The Hiring Problem

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The hiring problem is a simple model of decision-making under uncertainty. It is closely related to the well-known Secretary Problem:

“A sequence of $n$ candidates is to be interviewed to fill a post. For each interviewed candidate we only learn about his/her relative rank among the candidates we’ve seen so far. After each interview, hire and finish, or discard and interview a new candidate. The $n$th candidate must be hired if we have reached that far.

The goal: devise an strategy that maximizes the probability of hiring the best of the $n$ candidates.”
The hiring problem

- Originally introduced by Broder et al. (SODA 2008)
- The candidates are modelled by a (potentially infinite) sequence of i.i.d. random variables $Q_i$ uniformly distributed in $[0, 1]$
- At step $i$ you either hire or discard candidate $i$ with score $Q_i$
- Decisions are irrevocable
- Goals: hire candidates at some reasonable rate, improve the “mean” quality of the company’s staff
The hiring problem

- Our model: a permutation $\sigma$ of length $n$, candidate $i$ has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, $s_i$ is the rank of the $i$th candidate relative to the candidates seen so far (i included)

Example

$$\sigma = 62817435$$
$$\sigma' =$$
$$S =$$
The hiring problem

- Our model: a permutation $\sigma$ of length $n$, candidate $i$ has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, $s_i$ is the rank of the $i$th candidate relative to the candidates seen so far ($i$ included)

Example

\[
\begin{align*}
\sigma &= 62817435 \\
\sigma' &= 1 \\
S &= 1
\end{align*}
\]
The hiring problem

- Our model: a permutation $\sigma$ of length $n$, candidate $i$ has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, $s_i$ is the rank of the $i$th candidate relative to the candidates seen so far ($i$ included)

Example

\[
\sigma = 62817435 \\
\sigma' = 21 \\
S = 11
\]
The hiring problem

- Our model: a permutation $\sigma$ of length $n$, candidate $i$ has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, $s_i$ is the rank of the $i$th candidate relative to the candidates seen so far ($i$ included)

Example

$$\sigma = 62817435$$
$$\sigma' = 213$$
$$S = 113$$
Our model: a permutation $\sigma$ of length $n$, candidate $i$ has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, $s_i$ is the rank of the $i$th candidate relative to the candidates seen so far ($i$ included).

Example

\[
\begin{align*}
\sigma &= 62817435 \\
\sigma' &= 3241 \\
S &= 1131
\end{align*}
\]
The hiring problem

- Our model: a permutation $\sigma$ of length $n$, candidate $i$ has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, $s_i$ is the rank of the $i$th candidate relative to the candidates seen so far ($i$ included)

Example

$\sigma = 62817435$

$\sigma' = 32514$

$S = 11314$
The hiring problem

- Our model: a permutation \( \sigma \) of length \( n \), candidate \( i \) has score \( \sigma(i) \); the permutation is actually presented as a sequence of unknown length \( S = s_1, s_2, s_3, \ldots \) with \( 1 \leq s_i \leq i + 1 \), \( s_i \) is the rank of the \( i \)th candidate relative to the candidates seen so far (\( i \) included)

Example

\[
\begin{align*}
\sigma &= 62817435 \\
\sigma' &= 426153 \\
S &= 113143
\end{align*}
\]
Our model: a permutation $\sigma$ of length $n$, candidate $i$ has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, $s_i$ is the rank of the $i$th candidate relative to the candidates seen so far ($i$ included)

Example

\[
\sigma = 62817435 \\
\sigma' = 5271643 \\
S = 1131433
\]
Our model: a permutation $\sigma$ of length $n$, candidate $i$ has score $\sigma(i)$; the permutation is actually presented as a sequence of unknown length $S = s_1, s_2, s_3, \ldots$ with $1 \leq s_i \leq i + 1$, $s_i$ is the rank of the $i$th candidate relative to the candidates seen so far ($i$ included).

Example

$$\sigma = 62817435$$
$$\sigma' = 62817435$$
$$S = 11314335$$
A hiring strategy is rank-based if and only if it only depends on the relative rank of the current candidate compared to the candidates seen so far.
Rank-based hiring

- Rank-based strategies modelize actual restrictions to measure qualities
- Many natural strategies are rank-based, e.g.,
  - above the best
  - above the $m$th best
  - above the median
  - above the $P\%$ best
- Assume only relative ranks of candidates are known, like the standard secretary problem
- Some hiring strategies are not rank-based, e.g., above the average, above a threshold.
Intermezzo: A crash course on generating functions and the symbolic method

– Excerpts from my short course “Analytic Combinatorics: A Primer”
Two basic counting principles

Let $A$ and $B$ be two finite sets.

**The Addition Principle**

If $A$ and $B$ are disjoint then

$$|A \cup B| = |A| + |B|$$

**The Multiplication Principle**

$$|A \times B| = |A| \times |B|$$
A combinatorial class is a pair \((\mathcal{A}, |\cdot|)\), where \(\mathcal{A}\) is a finite or denumerable set of values (combinatorial objects, combinatorial structures), \(|\cdot| : \mathcal{A} \rightarrow \mathbb{N}\) is the size function and for all \(n \geq 0\)

\[
\mathcal{A}_n = \{x \in \mathcal{A} \mid |x| = n\} \text{ is finite}
\]
Combinatorial classes

**Example**

- $\mathcal{A} =$ all finite strings from a binary alphabet; $|s|$ = the length of string $s$
- $\mathcal{B} =$ the set of all permutations; $|\sigma|$ = the order of the permutation $\sigma$
- $\mathcal{C}_n =$ the partitions of the integer $n$; $|p| = n$ if $p \in \mathcal{C}_n$
Labelled and unlabelled classes

- In **unlabelled** classes, objects are made up of indistinguishable atoms; an atom is an object of size 1.
- In **labelled** classes, objects are made up of distinguishable atoms; in an object of size $n$, each of its $n$ atoms bears a distinct label from $\{1, \ldots, n\}$.
Let $a_n = \# A_n$ be the number of objects of size $n$ in $A$. Then the formal power series

$$A(z) = \sum_{n \geq 0} a_n z^n = \sum_{\alpha \in A} z^{\left|\alpha\right|}$$

is the (ordinary) generating function of the class $A$. The coefficient of $z^n$ in $A(z)$ is denoted $[z^n] A(z)$:

$$[z^n] A(z) = [z^n] \sum_{n \geq 0} a_n z^n = a_n$$
Ordinary generating functions (OGFs) are mostly used to enumerate unlabelled classes.

### Example

\[ \mathcal{L} = \{ w \in (0 + 1)^* \mid w \text{ does not contain two consecutive 0's} \} \]

\[ = \{ \epsilon, 0, 1, 01, 10, 11, 010, 011, 101, 110, 111, \ldots \} \]

\[ L(z) = z^{\mid \epsilon \mid} + z^{\mid 0 \mid} + z^{\mid 1 \mid} + z^{\mid 01 \mid} + z^{\mid 10 \mid} + z^{\mid 11 \mid} + \ldots \]

\[ = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \ldots \]

Exercise: Can you guess the value of \( L_n = [z^n]L(z) \)?
Counting generating functions

Definition

Let \( a_n = \# A_n \) = the number of objects of size \( n \) in \( A \). Then the formal power series

\[
\hat{A}(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!} = \sum_{\alpha \in A} \frac{z^{\lvert \alpha \rvert}}{\lvert \alpha \rvert!}
\]

is the exponential generating function of the class \( A \).
Counting generating functions

Exponential generating functions (EGFs) are used to enumerate labelled classes.

Example

\[ C = \text{circular permutations} = \{ \varepsilon, 1, 12, 123, 132, 1234, 1243, 1324, 1342, 1423, 1432, 12345, \ldots \} \]

\[ \hat{C}(z) = \frac{1}{0!} + \frac{z}{1!} + \frac{z^2}{2!} + 2\frac{z^3}{3!} + 6\frac{z^4}{4!} + \cdots \]

\[ c_n = n! \cdot [z^n] \hat{C}(z) = (n - 1)!, \quad n > 0 \]
Let $C = A + B$, the disjoint union of the unlabelled classes $A$ and $B$ ($A \cap B = \emptyset$). Then

$$C(z) = A(z) + B(z)$$

And

$$c_n = [z^n]C(z) = [z^n]A(z) + [z^n]B(z) = a_n + b_n$$
Let $C = A \times B$, the Cartesian product of the unlabelled classes $A$ and $B$. The size of $(\alpha, \beta) \in C$, where $a \in A$ and $\beta \in B$, is the sum of sizes: $|(\alpha, \beta)| = |\alpha| + |\beta|$. Then

$$C(z) = A(z) \cdot B(z)$$

Proof.

$$C(z) = \sum_{\gamma \in C} z^{\mid \gamma \mid} = \sum_{(\alpha, \beta) \in A \times B} z^{\mid \alpha \mid + \mid \beta \mid} = \sum_{\alpha \in A} \sum_{\beta \in B} z^{\mid \alpha \mid} \cdot z^{\mid \beta \mid}$$

$$= \left( \sum_{\alpha \in A} z^{\mid \alpha \mid} \right) \cdot \left( \sum_{\beta \in B} z^{\mid \beta \mid} \right) = A(z) \cdot B(z)$$
The $n$th coefficient of the OGF for a Cartesian product is the convolution of the coefficients $\{a_n\}$ and $\{b_n\}$:

$$c_n = [z^n]C(z) = [z^n]A(z) \cdot B(z)$$

$$= \sum_{k=0}^{n} a_k b_{n-k}$$
Sequences

Let $A$ be a class without any empty object ($A_0 = \emptyset$). The class $C = \text{Seq}(A)$ denotes the class of sequences of $A$’s.

$$C = \{(\alpha_1, \ldots, \alpha_k) | k \geq 0, \alpha_i \in A\}$$

$$= \{\varepsilon\} + A + (A \times A) + (A \times A \times A) + \cdots = \{\varepsilon\} + A \times C$$

Then

$$C(z) = \frac{1}{1 - A(z)}$$

Proof.

$$C(z) = 1 + A(z) + A^2(z) + A^3(z) + \cdots = 1 + A(z) \cdot C(z)$$
Disjoint unions of labelled classes are defined as for unlabelled classes and \( \hat{C}(z) = \hat{A}(z) + \hat{B}(z) \), for \( C = A + B \). Also, \( c_n = a_n + b_n \).

To define labelled products, we must take into account that for each pair \((\alpha, \beta)\) where \(|\alpha| = k\) and \(|\alpha| + |\beta| = n\), we construct \( \binom{n}{k} \) distinct pairs by consistently relabelling the atoms of \( \alpha \) and \( \beta \):

\[
\alpha = (2, 1, 4, 3), \quad \beta = (1, 3, 2)
\]

\[
\alpha \times \beta = \{(2, 1, 4, 3, 5, 7, 6), (2, 1, 5, 3, 4, 7, 6), \ldots, (5, 4, 7, 6, 1, 3, 2)\}
\]

\[
\#(\alpha \times \beta) = \binom{7}{4} = 35
\]

The size of an element in \( \alpha \times \beta \) is \(|\alpha| + |\beta| \).
Labelled products

For a class $C$ that is labelled product of two labelled classes $A$ and $B$

$$C = A \times B = \bigcup_{\alpha \in A} \bigcup_{\beta \in B} \alpha \times \beta$$

the following relation holds for the corresponding EGFs

$$\hat{C}(z) = \sum_{\gamma \in C} \frac{z^{|\gamma|!}}{|\gamma|!} = \sum_{\alpha \in A} \sum_{\beta \in B} \left( \frac{|\alpha| + |\beta|}{|\alpha|} \right) \frac{z^{|\alpha|+|\beta|}}{(|\alpha| + |\beta|)!}$$

$$= \sum_{\alpha \in A} \sum_{\beta \in B} \frac{1}{|\alpha|!|\beta|!} z^{|\alpha|+|\beta|} = \left( \sum_{\alpha \in A} \frac{z^{|\alpha|}}{|\alpha|!} \right) \cdot \left( \sum_{\beta \in B} \frac{z^{|\beta|}}{|\beta|!} \right)$$

$$= \hat{A}(z) \cdot \hat{B}(z)$$
Labelled products

The $n$th coefficient of $\hat{C}(z) = \hat{A}(z) \cdot \hat{B}(z)$ is also a convolution

$$c_n = [z^n] \hat{C}(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$
Sequences of labelled object are defined as in the case of unlabelled objects. The construction $\mathcal{C} = \text{Seq}(\mathcal{A})$ is well defined if $\mathcal{A}_0 = \emptyset$. If $\mathcal{C} = \text{Seq}(\mathcal{A}) = \{\varepsilon\} + \mathcal{A} \times \mathcal{C}$ then

$$\hat{\mathcal{C}}(z) = \frac{1}{1 - \hat{A}(z)}$$

**Example**

Permutations are labelled sequences of atoms, $\mathcal{P} = \text{Seq}(\mathcal{Z})$. Hence,

$$\hat{\mathcal{P}}(z) = \frac{1}{1 - z} = \sum_{n \geq 0} z^n$$

$$n! \cdot [z^n] \hat{\mathcal{P}}(z) = n!$$
### A dictionary of admissible unlabelled operators

<table>
<thead>
<tr>
<th>Class</th>
<th>OGF</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$</td>
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</tr>
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<td>$z$</td>
<td>Atomic</td>
</tr>
<tr>
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<td>Disjoint union</td>
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<td>Product</td>
</tr>
<tr>
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<td>$\frac{1}{1-A(z)}$</td>
<td>Sequence</td>
</tr>
<tr>
<td>$\Theta A$</td>
<td>$\Theta A(z) = z A'(z)$</td>
<td>Marking</td>
</tr>
<tr>
<td>$\text{MSet}(A)$</td>
<td>$\exp \left( \sum_{k&gt;0} A(z^k)/k \right)$</td>
<td>Multiset</td>
</tr>
<tr>
<td>$\text{PSet}(A)$</td>
<td>$\exp \left( \sum_{k&gt;0} (-1)^k A(z^k)/k \right)$</td>
<td>Powerset</td>
</tr>
<tr>
<td>$\text{Cycle}(A)$</td>
<td>$\sum_{k&gt;0} \frac{\phi(k)}{k} \ln \frac{1}{1-A(z^k)}$</td>
<td>Cycle</td>
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## A dictionary of admissible labelled operators

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<td>$\exp(\hat{A}(z))$</td>
<td>Set</td>
</tr>
<tr>
<td>Cycle$(A)$</td>
<td>$\ln \left( \frac{1}{1 - \hat{A}(z)} \right)$</td>
<td>Cycle</td>
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</table>
We need often to study some characteristic of combinatorial structures, e.g., the number of left-to-right maxima in a permutation, the height of a rooted tree, the number of complex components in a graph, etc.

Suppose $X : \mathcal{A}_n \rightarrow \mathbb{N}$ is a characteristic under study. Let

$$a_{n,k} = \#\{\alpha \in \mathcal{A} \mid |\alpha| = n, X(\alpha) = k\}$$

We can view the restriction $X_n : \mathcal{A}_n \rightarrow \mathbb{N}$ as a random variable. Then under the usual uniform model

$$\mathbb{P}[X_n = k] = \frac{a_{n,k}}{a_n}$$
Bivariate generating functions

Define

\[ A(z, u) = \sum_{n,k \geq 0} a_{n,k} z^n u^k \]

\[ = \sum_{\alpha \in \mathcal{A}} \alpha! u^X(\alpha) z^{\alpha} \]

Then \( a_{n,k} = [z^n u^k] A(z, u) \) and

\[ \mathbb{P}[X_n = k] = \frac{[z^n u^k] A(z, u)}{[z^n] A(z, 1)} \]
Bivariate generating functions

We can also define

\[ B(z, u) = \sum_{n,k \geq 0} \mathbb{P}[X_n = k] z^n u^k \]

\[ = \sum_{\alpha \in A} \mathbb{P}[\alpha] z^{\alpha_1} u^{X(\alpha)} \]

and thus \( B(z, u) \) is a generating function whose coefficient of \( z^n \) is the probability generating function of the r.v. \( X_n \)

\[ B(z, u) = \sum_{n \geq 0} P_n(u) z^n \]

\[ P_n(u) = [z^n] B(z, u) = \sum_{k \geq 0} \mathbb{P}[X_n = k] u^k \]
If $P(u)$ is the probability generating function of a random variable $X$ then

\begin{align*}
P(1) &= 1, \\
P'(1) &= E[X], \\
P''(1) &= E[X^2] = E[X(X - 1)], \\
\mathbb{V}[X] &= P''(1) + P'(1) - (P'(1))^2
\end{align*}
Bivariate generating functions

We can study the moments of $X_n$ by successive differentiation of $B(z, u)$ (or $A(z, u)$). For instance,

$$B(z) = \sum_{n \geq 0} \mathbb{E}[X_n] z^n = \frac{\partial B}{\partial u} \bigg|_{u=1}$$

For the $r$th factorial moments of $X_n$

$$B^{(r)}(z) = \sum_{n \geq 0} \mathbb{E}[X_n^r] z^n = \frac{\partial^r B}{\partial u^r} \bigg|_{u=1}$$

$$X_n^r = X_n(X_n - 1) \cdots (X_n - r + 1)$$
Consider the following specification for permutations

\[ \mathcal{P} = \{\emptyset\} + \mathcal{P} \times \mathbb{Z} \]

The BGF for the probability that a random permutation of size \( n \) has \( k \) left-to-right maxima is

\[ M(z, u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{\lvert \sigma \rvert}}{\lvert \sigma \rvert!} u^{X(\sigma)}, \]

where \( X(\sigma) = \# \) of left-to-right maxima in \( \sigma \)
The number of left-to-right maxima in a permutation

With the recursive decomposition of permutations and since the last element of a permutation of size $n$ is a left-to-right maxima iff its label is $n$

$$M(z, u) = \sum_{\sigma \in \mathcal{P}} \sum_{1 \leq j \leq |\sigma|+1} \frac{z^{|\sigma|+1}}{(|\sigma| + 1)!} u^{X(\sigma)+[j=|\sigma|+1]}$$

$[P] = 1$ if $P$ is true, $[P] = 0$ otherwise.
The number of left-to-right maxima in a permutation

\[ M(z, u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|+1}}{(|\sigma| + 1)!} u^{X(\sigma)} \sum_{1 \leq j \leq |\sigma|+1} u^{[j=|\sigma|+1]} \]

Taking derivatives w.r.t. \( z \)

\[ \frac{\partial}{\partial z} M = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{X(\sigma)} (|\sigma| + u) = z \frac{\partial}{\partial z} M + uM \]

Hence,

\[ (1 - z) \frac{\partial}{\partial z} M(z, u) - uM(z, u) = 0 \]
The number of left-to-right maxima in a permutation

Solving, since \( M(0, u) = 1 \)

\[
M(z, u) = \left( \frac{1}{1 - z} \right)^u = \sum_{n,k \geq 0} \binom{n}{k} \frac{z^n}{n!} u^k
\]

where \( \binom{n}{k} \) denote the (signless) Stirling numbers of the first kind, also called Stirling cycle numbers.

Taking the derivative w.r.t. \( u \) and setting \( u = 1 \)

\[
m(z) = \frac{\partial}{\partial z} M(z, u) \bigg|_{u=1} = \frac{1}{1 - z} \ln \frac{1}{1 - z}
\]

Thus the average number of left-to-right maxima in a random permutation of size \( n \) is

\[
[z^n] m(z) = \mathbb{E}[X_n] = H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + \gamma + O(1/n)
\]

\[
\frac{1}{1 - z} \ln \frac{1}{1 - z} = \sum_{\ell} \sum_{m > 0} z^\ell \frac{z^m}{m} = \sum_{n \geq 0} z^n \sum_{k=1}^{n} \frac{1}{k}
\]
...back to bussiness now!
The recursive decomposition of permutations

\[ \mathcal{P} = \varepsilon + \mathcal{P} \times Z \]

is the natural choice for the analysis of rank-based strategies, with \( \times \) denoting the labelled product.

For each \( \sigma \) in \( \mathcal{P} \), \( \{\sigma\} \times Z \) is the set of \( |\sigma| + 1 \) permutations

\[ \{\sigma \star 1, \sigma \star 2, \ldots, \sigma \star (n + 1)\}, \quad n = |\sigma| \]

\( \sigma \star j \) denotes the permutation one gets after relabelling \( j \), \( j + 1 \), \ldots, \( n = |\sigma| \) in \( \sigma \) to \( j + 1 \), \( j + 2 \), \ldots, \( n + 1 \) and appending \( j \) at the end

\[ 32451 \star 3 = 425613 \]
\[ 32451 \star 2 = 435612 \]
Rank-based hiring

- $\mathcal{H}(\sigma) =$ the set of candidates hired in permutation $\sigma$
- $h(\sigma) = \#\mathcal{H}(\sigma)$
- Let $X_j(\sigma) = 1$ if candidate with score $j$ is hired after $\sigma$ and $X_j(\sigma) = 0$ otherwise.
- $h(\sigma \star j) = h(\sigma) + X_j(\sigma)$
Theorem

Let $H(z, u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{\vert \sigma \vert}}{\vert \sigma \vert !} u^{h(\sigma)}$. Then

$$(1 - z) \frac{\partial}{\partial z} H(z, u) - H(z, u) = (u - 1) \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{\vert \sigma \vert}}{\vert \sigma \vert !} u^{h(\sigma)},$$

where $X(\sigma)$ the number of $j$ such that $X_j(\sigma) = 1$. 
Rank-based hiring

We can write \( h(\sigma) = 0 \) if \( \sigma \) is the empty permutation and \( h(\sigma \star j) = h(\sigma) + X_j(\sigma) \).

\[
H(z, u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{\mid\sigma\mid}}{|\sigma|!} u^{h(\sigma)} = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_n} \frac{z^{\mid\sigma\mid}}{|\sigma|!} u^{h(\sigma)}
\]

\[
= 1 + \sum_{n>0} \sum_{1 \leq j \leq n} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{\mid\sigma\star j\mid}}{|\sigma \star j|!} u^{h(\sigma \star j)}
\]

\[
= 1 + \sum_{n>0} \sum_{1 \leq j \leq n} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{\mid\sigma\mid+1}}{(\mid\sigma\mid + 1)!} u^{h(\sigma)+X_j(\sigma)}
\]

\[
= 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{\mid\sigma\mid+1}}{(\mid\sigma\mid + 1)!} u^{h(\sigma)} \sum_{1 \leq j \leq n} u^{X_j(\sigma)}.
\]
Since $X_j(\sigma)$ is either 0 or 1 for all $j$ and all $\sigma$, we have

$$\sum_{1 \leq j \leq n} u^{X_j(\sigma)} = (|\sigma| + 1 - X(\sigma)) + uX(\sigma),$$

where $X(\sigma) = \sum_{1 \leq j \leq |\sigma| + 1} X_j(\sigma)$.

$$H(z, u) = 1 + \sum_{n > 0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma| + 1)!} u^{h(\sigma)} \left((|\sigma| + 1 - X(\sigma)) + uX(\sigma)\right).$$

The theorem follows after differentiation and a few additional algebraic manipulations.
A hiring strategy is **pragmatic** if and only if

- Whenever it would hire a candidate with score $j$, it would hire a candidate with a larger score
  
  $$X_j(\sigma) = 1 \implies X_{j'}(\sigma) = 1 \quad \text{for all } j' \geq j$$

- The number of scores it would potentially hire increases at most by one if and only if the candidate in the previous step was hired
  
  $$X(\sigma \star j) \leq X(\sigma) + X_j(\sigma)$$
Pragmatic strategies

- The first condition is very natural and reasonable; the second one is technically necessary for several results we discuss later.
- Above the best, above the \( m \)th best, above the \( P \)% best, … are all pragmatic.
For any pragmatic hiring strategy and any permutation $\sigma$, the $X(\sigma)$ best candidates of $\sigma$ have been hired (and possibly others).
Pragmatic strategies
Let $r_n$ denote the rank of the last hired candidate in a random permutation, and

$$g_n = 1 - \frac{r_n}{n}$$

is called the gap.

**Theorem**

*For any pragmatic hiring strategy,*

$$\mathbb{E}[g_n] = \frac{1}{2n}(\mathbb{E}[X_n] - 1),$$

*where* $\mathbb{E}[X_n] = [z^n] \sum_{\sigma \in \mathcal{P}} X(\sigma)z^{|\sigma|}/|\sigma|!$. 


Candidate $i$ is hired if and only if her score is above the score of the best currently hired candidate.

- $X(\sigma) = 1$
- $\mathcal{H}(\sigma) = \{i : i \text{ is a left-to-right maximum}\}$
- $\mathbb{E}[h_n] = [z^n] \frac{\partial H}{\partial u} \bigg|_{u=1} = \ln n + O(1)$
- Variance of $h_n$ is also $\ln n + O(1)$ and after proper normalization $h^*_n$ converges to $\mathcal{N}(0,1)$
Candidate $i$ is hired if and only if her score is above the score of the $m$th best currently hired candidate.

- $X(\sigma) = |\sigma| + 1$ if $|\sigma| < m$; $X(\sigma) = m$ if $|\sigma| \geq m$
- $\mathbb{E}[h_n] = [z^n] \frac{\partial H}{\partial u} \bigg|_{u=1} = m \ln n + O(1)$ for fixed $m$
- Variance of $h_n$ is also $m \ln n + O(1)$ and after proper normalization $h_n^*$ converges to $\mathcal{N}(0, 1)$
- The case of arbitrary $m$ can be studied by introducing
  $H(z, u, v) = \sum_{m \geq 1} v^m H^{(m)}(z, u)$, where $H^{(m)}(z, u)$ is the GF that corresponds to a given particular $m$.
- We can show that
  $\mathbb{E}[h_n] = m(H_n - H_m + 1) \sim m \ln(n/m) + m + O(1)$, with $H_n$ the $n$th harmonic number
Candidate $i$ is hired if and only if her score is above the score of the median of the scores of currently hired candidates.

- $X(\sigma) = \lceil (h(\sigma) + 1)/2 \rceil$
- $\sqrt{\frac{n}{\pi}}(1 + O(n^{-1})) \leq \mathbb{E}[h_n] \leq 3\sqrt{\frac{n}{\pi}}(1 + O(n^{-1}))$
- This result follows easily by using previous theorem with $X_L(\sigma) = (h(\sigma) + 1)/2$ and $X_U(\sigma) = (h(\sigma) + 3)/2$ to lower and upper bound
Hiring above the median

\( n \in \{1000, \ldots, 10000\} \), \( M = 100 \) random permutations for each \( n \)

In red: lower bound (using \( X_L \)); in green: upper bound (using \( X_U \)); in yellow: simulation
Other quantities, e.g. time of the last hiring, etc. can also be analyzed using techniques from analytic combinatorics.

We have also analyzed hiring above the $P\%$ best candidate with the same machinery, actually we have explicit solutions for $H(z, u)$.

We have extensions of these results to cope with randomized hiring strategies.

Many variants of the problem are interesting and natural; for instance, include firing policies.
Thanks for your attention!