Probabilistic analysis of algorithms: What's it good for?

Conrado Martínez

Univ. Politècnica de Catalunya, Spain

University of Cape Town
February 2008

Introduction

Probabilistic analysis of algorithms is the right tool when

- We want to analyze "typical" behavior of algorithms
- We want to compare algorithms with asymptotically equivalent performances
- We want to analyze randomized algorithms (essential!)
- We want to have some mathematical fun :)

Introduction

Probabilistic analysis of algorithms is the right tool when

- We want to analyze "typical" behavior of algorithms
- We want to compare algorithms with asymptotically equivalent performances
- We want to analyze randomized algorithms (essential!)
- We want to have some mathematical fun :)

Introduction

Probabilistic analysis of algorithms is the right tool when

- We want to analyze "typical" behavior of algorithms
- We want to compare algorithms with asymptotically equivalent performances
- We want to analyze randomized algorithms (essential!)
- We want to have some mathematical fun :)

Introduction

Probabilistic analysis of algorithms is the right tool when

- We want to analyze "typical" behavior of algorithms
- We want to compare algorithms with asymptotically equivalent performances
- We want to analyze randomized algorithms (essential!)
- We want to have some mathematical fun :)

Introduction

A few well known examples:

- Quicksort
- Find, a.k.a. Quickselect
- Hashing
- Simplex
- Randomized data structures
- and many more . . .

Introduction

A few well known examples:

- Quicksort
- Find, a.k.a. Quickselect
- Hashing
- Simplex
- Randomized data structures
- and many more . . .

Introduction

A few well known examples:

- Quicksort
- Find, a.k.a. Quickselect
- Hashing
- Simplex
- Randomized data structures
- and many more . . .

Introduction

A few well known examples:

- Quicksort
- Find, a.k.a. Quickselect
- Hashing
- Simplex
- Randomized data structures
- and many more ...

Introduction

A few well known examples:

- Quicksort
- Find, a.k.a. Quickselect
- Hashing
- Simplex
- Randomized data structures
- and many more ...

Introduction

A few well known examples:

- Quicksort
- Find, a.k.a. Quickselect
- Hashing
- Simplex
- Randomized data structures
- and many more . . .

Introduction

It was indeed very difficult for me to make a choice of examples . . .

Introduction

…even if I restricted myself to those few that I've worked out myself!

- Randomized binary search trees
- Optimal sampling for quicksort and quickselect
- Adaptive sampling for quickselect
- Updates and associative queries in relaxed K-d trees
- Exhaustive and random generation of combinatorial objects
- Partial sorting
- Probabilistic analysis of binary search trees, skip lists, …

Example #1: Generating random derangements

Le Problème des Derangements:
"A number of gentlemen, say n, surrender their top hats in the cloakroom and proceed to the evening's enjoyment. After wining and dining (and wining some more), they stumble back to the cloakroom and confusedly take the first top-hat they see. What is the probability that no gentleman gets his own hat?"

Derangements

- A derangement is a permutation without fixed points: $\pi(i) \neq i$ for any i, $1 \leq i \leq n$
- The number D_n of derangements of size n is

$$D_n = n! \cdot \left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right] = \left\lfloor \frac{n!+1}{e} \right\rfloor .$$

- As $n \to \infty$, $D_n/n! \sim 1/e \approx 0.36788$. In fact, e^{-1} is a extremely good approximation to the probability that a random permutation is a derangement for $n \geq 10$.

Derangements

- A derangement is a permutation without fixed points: $\pi(i) \neq i$ for any i, $1 \leq i \leq n$
- The number D_n of derangements of size n is

$$D_n = n! \cdot \left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right] = \left\lfloor \frac{n!+1}{e} \right\rfloor .$$

- As $n \to \infty$, $D_n/n! \sim 1/e \approx 0.36788$. In fact, e^{-1} is a extremely good approximation to the probability that a random permutation is a derangement for $n \geq 10$.

Derangements

- A derangement is a permutation without fixed points: $\pi(i) \neq i$ for any i, $1 \leq i \leq n$
- The number D_n of derangements of size n is

$$D_n = n! \cdot \left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right] = \left\lfloor \frac{n!+1}{e} \right\rfloor .$$

- As $n \to \infty$, $D_n/n! \sim 1/e \approx 0.36788$. In fact, e^{-1} is a extremely good approximation to the probability that a random permutation is a derangement for $n \geq 10$.

Excursion: Fisher-Yates' shuffle

```
procedure RandomPermutation($n$)
    for $i \leftarrow 1$ to $n$ do $A[i] \leftarrow i$
    for $i \leftarrow n$ downto 1 do
        $j \leftarrow$ Uniform$(1, i)$
        $A[i] \leftrightarrow A[j]$
    return $A$
```

Excursion: Sattolo's algorithm

```
procedure RandomCyclicPermutation($n$)
    for $i \leftarrow 1$ to $n$ do $A[i] \leftarrow i$
    for $i \leftarrow n$ downto $1$ do
        $j \leftarrow$ Uniform($1, i - 1$)
        $A[i] \leftrightarrow A[j]$
    return $A$
```

Excursion: The rejection method

```
Require: $n \neq 1$
  procedure RandomDerangement($n$)
      repeat
          $A \leftarrow$ RandomPermutation($n$)
      until Is-Derangement($A$) return  $A$
```

$$\mathbb{P}[A \text{ is a derangement}] \approx \frac{1}{e}$$

$$\mathbb{E}[\# \text{ of calls to Random}] = e \cdot n + O(1)$$

Excursion: The rejection method

```
Require: $n \neq 1$
  procedure RandomDerangement($n$)
      repeat
          $A \leftarrow$ RandomPermutation($n$)
      until Is-Derangement($A$) return $A$
```

$$\mathbb{P}[A \text{ is a derangement}] \approx \frac{1}{e}$$

$$\mathbb{E}[\# \text{ of calls to Random}] = e \cdot n + O(1)$$

A recurrence for the number of derangements

$$D_0 = 1, D_1 = 0$$

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

A recurrence for the number of derangements

$$D_0 = 1, D_1 = 0$$
$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

Choice #1: n belongs to a cycle of length > 2.
The derangement of size n is built by constructing a derangement of size $n-1$ and then n is inserted into any of the cycles (of length ≥ 2); there are $(n-1)$ possible ways to do that

A recurrence for the number of derangements

$$D_0 = 1, D_1 = 0$$
$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

Choice #2: n belongs to a cycle of length 2.
The derangement of size n is built by constructing a cycle of size 2 with n and some j, $1 \leq j \leq n-1$; then we build a derangement of size $n-2$ with the remaining elements

The recursive method

```
$C \leftarrow \{1, 2, \ldots, n\}$
RandomDerangement-Rec($n, C$)
```

```
Require: $n \neq 1$
  procedure RandomDerangement-Rec($n$, $C$)
      if $n \leq 1$ then return
      $j \leftarrow$ a random element from $C$
      $p \leftarrow$ Uniform$(0, 1)$
      if $p < (n-1)D_{n-2}/D_n$ then
          RandomDerangement-Rec($n - 2, C \setminus \{j, n\}$)
          $\pi(n) \leftarrow j; \pi(j) \leftarrow n$
      else
          RandomDerangement-Rec($n - 1, C \setminus \{n\}$)
          $\pi(n) \leftarrow \pi(j); \pi(j) \leftarrow n$
```

Our algorithm

```
Require: $n \neq 1$
  procedure RandomDerangement($n$)
    for $i \leftarrow 1$ to $n$ do $A[i] \leftarrow i; mark[i] \leftarrow$ false
    $i \leftarrow n; u \leftarrow n$
    while $u \geq 2$ do
      if $\neg mark[i]$ then
        $j \leftarrow$ a random unmarked element in $A[1..i-1]$

        $A[i] \leftrightarrow A[j]$

        if $j$ has to close a cycle then
          $mark[j] \leftarrow$ true; $u \leftarrow u - 1$
        $u \leftarrow u - 1$
      $i \leftarrow i - 1$
    return $A$
```

Our algorithm

```
Require: $n \neq 1$
  procedure RandomDerangement($n$)
    for $i \leftarrow 1$ to $n$ do $A[i] \leftarrow i; mark[i] \leftarrow$ false
    $i \leftarrow n; u \leftarrow n$
    while $u \geq 2$ do
      if $\neg mark[i]$ then
        repeat $j \leftarrow$ Random$(1, i-1)$
        until $\neg mark[j]$
        $A[i] \leftrightarrow A[j]$
        $p \leftarrow$ Uniform$(0, 1)$
        if $p < (u-1)D_{u-2}/D_u$
          $mark[j] \leftarrow$ true$; u \leftarrow u - 1$
        $u \leftarrow u - 1$
      $i \leftarrow i - 1$
    return $A$
```

The analysis

- # of marked elements = # of cycles (C_n)
- # of iterations = # of calls to Uniform = $n - C_n$
- G = # of calls to Random
- G_i = # of calls to Random at iteration i
-

$$\begin{aligned}\mathbb{E}[\text{cost}] &= n - \mathbb{E}[C_n] + \mathbb{E}[G] \\ &= n - \mathbb{E}[C_n] + \sum_{1<i\leq n} \mathbb{E}[G_i]\end{aligned}$$

The analysis

- # of marked elements = # of cycles (C_n)
- # of iterations = # of calls to Uniform = $n - C_n$
- G = # of calls to Random
- G_i = # of calls to Random at iteration i

$$\begin{aligned}\mathbb{E}[\text{cost}] &= n - \mathbb{E}[C_n] + \mathbb{E}[G] \\ &= n - \mathbb{E}[C_n] + \sum_{1<i\leq n} \mathbb{E}[G_i]\end{aligned}$$

The analysis

- # of marked elements = # of cycles (C_n)
- # of iterations = # of calls to Uniform = $n - C_n$
- G = # of calls to Random
- G_i = # of calls to Random at iteration i

$$\begin{aligned}\mathbb{E}[\text{cost}] &= n - \mathbb{E}[C_n] + \mathbb{E}[G] \\ &= n - \mathbb{E}[C_n] + \sum_{1<i\leq n} \mathbb{E}[G_i]\end{aligned}$$

The analysis

- \# of marked elements = # of cycles (C_n)
- \# of iterations = # of calls to Uniform = $n - C_n$
- G = # of calls to Random
- G_i = # of calls to Random at iteration i
-

$$\begin{aligned}\mathbb{E}[\text{cost}] &= n - \mathbb{E}[C_n] + \mathbb{E}[G] \\ &= n - \mathbb{E}[C_n] + \sum_{1<i\leq n} \mathbb{E}[G_i]\end{aligned}$$

The analysis

- # of marked elements = # of cycles (C_n)
- # of iterations = # of calls to Uniform = $n - C_n$
- G = # of calls to Random
- G_i = # of calls to Random at iteration i
-

$$\begin{aligned}\mathbb{E}[\text{cost}] &= n - \mathbb{E}[C_n] + \mathbb{E}[G] \\ &= n - \mathbb{E}[C_n] + \sum_{1<i\leq n} \mathbb{E}[G_i]\end{aligned}$$

The analysis

The computation of $\mathbb{E}[C_n]$ can be done via standard generating function techniques:

$$C(z,v) = \sum_{A \in \mathcal{D}} \frac{z^{|A|}}{|A|!} v^{\#\ \text{cycles}(A)}$$

$$= \exp\left(v\left(\log\frac{1}{1-z} - z\right)\right) = e^{-vz}\frac{1}{(1-z)^v}$$

$$\mathbb{E}\left[v^{C_n}\right] = \frac{n!}{D_n}[z^n]C(z,v) = \frac{e^{1-v}}{(v-1)!}n^{v-1}(1+O(n^{-1+\epsilon}))$$

$$\mathbb{E}[C_n] = \log n + O(1), \quad \mathbb{V}[C_n] = \log n + O(1)$$

$$\frac{C_n - \log n}{\sqrt{\log n}} \to \mathcal{N}(0,1)$$

The analysis

The computation of $\mathbb{E}[C_n]$ can be done via standard generating function techniques:

$$C(z,v) = \sum_{A \in \mathcal{D}} \frac{z^{|A|}}{|A|!} v^{\#\ \text{cycles}(A)}$$

$$= \exp\left(v\left(\log\frac{1}{1-z} - z\right)\right) = e^{-vz}\frac{1}{(1-z)^{v}}$$

$$\mathbb{E}\left[v^{C_n}\right] = \frac{n!}{D_n}[z^n]C(z,v) = \frac{e^{1-v}}{(v-1)!}n^{v-1}(1+O(n^{-1+\epsilon}))$$

$$\mathbb{E}[C_n] = \log n + O(1), \quad \mathbb{V}[C_n] = \log n + O(1)$$

$$\frac{C_n - \log n}{\sqrt{\log n}} \to \mathcal{N}(0,1)$$

The analysis

The computation of $\mathbb{E}[C_n]$ can be done via standard generating function techniques:

$$C(z,v) = \sum_{A \in \mathcal{D}} \frac{z^{|A|}}{|A|!} v^{\#\ \text{cycles}(A)}$$

$$= \exp\left(v\left(\log\frac{1}{1-z} - z\right)\right) = e^{-vz}\frac{1}{(1-z)^v}$$

$$\mathbb{E}\left[v^{C_n}\right] = \frac{n!}{D_n}[z^n]C(z,v) = \frac{e^{1-v}}{(v-1)!}n^{v-1}(1+O(n^{-1+\epsilon}))$$

$$\mathbb{E}[C_n] = \log n + O(1), \qquad \mathbb{V}[C_n] = \log n + O(1)$$

$$\frac{C_n - \log n}{\sqrt{\log n}} \to \mathcal{N}(0,1)$$

The analysis

The computation of $\mathbb{E}[C_n]$ can be done via standard generating function techniques:

$$C(z,v) = \sum_{A \in \mathcal{D}} \frac{z^{|A|}}{|A|!} v^{\#\text{ cycles}(A)}$$

$$= \exp\left(v\left(\log\frac{1}{1-z} - z\right)\right) = e^{-vz}\frac{1}{(1-z)^v}$$

$$\mathbb{E}\left[v^{C_n}\right] = \frac{n!}{D_n}[z^n]C(z,v) = \frac{e^{1-v}}{(v-1)!}n^{v-1}(1+O(n^{-1+\epsilon}))$$

$$\mathbb{E}[C_n] = \log n + O(1), \quad \mathbb{V}[C_n] = \log n + O(1)$$

$$\frac{C_n - \log n}{\sqrt{\log n}} \to \mathcal{N}(0,1)$$

The analysis

The computation of $\mathbb{E}[C_n]$ can be done via standard generating function techniques:

$$C(z,v) = \sum_{A\in\mathcal{D}} \frac{z^{|A|}}{|A|!} v^{\#\text{ cycles}(A)}$$

$$= \exp\left(v\left(\log\frac{1}{1-z} - z\right)\right) = e^{-vz}\frac{1}{(1-z)^v}$$

$$\mathbb{E}\left[v^{C_n}\right] = \frac{n!}{D_n}[z^n]C(z,v) = \frac{e^{1-v}}{(v-1)!}n^{v-1}(1+O(n^{-1+\epsilon}))$$

$$\mathbb{E}[C_n] = \log n + O(1), \quad \mathbb{V}[C_n] = \log n + O(1)$$

$$\frac{C_n - \log n}{\sqrt{\log n}} \to \mathcal{N}(0,1)$$

The analysis

- M_i indicator variable for the event "$A[i]$ gets marked"
- $M_i = 1 \implies G_i = 0$
-

$$\mathbb{E}[G] = \sum_{1<i\leq n} \mathbb{E}[G_i \mid M_i = 0] \cdot \mathbb{P}[M_i = 0]$$

The analysis

- M_i indicator variable for the event "$A[i]$ gets marked"
- $M_i = 1 \implies G_i = 0$
- $$\mathbb{E}[G] = \sum_{1<i\leq n} \mathbb{E}[G_i \mid M_i = 0] \cdot \mathbb{P}[M_i = 0]$$

The analysis

- M_i indicator variable for the event "$A[i]$ gets marked"
- $M_i = 1 \implies G_i = 0$
- $$\mathbb{E}[G] = \sum_{1<i\leq n} \mathbb{E}[G_i \mid M_i = 0] \cdot \mathbb{P}[M_i = 0]$$

The analysis

- U_i = # of unmarked elements in $A[1..i]; U_n = n$
- B_{i+1} = # of marked elements in $A[1..i]; B_{n+1} = 0$
- $U_i + B_{i+1} = i$
- If $A[i]$ is not marked then G_i is geometrically distributed with probability of success $(U_i - 1)/(i - 1) = (i - 1 - B_{i+1})/(i - 1)$; hence

$$\mathbb{E}[G_i \mid M_i = 0] = \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \,\middle|\, M_i = 0\right]$$

The analysis

- $U_i = \#$ of unmarked elements in $A[1..i]; U_n = n$
- $B_{i+1} = \#$ of marked elements in $A[1..i]; B_{n+1} = 0$
- $U_i + B_{i+1} = i$
- If $A[i]$ is not marked then G_i is geometrically distributed with probability of success $(U_i - 1)/(i - 1) = (i - 1 - B_{i+1})/(i - 1)$; hence

$$\mathbb{E}[G_i \mid M_i = 0] = \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \mid M_i = 0\right]$$

The analysis

- U_i = # of unmarked elements in $A[1..i]; U_n = n$
- B_{i+1} = # of marked elements in $A[1..i]; B_{n+1} = 0$
- $U_i + B_{i+1} = i$
- If $A[i]$ is not marked then G_i is geometrically distributed with probability of success $(U_i - 1)/(i-1) = (i-1-B_{i+1})/(i-1)$; hence

$$\mathbb{E}[G_i \mid M_i = 0] = \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \mid M_i = 0\right]$$

The analysis

- U_i = # of unmarked elements in $A[1..i]; U_n = n$
- B_{i+1} = # of marked elements in $A[1..i]; B_{n+1} = 0$
- $U_i + B_{i+1} = i$
- If $A[i]$ is not marked then G_i is geometrically distributed with probability of success $(U_i - 1)/(i-1) = (i - 1 - B_{i+1})/(i-1)$; hence

$$\mathbb{E}[G_i \mid M_i = 0] = \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \mid M_i = 0\right]$$

The analysis

- $B_{i+1} \leq C_n$
- $0 \leq B_{i+1} \leq i$
- $U_i \neq 1$ and $B_{i+1} \neq i - 1$ for all $1 \leq i \leq n$
- If $M_i = 0$ then $B_{i+1} < i - 1$

The analysis

- $B_{i+1} \leq C_n$
- $0 \leq B_{i+1} \leq i$
- $U_i \neq 1$ and $B_{i+1} \neq i - 1$ for all $1 \leq i \leq n$
- If $M_i = 0$ then $B_{i+1} < i - 1$

The analysis

- $B_{i+1} \leq C_n$
- $0 \leq B_{i+1} \leq i$
- $U_i \neq 1$ and $B_{i+1} \neq i - 1$ for all $1 \leq i \leq n$
- If $M_i = 0$ then $B_{i+1} < i - 1$

The analysis

- $B_{i+1} \leq C_n$
- $0 \leq B_{i+1} \leq i$
- $U_i \neq 1$ and $B_{i+1} \neq i - 1$ for all $1 \leq i \leq n$
- If $M_i = 0$ then $B_{i+1} < i - 1$

The analysis

$$\begin{aligned}
\mathbb{E}[G] &= \sum_{1<i\leq n} \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \mid M_i = 0\right] \cdot \mathbb{P}[M_i = 0] \\
&\leq \sum_{1<i\leq n} \mathbb{E}\left[\min\left\{i-1, \frac{i-1}{i-1-C_n}\right\}\right] \\
&\leq \sum_{1\leq k\leq\lfloor n/2\rfloor} \mathbb{P}[C_n = k]\left(\sum_{i=1}^{k+1}(i-1) + \sum_{i=k+2}^{\lfloor n/2\rfloor}\frac{i-1}{i-1-k}\right) \\
&= n-1-\mathbb{E}[C_n] + \frac{1}{2}\mathbb{E}\left[{C_n}^2\right] + O(\mathbb{E}[C_n\log(n-C_n)]) \\
&= n + O\left(\mathbb{E}\left[{C_n}^2\right]\right) + O(\log n\cdot\mathbb{E}[C_n]) = n + O(\log^2 n)
\end{aligned}$$

The analysis

$$
\begin{aligned}
\mathbb{E}[G] &= \sum_{1<i\leq n} \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \mid M_i = 0\right] \cdot \mathbb{P}[M_i = 0] \\
&\leq \sum_{1<i\leq n} \mathbb{E}\left[\min\left\{i-1, \frac{i-1}{i-1-C_n}\right\}\right] \\
&\leq \sum_{1\leq k\leq \lfloor n/2\rfloor} \mathbb{P}[C_n = k] \left(\sum_{i=1}^{k+1}(i-1) + \sum_{i=k+2}^{\lfloor n/2\rfloor} \frac{i-1}{i-1-k}\right) \\
&= n - 1 - \mathbb{E}[C_n] + \frac{1}{2}\,\mathbb{E}\left[{C_n}^2\right] + O(\mathbb{E}[C_n \log(n - C_n)]) \\
&= n + O(\mathbb{E}\left[{C_n}^2\right]) + O(\log n \cdot \mathbb{E}[C_n]) = n + O(\log^2 n)
\end{aligned}
$$

The analysis

$$\begin{aligned}
\mathbb{E}[G] &= \sum_{1<i\le n} \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \,\middle|\, M_i = 0\right] \cdot \mathbb{P}[M_i = 0] \\
&\le \sum_{1<i\le n} \mathbb{E}\left[\min\left\{i-1, \frac{i-1}{i-1-C_n}\right\}\right] \\
&\le \sum_{1\le k\le \lfloor n/2\rfloor} \mathbb{P}[C_n = k]\left(\sum_{i=1}^{k+1}(i-1) + \sum_{i=k+2}^{\lfloor n/2\rfloor}\frac{i-1}{i-1-k}\right) \\
&= n - 1 - \mathbb{E}[C_n] + \frac{1}{2}\,\mathbb{E}\left[{C_n}^2\right] + O(\mathbb{E}[C_n \log(n - C_n)]) \\
&= n + O(\mathbb{E}\left[{C_n}^2\right]) + O(\log n \cdot \mathbb{E}[C_n]) = n + O(\log^2 n)
\end{aligned}$$

The analysis

$$\begin{aligned}
\mathbb{E}[G] &= \sum_{1<i\le n} \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \mid M_i = 0\right] \cdot \mathbb{P}[M_i = 0] \\
&\le \sum_{1<i\le n} \mathbb{E}\left[\min\left\{i-1, \frac{i-1}{i-1-C_n}\right\}\right] \\
&\le \sum_{1\le k\le \lfloor n/2\rfloor} \mathbb{P}[C_n = k] \left(\sum_{i=1}^{k+1}(i-1) + \sum_{i=k+2}^{\lfloor n/2\rfloor} \frac{i-1}{i-1-k}\right) \\
&= n - 1 - \mathbb{E}[C_n] + \frac{1}{2}\,\mathbb{E}\left[{C_n}^2\right] + O(\mathbb{E}[C_n \log(n - C_n)]) \\
&= n + O(\mathbb{E}\left[{C_n}^2\right]) + O(\log n \cdot \mathbb{E}[C_n]) = n + O(\log^2 n)
\end{aligned}$$

The analysis

$$\begin{aligned}
\mathbb{E}[G] &= \sum_{1<i\leq n} \mathbb{E}\left[\frac{i-1}{i-1-B_{i+1}} \mid M_i = 0\right] \cdot \mathbb{P}[M_i = 0] \\
&\leq \sum_{1<i\leq n} \mathbb{E}\left[\min\left\{i-1, \frac{i-1}{i-1-C_n}\right\}\right] \\
&\leq \sum_{1\leq k\leq \lfloor n/2\rfloor} \mathbb{P}[C_n = k] \left(\sum_{i=1}^{k+1}(i-1) + \sum_{i=k+2}^{\lfloor n/2\rfloor} \frac{i-1}{i-1-k}\right) \\
&= n - 1 - \mathbb{E}[C_n] + \frac{1}{2}\,\mathbb{E}\left[{C_n}^2\right] + O(\mathbb{E}[C_n \log(n - C_n)]) \\
&= n + O(\mathbb{E}\left[{C_n}^2\right]) + O(\log n \cdot \mathbb{E}[C_n]) = n + O(\log^2 n)
\end{aligned}$$

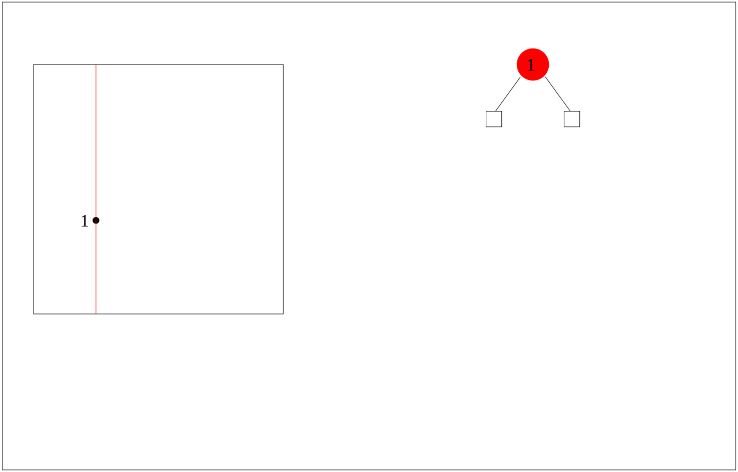
The analysis

Since we also have $\mathbb{E}[G] \geq n - \mathbb{E}[C_n]$, we have finally

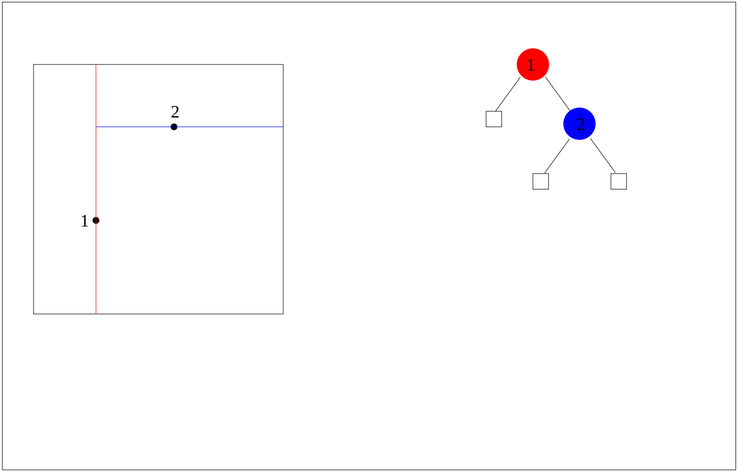
$$\mathbb{E}[\text{cost}] = n - \mathbb{E}[C_n] + \mathbb{E}[G] = 2n + O(\log^2 n)$$

Example #2: Updating K-d trees

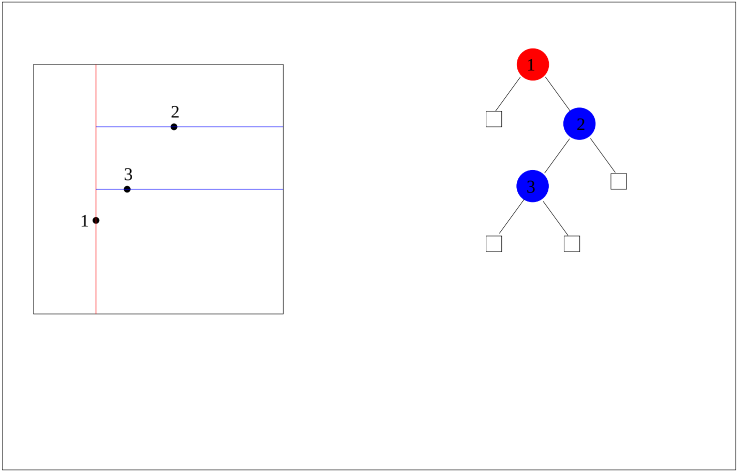
Example #2: Updating K-d trees



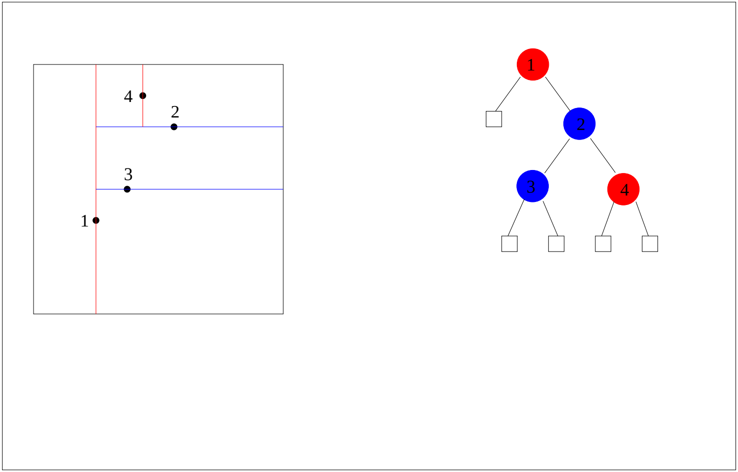
Example #2: Updating K-d trees



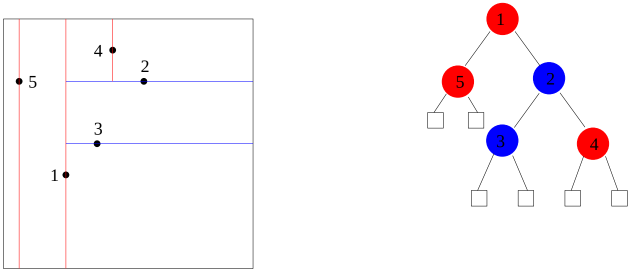
Example #2: Updating K-d trees



Example #2: Updating K-d trees



Example #2: Updating K-d trees



Insertion in relaxed K-d trees

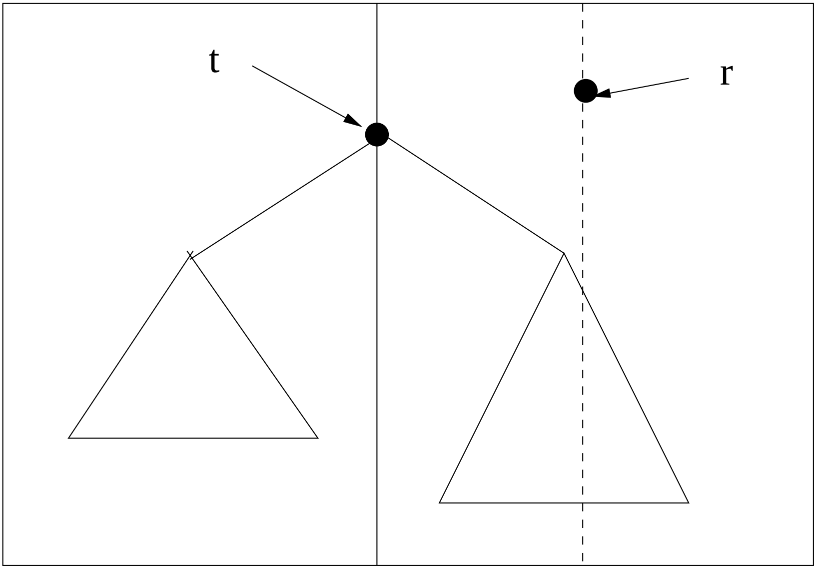
```
rkdt insert(rkdt t, const Elem& x) {
     int n = size(t);
     int u = random(0,n);
     if (u == n)
        return insert_at_root(t, x);
     else { // t cannot be empty
        int i = t -> discr;
        if (x[i] < t -> key[i])
           t -> left = insert(t -> left, x);
        else
           t -> right = insert(t -> right, x);
        return t;
     }
}
```

Deletion in relaxed K-d trees

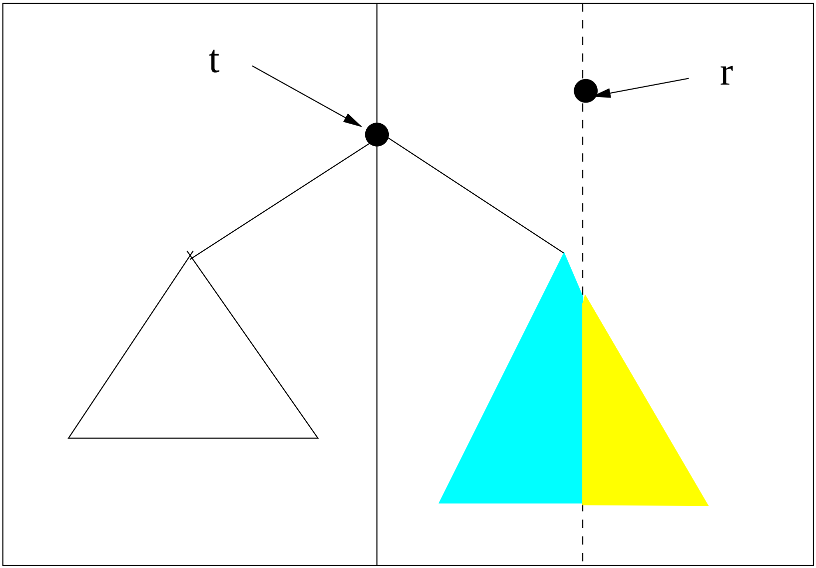
```
rkdt delete(rkdt t, const Elem& x) {
     if (t == NULL) return NULL;
     if (t -> key == x)
        return join(t -> left, t -> right);

     int i = t -> discr;
     if (x -> key[i] < t -> key[i])
        t -> left = delete(t -> left, x);
     else
        t -> right = delete(t -> right, x);
     return t;
}
```

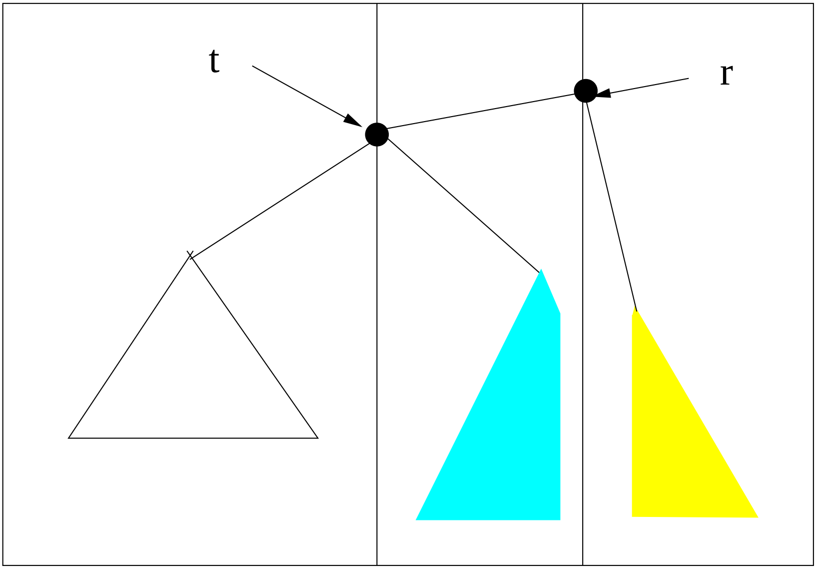
Split: Case #1



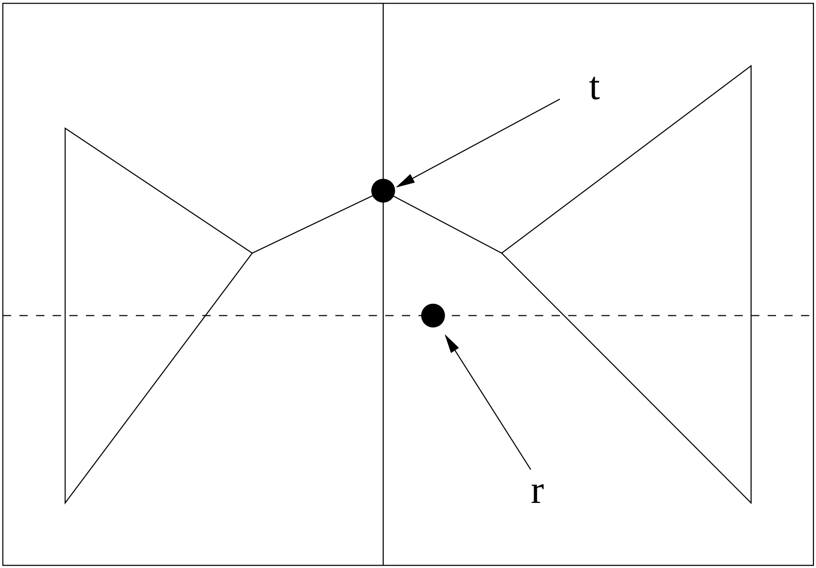
Split: Case #1



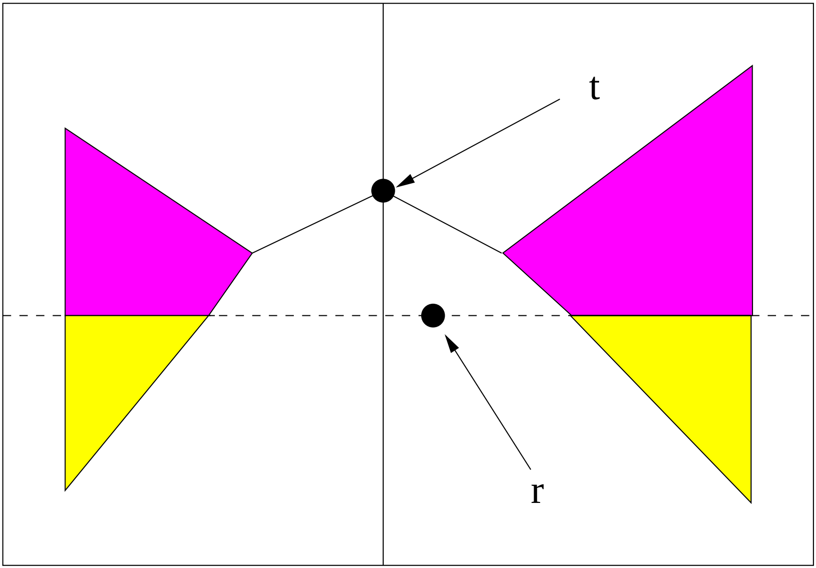
Split: Case #1



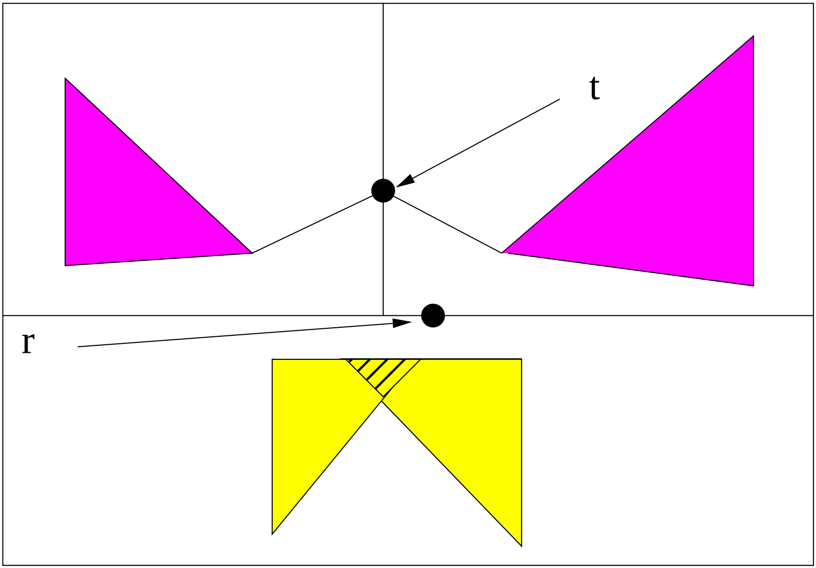
Split: Case #2



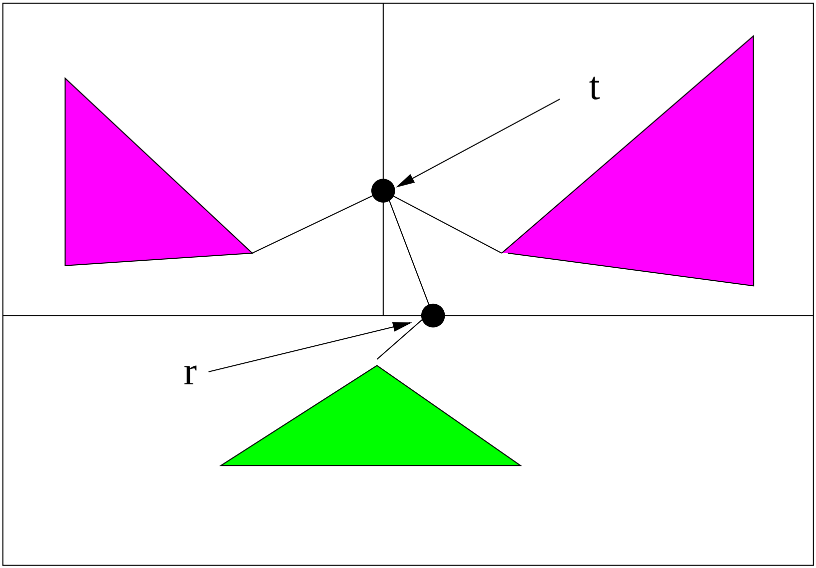
Split: Case #2



Split: Case #2



Split: Case #2



Analysis of split/join

- s_n = avg. number of visited nodes in a split
- m_n = avg. number of visited nodes in a join
- $$s_n = 1 + \frac{2}{nK} \sum_{0 \le j < n} \frac{j+1}{n+1} s_j + \frac{2(K-1)}{nK} \sum_{0 \le j < n} s_j + \frac{K-1}{K} \sum_{0 \le j < n} \pi_{n,j} m_j,$$

  where $\pi_{n,j}$ is probability of joining two trees with total size j.

Analysis of split/join

- s_n = avg. number of visited nodes in a split
- m_n = avg. number of visited nodes in a join
- $$s_n = 1 + \frac{2}{nK} \sum_{0 \le j < n} \frac{j+1}{n+1} s_j + \frac{2(K-1)}{nK} \sum_{0 \le j < n} s_j + \frac{K-1}{K} \sum_{0 \le j < n} \pi_{n,j} m_j,$$

  where $\pi_{n,j}$ is probability of joining two trees with total size j.

Analysis of split/join

- s_n = avg. number of visited nodes in a split
- m_n = avg. number of visited nodes in a join
- $$s_n = 1 + \frac{2}{nK} \sum_{0 \le j < n} \frac{j+1}{n+1} s_j + \frac{2(K-1)}{nK} \sum_{0 \le j < n} s_j + \frac{K-1}{K} \sum_{0 \le j < n} \pi_{n,j} m_j,$$

  where $\pi_{n,j}$ is probability of joining two trees with total size j.

Analysis of split/join

- The recurrence for s_n is

$$s_n = 1 + \frac{2}{nK} \sum_{0 \le j < n} \frac{j+1}{n+1} s_j + \frac{2(K-1)}{nK} \sum_{0 \le j < n} s_j + \frac{2(K-1)}{nK} \sum_{0 \le j < n} \frac{n-j}{n+1} m_j,$$

  with $s_0 = 0$.

- The recurrence for m_n has exactly the same shape with the rôles of s_n and m_n interchanged; it easily follows that $s_n = m_n$.

Analysis of split/join

- Define
$$S(z) = \sum_{n \geq 0} s_n z^n$$
- The recurrence for s_n translates to

$$z \frac{d^2 S}{dz^2} + 2 \frac{1-2z}{1-z} \frac{dS}{dz} - 2 \left(\frac{3K-2}{K} - z \right) \frac{S(z)}{(1-z)^2} = \frac{2}{(1-z)^3},$$

with initial conditions $S(0) = 0$ and $S'(0) = 1$.

Analysis of split/join

- The homogeneous second order linear ODE is of hypergeometric type.
- An easy particular solution of the ODE is

$$-\frac{1}{2}\left(\frac{K}{K-1}\right)\frac{1}{1-z}$$

Analysis of split/join

Theorem

The generating function $S(z)$ of the expected cost of split is, for any $K \geq 2$,

$$S(z) = \frac{1}{2}\frac{1}{1-\frac{1}{K}}\left[(1-z)^{-\alpha}\cdot {}_2F_1\left(\begin{matrix}1-\alpha, 2-\alpha\\ 2\end{matrix}\;\middle|\; z\right) - \frac{1}{1-z}\right],$$

where $\alpha = \alpha(K) = \frac{1}{2}\left(1+\sqrt{17-\frac{16}{K}}\right)$.

Analysis of split/join

Theorem

The expected cost s_n of splitting a relaxed K-d tree of size n is

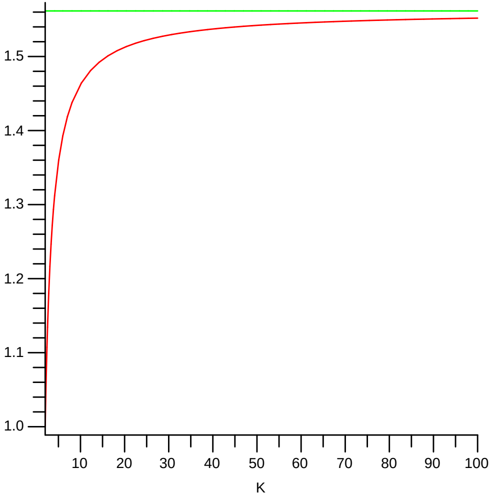
$$s_n = \eta(K)\, n^{\phi(K)} + o(n),$$

with

$$\eta = \frac{1}{2}\frac{1}{1-\frac{1}{K}}\frac{\Gamma(2\alpha-1)}{\alpha\Gamma^3(\alpha)},$$

$$\phi = \alpha - 1 = \frac{1}{2}\left(\sqrt{17-\frac{16}{K}}-1\right).$$

Analysis of split/join



Plot of $\phi(K)$

The cost of insertions and deletions

- The recurrence for the expected cost of an insertion is

$$I_n = \frac{\mathcal{I}_n}{n+1} + \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{2}{n}\sum_{0\le j<n}\frac{j+1}{n+1}I_j\right)$$
$$= \frac{\mathcal{I}_n}{n+1} + 1 + \mathcal{O}\left(\frac{1}{n}\right) + \frac{2}{n+1}\sum_{0\le j<n}\frac{j+1}{n+1}I_j.$$

  with $\mathcal{I}_n$ the average cost of an insertion at root
- The expected cost of deletions D_n satisfies a similar recurrence; it is asymptotically equivalent to the average cost of insertions
- We substitute $\mathcal{I}_n$ by the costs obtained previously (s_n)

The cost of insertions and deletions

Theorem

Let I_n and D_n denote the average cost of a randomized insertion and randomized deletion in a random relaxed K-d tree of size n using split and join. Then

1. if $K = 2$ then $I_n \sim D_n = 4 \ln n + \mathcal{O}(1)$.
2. if $K > 2$ then

$$I_n \sim D_n = \eta \frac{\phi - 1}{\phi + 1} n^{\phi - 1} + \mathcal{O}(\log n),$$

where $\mathcal{I}_n = \eta \, n^{\phi} + \mathcal{O}(1)$.

The cost of insertions and deletions

Theorem

Let I_n and D_n denote the average cost of a randomized insertion and randomized deletion in a random relaxed K-d tree of size n using split and join. Then

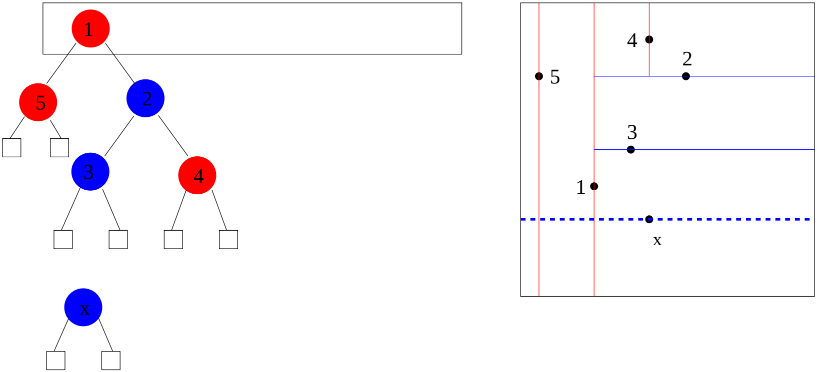
1. if $K = 2$ then $I_n \sim D_n = 4 \ln n + \mathcal{O}(1)$.
2. if $K > 2$ then

$$I_n \sim D_n = \eta \frac{\phi - 1}{\phi + 1} n^{\phi - 1} + \mathcal{O}(\log n),$$

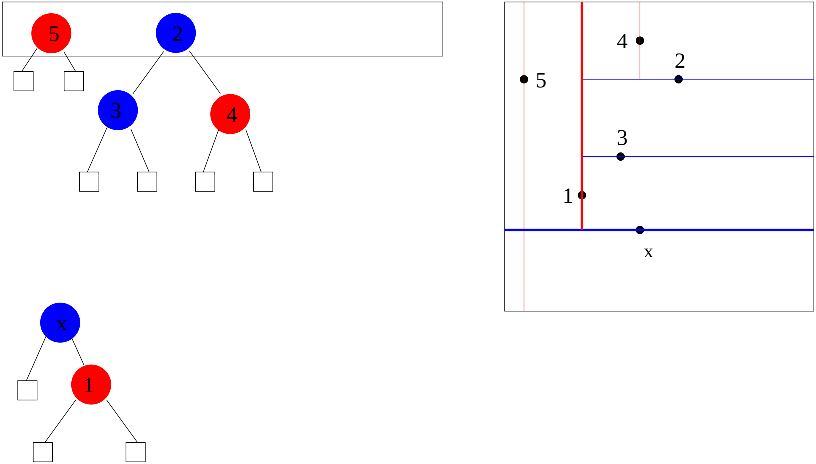
   where $\mathcal{I}_n = \eta \, n^{\phi} + \mathcal{O}(1)$.

Note that for $K > 2$, $\phi(K) > 1$!

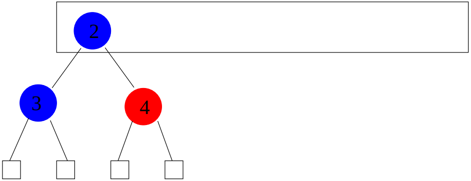
Copy-based insertions

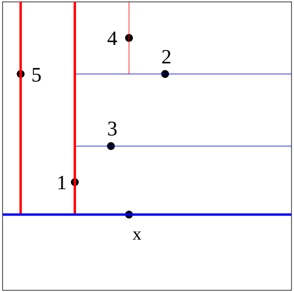


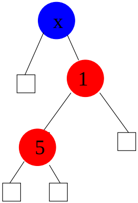
Copy-based insertions



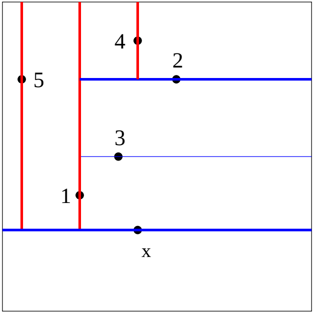
Copy-based insertions

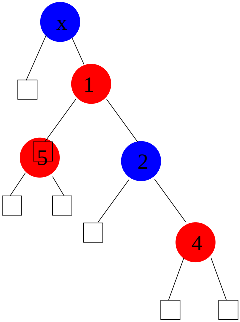




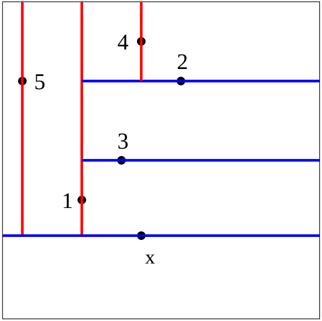


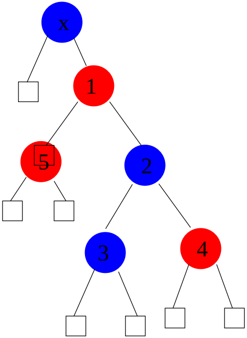
Copy-based insertions





Copy-Based insertions





Excursion: Partial match

Given a query $q = (q_0, \ldots, q_{K_1})$ where each $q_i \in [0, 1]$ or $q_i = *$, find all elements x in the K-d tree such that $x_i = q_i$ whenever $q_i \neq *$.

Partial match

```
void partial_match(rkdt t, query q) {
   if (t == NULL) return;
   if (matches(t -> key, q))
      report(t-> key);
   int i = t -> discr;
   if (q[i] == '*') {
      partial_match(t -> left, q);
      partial_match(t -> right, q);
   } else if (q[i] < t -> key) {
      partial_match(t -> left, q);
   } else {
      partial_match(t -> left, q);
   }
}
```

Analysis of copy-based updates

The cost of building T using copy-based insertion of a key x:

$$C(T) = P(T) + \frac{1}{K}\frac{|L|+1}{|T|+1}C(L) + \frac{1}{K}\frac{|R|+1}{|T|+1}C(R) + \frac{K-1}{K}\left(C(L)+C(R)\right),$$

where $P(T)$ denotes the number of nodes visited by a partial match in $T - \{x\}$ with query $q = (x_0, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_{K-1})$

Analysis of copy-based updates

The cost of making an insertion at root into a tree of size n:

$$C_n = P_n + \frac{2}{nK} \sum_{0 \le k < n} \frac{k+1}{n+1} C_k + \frac{2(K-1)}{nK} \sum_{0 \le k < n} C_k .$$

with P_n the expected cost of a partial match in a random relaxed K-d tree of size n with only one specified coordinate out of K coordinates

Analysis of copy-based updates

Theorem (Duch et al. 1998, Martínez et al. 2001)

The expected cost P_n (measured as the number of key comparisons) of a partial match query with s out of K attributes specified, $0 < s < K$, in a randomly built relaxed K-d tree of size n is

$$P_n = \beta(s/K) \cdot n^{\rho(s/K)} + \mathcal{O}(1),$$

where

$$\rho = \rho(x) = \left(\sqrt{9-8x} - 1\right)/2,$$

$$\beta(x) = \frac{\Gamma(2\rho+1)}{(1-x)(\rho+1)\Gamma^3(\rho+1)},$$

and $\Gamma(x)$ is Euler's Gamma function.

Analysis of copy-based updates

We will use Roura's Continuous Master Theorem to solve recurrences of the form:

$$F_n = t_n + \sum_{0 \le j < n} w_{n,j} F_j, \qquad n \ge n_0,$$

where t_n is the so-called toll function and the quantities $w_{n,j} \ge 0$ are called weights

Excursion: Roura's Continuous Master Theorem

Theorem (Roura 2001)

Let $t_n \sim C n^a \log^b n$ for some constants C, $a \geq 0$ and $b > -1$, and let $\omega(z)$ be a real function over $[0,1]$ such that

$$\sum_{0 \leq j < n} \left| w_{n,j} - \int_{j/n}^{(j+1)/n} \omega(z)\, dz \right| = \mathcal{O}(n^{-d})$$

for some constant $d > 0$. Let $\phi(x) = \int_0^1 z^x\, \omega(z)\, dz$, and define $\mathcal{H} = 1 - \phi(a)$. Then

1. If $\mathcal{H} > 0$ then $F_n \sim t_n \,/\, \mathcal{H}$.
2. If $\mathcal{H} = 0$ then $F_n \sim t_n \ln n \,/\, \mathcal{H}'$, where $\mathcal{H}' = -(b+1) \int_0^1 z^a \ln z\, \omega(z)\, dz$.
3. If $\mathcal{H} < 0$ then $F_n = \Theta(n^\alpha)$, where α is the unique real solution of $\phi(x) = 1$.

Analysis of copy-based updates

Applying the CMT to our recurrence we have

- $\omega(z) = \frac{2z}{K} + \frac{2(K-1)}{K}$
- $t_n = P_n \implies a = \varrho = \rho(1/K) = (\sqrt{9 - 8/K} - 1)/2$

Thus $\mathcal{H} = 0$

Analysis of copy-based updates

Applying the CMT to our recurrence we have

- $\omega(z) = \frac{2z}{K} + \frac{2(K-1)}{K}$
- $t_n = P_n \implies a = \varrho = \rho(1/K) = (\sqrt{9 - 8/K} - 1)/2$

Thus $\mathcal{H} = 0$

We have to compute $\mathcal{H}'$ with $b = 0$

$$\mathcal{H}' = -(b+1) \int_0^1 z^a \omega(z) \ln z \, dz$$

and get

$$\mathcal{H}' = 2 \frac{K\varrho^2 + (4K-2)\varrho + 4K - 3}{K(\varrho+2)^2(\varrho+1)^2}.$$

Analysis of copy-based updates

Theorem

The average cost C_n of copy-based insertion at root of a random relaxed K-d tree is

$$C_n = \gamma \cdot n^{\varrho} \ln n + o(n \ln n),$$

where

$$\varrho = \varrho(K) = \rho(1/K) = \left(\sqrt{9 - 8/K} - 1\right)/2,$$

$$\gamma = \frac{\beta(1/K)}{\mathcal{H}'} = \frac{\Gamma(2\varrho + 1)K(\varrho + 2)^2(\varrho + 1)}{2(1 - \frac{1}{K})\Gamma^3(\varrho + 1)(K\varrho^2 + (4K - 2)\varrho + (4K - 3))}.$$

The average cost C_n' of copy-based deletion of the root of a random relaxed K-d tree of size $n + 1$ is C_n.

The cost of insertions and deletions (2)

Theorem

For any fixed dimension $K \geq 2$, the average cost of a randomized insertion or deletion in random relaxed K-d tree of size n using copy-based updates is

$$I_n \sim D_n = 2 \ln n + \Theta(1).$$

The cost of insertions and deletions (2)

Theorem

For any fixed dimension $K \geq 2$, the average cost of a randomized insertion or deletion in random relaxed K-d tree of size n using copy-based updates is

$$I_n \sim D_n = 2\ln n + \Theta(1).$$

The "reconstruction" phase has constant cost on the average!

Example #3: Partial sorting

- Partial sorting: Given an array A of n elements and a value $1 \leq m \leq n$, rearrange A so that its first m positions contain the m smallest elements in ascending order
- For $m = \Theta(n)$ it might be OK to sort the array; otherwise, we are doing too much work

Example #3: Partial sorting

- Partial sorting: Given an array A of n elements and a value $1 \leq m \leq n$, rearrange A so that its first m positions contain the m smallest elements in ascending order
- For $m = \Theta(n)$ it might be OK to sort the array; otherwise, we are doing too much work

A few common solutions

- Idea #1: Partial heapsort
  - Build a heap with the n elements and perform m extractions of the heap's minimum
  - The worst-case cost is $\Theta(n + m \log n)$
  - This the "traditonal" implementation of C++ STL's `partial_sort`

A few common solutions

- Idea #1: Partial heapsort
  - Build a heap with the n elements and perform m extractions of the heap's minimum
  - The worst-case cost is $\Theta(n + m \log n)$
  - This the "traditonal" implementation of C++ STL's `partial_sort`

A few common solutions

- Idea #1: Partial heapsort
    - Build a heap with the n elements and perform m extractions of the heap's minimum
    - The worst-case cost is $\Theta(n + m \log n)$
    - This the "traditonal" implementation of C++ STL's `partial_sort`

A few common solutions

- Idea #2: On-line selection
  - Build a heap with the m first elements; then scan the remaining $n - m$ elements and update the heap as needed; finally extract the m elements from the heap
  - The worst-case cost is $\Theta(n \log m)$
  - Not very attractive unless m is very small or if used in on-line settings

A few common solutions

- Idea #2: On-line selection
  - Build a heap with the m first elements; then scan the remaining $n - m$ elements and update the heap as needed; finally extract the m elements from the heap
  - The worst-case cost is $\Theta(n \log m)$
  - Not very attractive unless m is very small or if used in on-line settings

A few common solutions

- Idea #2: On-line selection
  - Build a heap with the m first elements; then scan the remaining $n - m$ elements and update the heap as needed; finally extract the m elements from the heap
  - The worst-case cost is $\Theta(n \log m)$
  - Not very attractive unless m is very small or if used in on-line settings

A few common solutions

- Idea #3: "Quickselsort"
  - Find the mth smallest element with quickselect, then quicksort the preceding $m-1$ elements
  - The average cost is $\Theta(n + m \log m)$
  - Uses two basic algorithms widely available (and highly tuned for performance in standard libraries)

A few common solutions

- Idea #3: "Quickselsort"
  - Find the mth smallest element with quickselect, then quicksort the preceding $m - 1$ elements
  - The average cost is $\Theta(n + m \log m)$
  - Uses two basic algorithms widely available (and highly tuned for performance in standard libraries)

A few common solutions

- Idea #3: "Quickselsort"
  - Find the mth smallest element with quickselect, then quicksort the preceding $m-1$ elements
  - The average cost is $\Theta(n + m \log m)$
  - Uses two basic algorithms widely available (and highly tuned for performance in standard libraries)

Partial quicksort

```
void partial_quicksort(vector<Elem>& A,
                      int i, int j, int m) {
    if (i < j) {
       int p = get_pivot(A, i, j);
       swap(A[p], A[l]);
       int k;
       partition(A, i, j, k);
       partial_quicksort(A, i, k - 1, m);
       if (k < m - 1)
          partial_quicksort(A, k + 1, j, m);
}   }
```

The analysis

- Probability that the selected pivot is the k-th of n elements: $\pi_{n,k}$
- Average number of comparisons $P_{n,m}$ to sort the m smallest elements out of n:

$$P_{n,m} = n - 1 + \sum_{k=m+1}^{n} \pi_{n,k} \cdot P_{k-1,m}$$
$$+ \sum_{k=1}^{m} \pi_{n,k} \cdot (P_{k-1,k-1} + P_{n-k,m-k})$$

The analysis

- Probability that the selected pivot is the k-th of n elements: $\pi_{n,k}$
- Average number of comparisons $P_{n,m}$ to sort the m smallest elements out of n:

$$P_{n,m} = n - 1 + \sum_{k=m+1}^{n} \pi_{n,k} \cdot P_{k-1,m} \\ + \sum_{k=1}^{m} \pi_{n,k} \cdot (P_{k-1,k-1} + P_{n-k,m-k})$$

The analysis

- For $m = n$, partial quicksort $\equiv$ quicksort; let q_n denote the average number of comparisons used by quicksort
- Hence,

$$\begin{aligned} P_{n,m} = n - 1 &+ \sum_{0 \le k < m} \pi_{n,k+1} \cdot q_k \\ &+ \sum_{k=m+1}^{n} \pi_{n,k} \cdot P_{k-1,m} + \sum_{k=1}^{m} \pi_{n,k} \cdot P_{n-k,m-k} \end{aligned} \tag{1}$$

The analysis

- For $m = n$, partial quicksort $\equiv$ quicksort; let q_n denote the average number of comparisons used by quicksort
- Hence,

$$P_{n,m} = n - 1 + \sum_{0 \le k < m} \pi_{n,k+1} \cdot q_k \\ + \sum_{k=m+1}^{n} \pi_{n,k} \cdot P_{k-1,m} + \sum_{k=1}^{m} \pi_{n,k} \cdot P_{n-k,m-k} \qquad (1)$$

The analysis

- The recurrence for $P_{n,m}$ is the same as for quickselect but the toll function is

$$t_{n,m} = n - 1 + \sum_{0 \le k < m} \pi_{n,k+1} \cdot q_k$$

- Up to now, everything holds no matter which pivot selection scheme do we use; for the standard variant we must take $\pi_{n,k} = 1/n$, for all $1 \le k \le n$

The analysis

- The recurrence for $P_{n,m}$ is the same as for quickselect but the toll function is

$$t_{n,m} = n - 1 + \sum_{0 \le k < m} \pi_{n,k+1} \cdot q_k$$

- Up to now, everything holds no matter which pivot selection scheme do we use; for the standard variant we must take $\pi_{n,k} = 1/n$, for all $1 \le k \le n$

The analysis: Generating functions

- Define the two BGFs

$$P(z,u) = \sum_{n\geq 0} \sum_{1\leq m\leq n} P_{n,m} z^n u^m$$

$$T(z,u) = \sum_{n\geq 0} \sum_{1\leq m\leq n} t_{n,m} z^n u^m$$

- Then the recurrence (1) translates to

$$\frac{\partial P}{\partial z} = \frac{P(z,u)}{1-z} + \frac{u\,P(z,u)}{1-uz} + \frac{\partial T}{\partial z} \qquad (2)$$

The analysis: Generating functions

- Define the two BGFs

$$P(z,u) = \sum_{n\geq 0} \sum_{1\leq m\leq n} P_{n,m} z^n u^m$$

$$T(z,u) = \sum_{n\geq 0} \sum_{1\leq m\leq n} t_{n,m} z^n u^m$$

- Then the recurrence (1) translates to

$$\frac{\partial P}{\partial z} = \frac{P(z,u)}{1-z} + \frac{u\,P(z,u)}{1-uz} + \frac{\partial T}{\partial z} \qquad (2)$$

The analysis: Generating functions

- Let $P(z,u) = F(z,u) + S(z,u)$, where $F(z,u)$ corresponds to the selection part of the toll function $(n-1)$ and $S(z,u)$ to the sorting part $(\sum_k q_k/n)$
- Let

$$T_F(z,u) = \sum_{n\geq 0}\sum_{1\leq m\leq n}(n-1)z^n u^m$$

$$T_S(z,u) = \sum_{n\geq 0}\sum_{1\leq m\leq n}\frac{1}{n}\left(\sum_{0\leq k<m} q_k\right) z^n u^m$$

The analysis: Generating functions

- Let $P(z,u) = F(z,u) + S(z,u)$, where $F(z,u)$ corresponds to the selection part of the toll function $(n-1)$ and $S(z,u)$ to the sorting part $(\sum_k q_k/n)$
- Let

$$T_F(z,u) = \sum_{n\geq 0} \sum_{1\leq m\leq n} (n-1) z^n u^m$$

$$T_S(z,u) = \sum_{n\geq 0} \sum_{1\leq m\leq n} \frac{1}{n} \left(\sum_{0\leq k<m} q_k \right) z^n u^m$$

The analysis: Generating functions

- Then, each of $F(z,u)$ and $S(z,u)$ satisfies a differential equation like (2) and

$$\begin{aligned} F(z,u) &= \frac{1}{(1-z)(1-zu)} \\ &\quad \times \left\{ \int (1-z)(1-zu)\frac{\partial T_F}{\partial z}\, dz + K_F \right\} \\ S(z,u) &= \frac{1}{(1-z)(1-zu)} \\ &\quad \times \left\{ \int (1-z)(1-zu)\frac{\partial T_S}{\partial z}\, dz + K_S \right\} \end{aligned}$$

The analysis: Generating functions

- $F(z,u)$ satisfies exactly the same differential equation as standard quickselect; it is well known (Knuth, 1971) that for $1 \le m \le n$,

$$F_{n,m} = [z^n u^m]F(z,u) = 2\Big(n + 3 + (n+1)H_n - (m+2)H_m - (n+3-m)H_{n+1-m}\Big)$$

The analysis: Generating functions

- To compute $S(z,u)$, we need first to determine $T_S(z,u)$
$$\frac{\partial T_S}{\partial z} = \frac{u}{1-z}\frac{Q(uz)}{1-uz}$$
where $Q(z) = \sum_{n\geq 0} q_n z^n$.
- With the toll function $n-1$, we solve the recurrence for quicksort to get
$$Q(z) = \frac{2}{(1-z)^2}\left(\ln\frac{1}{1-z} - z\right)$$

The analysis: Generating functions

- To compute $S(z,u)$, we need first to determine $T_S(z,u)$
$$\frac{\partial T_S}{\partial z} = \frac{u}{1-z}\frac{Q(uz)}{1-uz}$$
where $Q(z) = \sum_{n\geq 0} q_n z^n$.
- With the toll function $n-1$, we solve the recurrence for quicksort to get
$$Q(z) = \frac{2}{(1-z)^2}\left(\ln\frac{1}{1-z} - z\right)$$

The analysis: Generating functions

- Hence,

$$\begin{aligned}
S(z,u) &= \frac{1}{(1-z)(1-uz)}\left\{\int u\,Q(uz)\,dz + K_S\right\} \\
&= \frac{2}{(1-uz)^2(1-z)}\ln\frac{1}{1-uz} \\
&\quad + \frac{2}{(1-z)(1-uz)}\ln\frac{1}{1-uz} \\
&\quad - 4\frac{uz}{(1-uz)^2(1-z)}
\end{aligned}$$

The analysis: Generating functions

- Extracting coefficients $S_{n,m} = [z^n u^m] S(z,u)$

$$S_{n,m} = 2(m+1)H_m - 6m + 2H_m$$

- And finally

$$P_{n,m} = 2n + 2(n+1)H_n - 2(n+3-m)H_{n+1-m} - 6m + 6$$

The analysis: Generating functions

- Extracting coefficients $S_{n,m} = [z^n u^m] S(z,u)$

$$S_{n,m} = 2(m+1)H_m - 6m + 2H_m$$

- And finally

$$\begin{aligned} P_{n,m} = {} & 2n + 2(n+1)H_n - 2(n+3-m)H_{n+1-m} \\ & - 6m + 6 \end{aligned}$$

Partial quicksort vs. quickselsort

- The average number of comparisons made by quickselsort is

$$Q_{n,m} = F_{n,m} + q_{m-1}$$

- Using partial quicksort we save

$$Q_{n,m} - P_{n,m} = 2m - 4H_m + 2$$

comparisons on the average

Partial quicksort vs. quickselsort

- The average number of comparisons made by quickselsort is

$$Q_{n,m} = F_{n,m} + q_{m-1}$$

- Using partial quicksort we save

$$Q_{n,m} - P_{n,m} = 2m - 4H_m + 2$$

comparisons on the average

Final remarks on partial quicksort

- Partial quicksort avoids some of the redundant comparisons, exchanges, ... made by quickselsort
- It is easily implemented
- It benefits from standard optimization techniques: sampling, recursion removal, recursion cutoff on small subfiles, improved partitioning schems, etc.
- The same idea can be applied to similar algorithms like radix sorting and quicksort for strings

Final remarks on partial quicksort

- Partial quicksort avoids some of the redundant comparisons, exchanges, ... made by quickselsort
- It is easily implemented
- It benefits from standard optimization techniques: sampling, recursion removal, recursion cutoff on small subfiles, improved partitioning schems, etc.
- The same idea can be applied to similar algorithms like radix sorting and quicksort for strings

Final remarks on partial quicksort

- Partial quicksort avoids some of the redundant comparisons, exchanges, ... made by quickselsort
- It is easily implemented
- It benefits from standard optimization techniques: sampling, recursion removal, recursion cutoff on small subfiles, improved partitioning schems, etc.
- The same idea can be applied to similar algorithms like radix sorting and quicksort for strings

Final remarks on partial quicksort

- Partial quicksort avoids some of the redundant comparisons, exchanges, . . . made by quickselsort
- It is easily implemented
- It benefits from standard optimization techniques: sampling, recursion removal, recursion cutoff on small subfiles, improved partitioning schems, etc.
- The same idea can be applied to similar algorithms like radix sorting and quicksort for strings

Concluding remarks

I hope I have convinced you about the usefulness of probabilistic analysis

- Provides useful information about typical behavior
- Necessary when analyzing randomized algorithms
- Allows meaningful comparisons between competitors of equivalent performance
- A source of beautiful and challenging mathematical problems!

Concluding remarks

I hope I have convinced you about the usefulness of probabilistic analysis

- Provides useful information about typical behavior
- Necessary when analyzing randomized algorithms
- Allows meaningful comparisons between competitors of equivalent performance
- A source of beautiful and challenging mathematical problems!

Concluding remarks

I hope I have convinced you about the usefulness of probabilistic analysis

- Provides useful information about typical behavior
- Necessary when analyzing randomized algorithms
- Allows meaningful comparisons between competitors of equivalent performance
- A source of beautiful and challenging mathematical problems!

Concluding remarks

I hope I have convinced you about the usefulness of probabilistic analysis

- Provides useful information about typical behavior
- Necessary when analyzing randomized algorithms
- Allows meaningful comparisons between competitors of equivalent performance
- A source of beautiful and challenging mathematical problems!

Credits

- Alois Panholzer and Helmut Prodinger: Generating random derangements
- Amalia Duch: Updating K-d trees

THANKS!!