_t(\mathcal{C})\right)}{\mathbf{Pr}\left[\mathcal{D}_{t-1} \wedge \mathcal{C}_t\right]} = \frac{\mathbf{E}\left(1[\mathcal{D}_{t-1}]L_t(\mathcal{C})\right)}{\mathbf{Pr}\left[\mathcal{D}_{t-1} \wedge \mathcal{C}_t\right]}.$$

□

To prove that $\mathbf{E}\left(L(\mathcal{C})\right) < +\infty$ and $\mathbf{E}\left(L(\mathcal{D})\right) < +\infty$ we use the following technical lemma.

Lemma 9. *Let $b = b(n)$ be the smallest natural number such that $(b-3)ms \geq 3\sqrt{2}/2$. Then, there exists $p = p(n) > 0$ such that for any fixed circle $\mathcal{R} \subset [0,1)^2$ of radius $r/2$, any $i \in \{1, \ldots, n\}$, any $t \in \mathbb{Z}$, and conditional upon any particular position of $X_{i,t}$ in the torus, the probability that $X_{i,t+bm} \in \mathcal{R}$ is at least p.*

The next lemma allows us to apply Lemma 8

Lemma 10. $\mathbf{E}\left(L(\mathcal{C})\right) < +\infty$ *and* $\mathbf{E}\left(L(\mathcal{D})\right) < +\infty$.

Theorem 1 follows from Lemma 10, Lemma 8 and Corollary 7.

4 Conclusion.

In this extended abstract, we have formally introduced the dynamic random geometric graph in order to study analytically the Random Walk model for MANETs, defined in [4]. We studied the expected length of the connectivity and disconnectivity periods, taking into account different step sizes s and different lengths m during which the angle remains invariant, always considering the static connectivity threshold $r = r_c$. We believe that a similar analysis can be performed for other values of $r > r_c$ as well. Also, it would be interesting to extend our results when the connectivity radii r_v are different for different vertices.

The *Random Walk* model simulates the behavior of a swarm of mobile vertices as sensors or robots, which move randomly to monitor an unknown territory or to search in it. There exist other models such as the *Random Way-point* model, where each vertex chooses randomly a fixed way-point (from a set of pre-determined way-points) and moves there, and when it arrives it chooses another and moves there (see [2]). A possible line of future research is to do a study similar to the one developed in this paper for this way-point model. We believe that the techniques developed in this paper will prove very useful to carry out that study.

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