Interval Sorting

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Dedicated to Brigitte Vallée

Joint work with:

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Introduction

The problem:

Input: An array $A[1..n]$ of n items drawn from a totally ordered domain; a set $I = \{[\ell_t, u_t] \mid 1 \leqslant t \leqslant p\}$ of p disjoint intervals with

$$1 \leqslant \ell_1 \leqslant u_1 < \ell_2 \leqslant u_2 < \cdots < \ell_p \leqslant u_p \leqslant n,$$

Output: The array A rearranged in such a way that

1. $A[\ell_t..u_t]$ contains the ℓ_tth,...,u_tth smallest elements of A in nondecreasing order, for all t, $1 \leqslant t \leqslant p$
2. $A[u_t + 1..\ell_{t+1} - 1]$ contains the $(u_t + 1)$th, ..., $(\ell_{t+1} - 1)$th smallest elements of A, for all t, $0 \leqslant t \leqslant p$ ($u_0 = 0$, $\ell_{p+1} = n + 1$)

Introduction

Example

$p = 2$, $I_1 = [5, 8]$, $I_2 = [12, 12]$

3	11	5	7	8	4	9	1	13	10	12	14	15	2	6

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3	1	4	2	5	6	7	8	9	11	10	12	15	13	14

⟵ gap ⟶ ⟵ block ⟶ ⟵ gap ⟶ ...

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The main interest of interval sorting is that it generalizes several related fundamental problems:

- Sorting: $p = 1, I = \{[1, n]\}$
- Selection of the jth: $p = 1, I = \{[j, j]\}$
- Multiple selection: $I = \{[j_1, j_1], [j_2, j_2], \ldots, [j_p, j_p]\}$
- Partial sorting: $p = 1, I = \{[1, m]\}, m < n$

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- Other instances of interval sorting might be useful:
  - Sort & filter: $p = 1, I = [\beta n, (1-\beta)n], \beta < 1/2$
  - Outliers: $p = 2, I = \{[1, k], [n-k+1, n]\}$
- Sorting A in (expected) time $\Theta(n \log n)$ solves the problem, but this is wasteful if $m = |I_1| + \ldots + |I_p| \ll n$

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What's ahead?

1. Chunksort: A simple divide & conquer algorithm for interval sorting
2. Average performance of chunksort
3. A simple lower bound for interval sorting
4. Intermezzo:
5. "Optimal" chunksort
6. Disgression: How far from optimal?

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Chunksort

procedure CHUNKSORT(A, i, j, I, r, s)
  if $i \geqslant j$ **then return** ▷ A contains one or no elements
  if $r \leqslant s$ **then**
    $pv \leftarrow$ SELECTPIVOT(A, i, j)
    PARTITION(A, pv, i, j, k)
    $t \leftarrow$ LOCATE(I, r, s, k)
▷ Locate the value t such that $\ell_t \leqslant k \leqslant u_t$ with $I_t = [\ell_t, u_t]$,
▷ or $u_t < k < \ell_{t+1}$
    if $u_t < k$ **then** ▷ k falls in the tth gap
      CHUNKSORT$(A, i, k-1, I, r, t)$
      CHUNKSORT$(A, k+1, j, I, t+1, s)$
    else ▷ k falls in the tth interval
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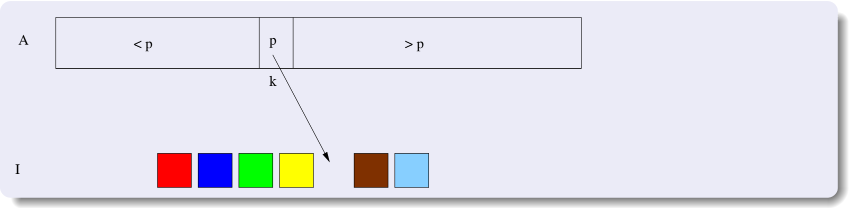
Chunksort: An example



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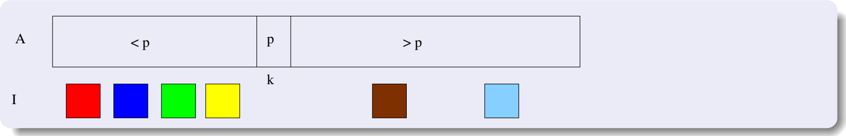

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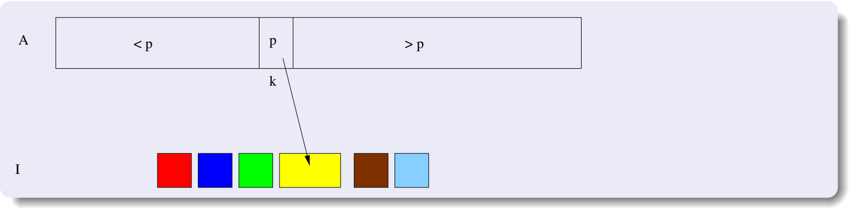

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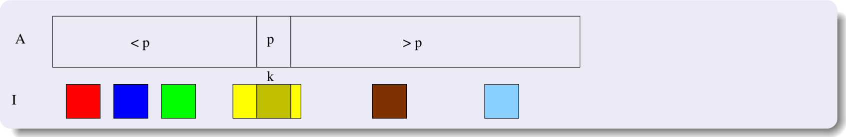

Chunksort

Example (Using chunksort to sort)

- $p = 1$, $I_1 = [1, n]$
- $1 \leqslant k \leqslant n \implies \ell_1 \leqslant k \leqslant u_1 \implies r = s = t = 1$

```
procedure CHUNKSORT(A, i, j, I, r, s)
    ...
    if u_t < k then ▷ k falls in the tth gap
        CHUNKSORT(A, i, k − 1, I, r, t)
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    else ▷ k falls in the tth interval
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Example (Using chunksort for selection)

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- $m < k \implies t = 1, u_1 < k$

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- $p = 1$, $I_1 = [m, m]$
- $k < m \implies t = 0, u_0 < k < \ell_1$

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Example (Using chunksort for partial sorting)

- $p = 1,\ I_1 = [1, m]$
- $1 \leqslant k \leqslant m \implies \ell_1 \leqslant k \leqslant u_1 \implies r = s = t = 1, k \leqslant u_1$

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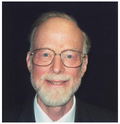
Quicksort: Average cost



C.A.R. Hoare

- Probability that the selected pivot is the k-th of n elements: $\pi_{n,k}$; for the basic variants here $\pi_{n,k} = 1/n$
- Average number of comparisons Q_n to sort n elements:

$$Q_n = n - 1 + \sum_{k=1}^{n} \pi_{n,k} \cdot (Q_{k-1} + Q_{n-k})$$

- Average number of comparisons Q_n to sort n elements (Hoare, 1962):

  $$Q_n = 2(n+1)H_n - 4n = 2n \ln n + (2\gamma - 4)n + 2 \ln n + O(1)$$

  where $H_n = \sum_{1 \leqslant k \leqslant n} 1/k = \ln n + O(1)$ is the n-th harmonic number.

Quickselect: Average cost

D.E. Knuth

- Average number of comparisons $C_{n,m}$ to select the m-th out of n:

$$C_{n,m} = n - 1 + \sum_{k=m+1}^{n} \pi_{n,k} \cdot C_{k-1,m} + \sum_{k=1}^{m-1} \pi_{n,k} \cdot C_{n-k,m-k}$$

- Average number of comparisons $C_{n,m}$ to select the m-th out of n elements (Knuth, 1971):

$$C_{n,m} = 2\Big(n + 3 + (n+1)H_n - (n+3-m)H_{n+1-m} - (m+2)H_m\Big)$$

Partial quicksort: Average cost

- Average number of comparisons $P_{n,m}$ to sort the m smallest elements out of n:

$$P_{n,m} = n - 1 + \sum_{k=m+1}^{n} \pi_{n,k} \cdot P_{k-1,m} + \sum_{k=1}^{m} \pi_{n,k} \cdot (P_{k-1,k-1} + P_{n-k,m-k})$$

- The solution is (Martínez, 2004):

$$P_{n,m} = 2n + 2(n+1)H_n - 2(n+3-m)H_{n+1-m} - 6m + 6$$

A Bit of Notation

- $I_t = [\ell_t, u_t]$: the tth interval, 1 $\leqslant t \leqslant p$
- $\bar{I}_t = [u_t + 1..\ell_{t+1} - 1]$: the tth gap, 0 $\leqslant t \leqslant p$
- $m_t = |I_t| = u_t - \ell_t + 1$: size of the tth interval
- $\overline{m}_t = |\bar{I}_t| = \ell_{t+1} - u_t - 1$: size of the tth gap
- $m = m_1 + \ldots + m_p$: # of elements to be sorted
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Chunksort: The recurrence

- We only count element comparisons
- Each partitioning stage needs $n - 1$ comparisons of the pivot with all the other elements
- We assume that pivots are chosen at random ($\pi_{n,k} = 1/n$)
- $C_{n;\{I_r,\ldots,I_s\}}$ = the average number of comparisons needed to do interval sort on n elements for the given set of intervals $\{I_r, \ldots, I_s\}$

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$$C_{n;\{I_r,\dots,I_s\}} = n-1 + \sum_{t=r-1}^{s} \sum_{k\in\overline{I}_t} \pi_{n,k}\big(C_{k-1;\{I_r,\dots,I_t\}} + C_{n-k;\{I_{t+1},\dots,I_s\}}\big)$$

$$+ \sum_{t=r}^{s} \sum_{k\in I_t} \pi_{n,k}\big(C_{k-1;\{I_r,\dots,I_t\}} + C_{n-k;\{I_t,\dots,I_s\}}\big),$$

How to solve the recurrence . . .

- We can solve this problem "iteratively", using generating functions
- First we have $p = 1$ and $I_1 = [i, j]$ and we translate the recurrence for $C_{n;\{[i,j]\}}$ into a functional equation for

$$C(z; x, y) = \sum_{n \geqslant 0} \sum_{1 \leqslant i \leqslant j \leqslant n} C_{n;\{[i,j]\}} z^n x^i y^j,$$

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How to solve the recurrence ...

- Then you can do a similar thing for $p = 2$, by introducing

$$C(z; x_1, y_1, x_2, y_2) = \sum_{n \geqslant 0} \sum_{1 \leqslant i \leqslant j \leqslant i' \leqslant j' \leqslant n} C_{n;\{[i,j],[i',j']\}} z^n \, x_1^i y_1^j x_2^{i'} y_2^{j'}$$

  which satisfies a similar ODE involving $C(z; x_r, y_r)$
- A pattern emerges here, so that one can obtain a general form for the ODE satisfied by $C(z; x_1, y_1, \ldots, x_p, y_p)$
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How to solve the recurrence . . .

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$$C(z; x_1, y_1, x_2, y_2) = \sum_{n \geqslant 0} \sum_{1 \leqslant i \leqslant j \leqslant i' \leqslant j' \leqslant n} C_{n;\{[i,j],[i',j']\}} z^n \, x_1^i y_1^j x_2^{i'} y_2^{j'}$$

  which satisfies a similar ODE involving $C(z; x_r, y_r)$
- A pattern emerges here, so that one can obtain a general form for the ODE satisfied by $C(z; x_1, y_1, \ldots, x_p, y_p)$
- Solve and extract the coefficients

. . . but how we actually did solve it

We guessed the solution from the known solutions for quicksort, quickselect, partial quicksort and multiple quickselect, some trial-and-error, and finally proved it by induction. . .

Chunksort: Average cost

Theorem

The average number of element comparisons $C_n := C_{n;\{I_1,\dots,I_p\}}$ *needed by chunksort given the intervals* $\{I_1, \dots, I_p\}$ *is*

$$\begin{aligned} C_n &= 2n + u_p - \ell_1 + 2(n+1)H_n - 7m - 2 + 15p \\ &\quad - 2(\ell_1 + 2)H_{\ell_1} - 2(n + 3 - u_p)H_{n+1-u_p} \\ &\quad - 2\sum_{k=1}^{p-1} (\overline{m}_k + 5)H_{\overline{m}_k+2}, \end{aligned}$$

1 Chunksort: A simple divide & conquer algorithm for interval sorting

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A simple lower bound for interval sorting

- $\Lambda(n, \mathbf{m}, \overline{\mathbf{m}})$ = minimum # of comparisons needed on average to solve interval sorting of intervals with sizes $\mathbf{m} = (m_1, \ldots, m_p)$ and gaps $\overline{\mathbf{m}} = (\overline{m}_0, \ldots, \overline{m}_p)$
- The two vectors $\mathbf{m}$, $\overline{\mathbf{m}}$ and the value n univocally determining the interval sorting instance
- Suppose we perform an optimal interval sort of the array of n elements, then we sort optimally the gaps; hence

$$\Lambda(n, \mathbf{m}, \overline{\mathbf{m}}) + \sum_{t=0}^{p} \log_2(\overline{m}_t!) \geqslant \log_2(n!)$$

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A simple lower bound for interval sorting

Lemma

$$\Lambda(n, \mathbf{m}, \overline{\mathbf{m}}) \geqslant \sum_{t=1}^{p} m_t \log_2 m_t + n\mathcal{H}\left(\{\overline{m}_0/n, m_1/n, \overline{m}_1/n, \ldots, m_p/n, \overline{m}_p/n\}\right) - m \log_2 e + o(n)$$

with $\mathcal{H}(\{q_t\}) = -\sum_t q_t \log_2 q_t$ *denoting the entropy of the discrete probability distribution* $\{q_t\}$ *and* $m = m_1 + \ldots + m_p$.

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Optimal quicksort

M. H. van Emden

- Using the median of a small sample as the pivot of each recursive call of quicksort improves the average cost of quicksort (Singleton's median-of-3, 1969)
- Van Emden (1970) and Hennequin (1989) have studied the performance of quicksort with median-of-$(2t+1)$ showing an steady improvement of performance

$$C_n^{(t)} = c_t n \log_2 n, \qquad c_0 = 2\ln 2 = 1.386, c_1 = 1.188, \ldots, c_\infty = 1$$

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- McGeoch and Tygar (1995) considered using the median of a variable-size sample for the first round, then fixed size samples on subsequent calls
- Martínez and Roura (2001) studied the use of variable-size sampling for quicksort and quickselect, showing that optimal expected performance can be achieved

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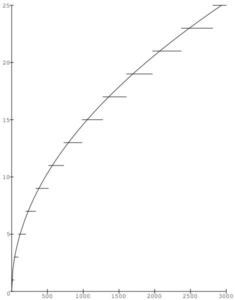
Optimal quicksort

Theorem (Martínez, Roura, 2001)

The expected performance of quicksort using as pivots the median of samples of size $s = s(n)$, such that $s \to \infty$ and $s/n \to 0$ as $n \to \infty$ is

$$n \log_2 n + \textit{lower order terms}$$

Optimal quicksort



- The lower order terms are minimized by choosing samples of size $\Theta(\sqrt{n})$
- The constant hidden in $\Theta(\sqrt{n})$ depends on the (linear) time algorithm used to find the median of the samples

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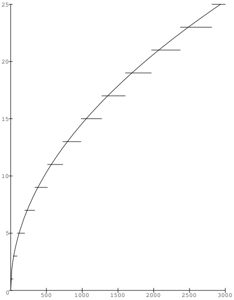

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Optimal quickselect

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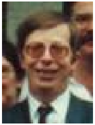

P. Kirschenhofer

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- Median-of-$(2t + 1)$ sampling can also be used for quickselect
- The improvements on the performance have been studied by several authors: Kirschenhofer, Prodinger, Martínez (1997), Grübel (1999), Martínez and Roura (2001)
- But ... is the median of the sample a good choice?

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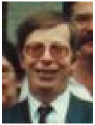

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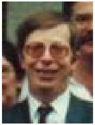

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Optimal quickselect

D. N. Panario

A. T. Viola

- In 2004, Martínez, Panario and Viola consider variants of quickselect where the rank r of the pivot within the sample of size s is proportional to the rank j of the sought element in the array n:

$$r \approx \frac{j}{n} \cdot s$$

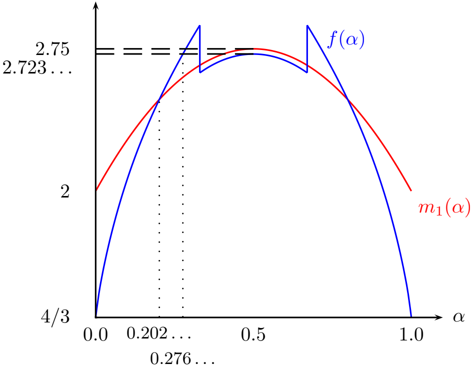
- More in general, they consider all variants where r is a function of $\alpha = j/n$

Optimal quickselect

- For all variants

$$C_{n,j} = f(\alpha) \cdot n + o(n), \alpha = j/n,$$

for instance, $f(\alpha) = m_0(\alpha) = 2 + 2\mathcal{H}(\alpha)$ for standard quickselect and $f(\alpha) = m_1(\alpha) = 2 + 3\alpha(1 - \alpha)$ for median-of-three



Optimal quickselect

- Optimal expected performance can be achieve with 3 basic "ingredients:"
  - Using variable-sample sizes $s = s(n)$ with $s \to \infty$, $s/n \to 0$
  - The rank of the pivot withis the sample must be $r \sim \alpha \cdot s$
  - If the souhgt element has rank $j > n/2$ take $r = \alpha \cdot s - \delta$; if $j < n/2$ then $r = \alpha \cdot s + \delta$, for some "small" δ, say $\delta = \sqrt{s}$
  - You want the chosen pivot to land very close to j on the correct side with high probability

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Optimal quickselect

Theorem (Martínez, Panario, Viola, 2004)

Any variant of quickselect using biased proportion-from-s with variable-size sampling has

$$f(\alpha) = 1 + \min(\alpha, 1 - \alpha)$$

Thus $C_{n,j} \sim n + \min(j, n - j) +$ *lower order terms*

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Optimal chunksort

The recipy for optimality:

1. Merge small gaps: replace two intervals separated by a gap of size $o(n)$ by a single interval
2. If there is only one interval to sort and it contains $m = n - o(n)$ elements pick a pivot whose rank is close to $n/2$; use the median of a large ($\sqrt{n}$) sample
3. If not, choose some endpoint $\ell_r, u_r, \ldots, \ell_s, u_s$, say ρ
   - If $\rho = \ell_t$, pick a pivot from a large sample with rank proportional to ρ and biased to land to the left of ρ
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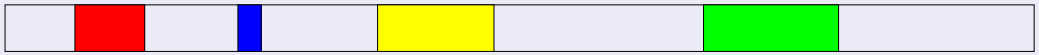

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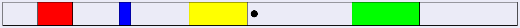

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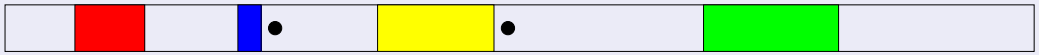

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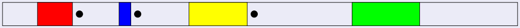

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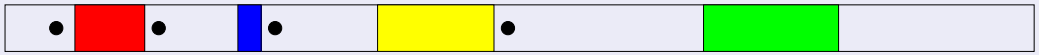

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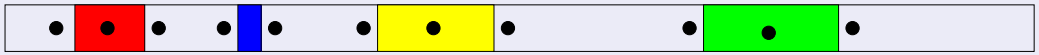

Optimal chunksort

- The problem is thus to find the optimal order $\Longrightarrow$ dynamic programming
- Given the collection of endpoints $\rho_i = u_{r-1}$, $\rho_{i+1} = \ell_r, \ldots,$ $\rho_{j-1} = u_s$, $\rho_j = \ell_{s+1}$ find the endpoint ρ_k such that minimizes $c(i,j)$:

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Optimal chunksort

F.F. Yao

- The dynamic programming to find the optimal order to “cut the bar” has cost $O(p^3)$; it is almost analogous to building an optimal search tree where the weights of the leaves are the sizes of the intervals
- The efficiency of the algorithm can be greatly improved to $O(p^2)$ using Knuth-Yao’s technique

Optimal chunksort

- We can use some heuristic to find a near-optimal solution to the “cut the bar” problem with cost $O(p \log p)$
- For instance, at each step, we can choose the endpoint ℓ_k or u_k which is closer to $(\rho_j - \rho_i)/2$; some care must be taken if we have ties, e.g., if $\ell_k = u_k$
- The analysis of the heuristic provides a useful upper bound on $c(0, 2p + 1)$, the optimal cost of the “cut the bar” phase
- The total cost of chunksort becomes

$$\sum_{t=1}^{p} m_t \log_2 m_t + c(0, 2p+1) + O(p\sqrt{n})$$

$$\leqslant \sum_{t=1}^{p} m_t \log_2 m_t + n \cdot H + n + \text{lower order terms}$$

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Optimal chunksort

- Together with the lower bound for Λ

$$\sum_{t=1}^{p} m_t \log_2 m_t + n \cdot H - m \log_2 e + o(n) \leqslant \Lambda(n, \mathbf{m}, \overline{\mathbf{m}})$$
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- The lower and upper bounds differ by $n + o(n)$ comparisons if $p \ll \sqrt{n}$ (which indeed is the case, as we collapsed all “small” gaps!)

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Optimal chunksort


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P. Sanders

- Kaligosi, Mehlhorn, Munro and Sanders (2005) have considered optimal multiple selection; they use similar techniques, but they propose an algorithm which picks a pivot close to the median for each recursive stage
- This yields a solution (for multiple selection) which is off by $O(n)$ comparisons from the optimal; our solution —which generalizes multiple selection— is off by at most $n + o(n)$ comparisons

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How far from optimal

1. The lower bound for $\Lambda(n, \mathbf{m}, \overline{\mathbf{m}})$ is not tight, for instance, for selection

$$\Lambda(n, \langle 1\rangle, \langle j-1, n-j\rangle) = n + \min(j-1, n-j) + \text{l.o.t.} \quad \leftarrow \text{on avg!}$$
$$\gg n\mathcal{H}\left(\{(j-1)/n, 1/n, (n-j)/n\}\right) + \text{l.o.t.}$$

2. The upper bound corresponds to the heuristic for “cutting the bar”, and isn’t tight either

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1. The lower bound for $\Lambda(n, \mathbf{m}, \overline{\mathbf{m}})$ is not tight, for instance, for selection

$$\Lambda(n, \langle 1 \rangle, \langle j{-}1, n{-}j \rangle) = n + \min(j{-}1, n{-}j) + \text{l.o.t.} \quad \leftarrow \text{on avg!}$$
$$\gg n\mathcal{H}\left(\{(j-1)/n, 1/n, (n-j)/n\}\right) + \text{l.o.t.}$$

2. The upper bound corresponds to the heuristic for "cutting the bar", and isn't tight either

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1. Chunksort: A simple divide & conquer algorithm for interval sorting
2. Average cost of chunksort
3. A simple lower bound for interval sorting
4. Intermezzo
5. "Optimal" chunksort
6. Disgression: How far from optimal?
7. Conclusions

Conclusions

- Interval sort's main interest is that it smoothly generalizes several fundamental problems: sorting, selection, multiple selection and partial sorting
- Chunksort (its basic variant) is a simple and elegant algorithm in the spirit of quicksort; its average performance is $\leqslant 2 + 2\ln 2 = 3.386$ times the optimal

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Merci beaucoup!