Chunksort: A Generalized Partial Sort Algorithm

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Introduction

The algorithm

The analysis

Conclusions
Generalized partial sorting

The problem:
- The input: An array $A$ of $n$ elements and $p$ intervals
  \[ I_1 = [l_1, u_1], I_2 = [l_2, u_2], \ldots, I_p = [l_p, u_p] \]
Generalized partial sorting

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Generalized partial sorting

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Example

$p = 2, I_1 = [5, 8], I_2 = [12, 12]$

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Generalized partial sorting

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\[
\begin{array}{cccccccccccc}
3 & 1 & 4 & 2 & 5 & 6 & 7 & 8 & 9 & 11 & 10 & 12 & 15 & 13 & 14 \\
\end{array}
\]

\leftarrow \text{gap} \rightarrow \quad \leftarrow \text{block} \rightarrow \quad \leftarrow \text{gap} \rightarrow \quad \ldots

Chunksort: A Generalized Partial Sort Algorithm
Generalized partial sorting

- Sorting the array solves the problem, but it might do much more work than actually needed, in particular, if \( m = |I_1| + |I_2| + \ldots + |I_p| = o(n) \)
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  - Multiple selection: \( I_1 = [j_1, j_1], \ldots, I_p = [j_p, j_p] \)
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  - Multiple selection: \( I_1 = [j_1, j_1], \ldots, I_p = [j_p, j_p] \)
  - Partial sorting: \( p = 1, I_1 = [1, m] \)
QuickSort and Relatives

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- They are simple, elegant, beautiful and practical divide-and-conquer solutions to sorting and selection
- Multiple quickselect uses the divide-and-conquer principle twice to solve the multiple selection problem (Prodinger, 1995)
- Partial quicksort is a slight variation of quicksort that efficiently solves the partial sorting problem (Martínez, 2004)
void quicksort(vector<Elem>& A, int i, int j) {
    if (i < j) {
        int p = select_pivot(A, i, j);
        swap(A[p], A[1]);
        int k;
        partition(A, i, j, k);
        quicksort(A, i, k - 1);
        quicksort(A, k + 1, j);
    }
}
Quickselect

Elem quickselect(vector&lt;Elem&gt;& A,
                   int i, int j, int m) {
    if (i >= j) return A[i];
    int p = select_pivot(A, i, j, m);
    swap(A[p], A[l]);
    int k;
    partition(A, i, j, k);
    if (m < k) quickselect(A, i, k - 1, m);
    else if (m > k) quickselect(A, k + 1, j, m);
    else return A[k];
}
Quicksort: The recurrence for average cost

- Probability that the selected pivot is the $k$-th of $n$ elements: $\pi_{n,k}$
- Average number of comparisons $Q_n$ to sort $n$ elements:

$$Q_n = n - 1 + \sum_{k=1}^{n} \pi_{n,k} \cdot (Q_{k-1} + Q_{n-k})$$
Quicksort: The average cost

- For the standard variant, the splitting probabilities are $\pi_{n,k} = 1/n$
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- For the standard variant, the splitting probabilities are $\pi_{n,k} = 1/n$
- Average number of comparisons $Q_n$ to sort $n$ elements (Hoare, 1962):

$$Q_n = 2(n + 1)H_n - 4n$$

$$= 2n \ln n + (2\gamma - 4)n + 2 \ln n + O(1)$$

where $H_n = \sum_{1 \leq k \leq n} 1/k = \ln n + O(1)$ is the $n$-th harmonic number.
Quickselect: The recurrence for average cost

Average number of comparisons $C_{n,j}$ to select the $j$-th out of $n$:

$$C_{n,j} = n - 1 + \sum_{k=j+1}^{n} \pi_{n,k} \cdot C_{k-1,m} + \sum_{k=1}^{j-1} \pi_{n,k} \cdot C_{n-k,m-k}$$
Quickselect: The average cost

- Average number of comparisons $C_{n,j}$ to select the $j$-th out of $n$ elements (Knuth, 1971):

$$C_{n,j} = 2(n + 3 + (n + 1)H_n - (n + 3 - j)H_{n+1-j} - (j + 2)H_j).$$
Quickselect: The average cost

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$$C_{n,j} = 2(n + 3 + (n + 1)H_n - (n + 3 - j)H_{n+1-j} - (j + 2)H_j).$$

- This is $\Theta(n)$ for any $j$, $1 \leq j \leq n$. 
Partial quicksort

void partial_quicksort(vector<Elem>& A, int i, int j, int m) {
    if (i < j) {
        int p = get_pivot(A, i, j);
        swap(A[p], A[1]);
        int k;
        partition(A, i, j, k);
        partial_quicksort(A, i, k - 1, m);
        if (k < m - 1)
            partial_quicksort(A, k + 1, j, m);
    }
}
Partial quicksort: The average cost

- Average number of comparisons $P_{n,m}$ to sort the $m$ smallest elements out of $n$:

$$P_{n,m} = n - 1 + \sum_{k=m+1}^{n} \pi_{n,k} \cdot P_{k-1,m}$$

$$+ \sum_{k=1}^{m} \pi_{n,k} \cdot (P_{k-1,k-1} + P_{n-k,m-k})$$
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- The solution is (Martínez, 2004):

$$P_{n,m} = 2n + 2(n + 1)H_n - 2(n + 3 - m)H_{n+1-m} - 6m + 6$$
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Chunksort: An example

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Chunksort: A Generalized Partial Sort Algorithm
Chunksort

void chunksort(vector<T>& A, vector<int>& I, int i, int j, int l, int u) {
    if (i >= j) return;
    if (l <= u) {
        int k; partition(A, i, j, k);
        int r = locate(I, l, u, k);
        // locate the value r such that I[r] ≤ k < I[r+1]
        if (r % 2 == 0) {
            // r = 2t
            chunksort(A, I, i, k - 1, l, r);
            chunksort(A, I, k + 1, j, r + 1, u);
        } else {
            // r = 2t-1
            chunksort(A, I, i, k - 1, l, r + 1);
            chunksort(A, I, k + 1, j, r, u);
        }
    }
}}

Chunksort: A Generalized Partial Sort Algorithm
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Example (Using chunksort for partial sorting)

If \( p = 1 \), \( I_1 = [1, m] \) then \( r = 1 \) whenever \( k < m \); hence we make the calls

- \( \text{chunksort}(A, I, i, k - 1, 1, 2); \)
- \( \text{chunksort}(A, I, k + 1, j, 1, 2); \)

If \( k \geq m \) then \( r = 2 \) and then we make the calls

- \( \text{chunksort}(A, I, i, k - 1, 1, 2); \)
- \( \text{chunksort}(A, I, k+1, j, 3, 2); \)
Chunksort

Example (Using chunksort for selection)

If $p = 1$, $I_1 = [j, j]$ then we will have $r = 0$ whenever $k < j$ so we call

chunksort(A, I, i, k - 1, 1, 0);
chunksort(A, I, k + 1, j, 1, 2);

If $k \geq j$ then $r = 2$ and then we make the calls

chunksort(A, I, i, k - 1, 1, 2);
chunksort(A, I, k + 1, j, 3, 2);
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2 The algorithm

3 The analysis

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Chunksort: A Generalized Partial Sort Algorithm
The recurrence

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- We only count element comparisons
- Each partitioning stage needs \( n - 1 \) comparisons of the pivot with all the other elements
- We assume that pivots are chosen at random \((\pi_{n,k} = 1/n)\)
- \( C_{i,j} (\{I_r, \ldots, I_s\}) = \) the average number of comparisons needed to process the subarray \( A[i..j] \)
  for the given set of intervals \( \{I_r, \ldots, I_s\} \), with \( i \leq l_r \) and \( u_s \leq j \)
The recurrence

\[
C_{i,j}(\{I_r, \ldots, I_s\}) = n - 1
\]

\[
+ \sum_{t=r}^{s-1} \left[ \sum_{l_t \leq k < u_t} \pi_{n,k}(C_{i,k-1}(\{I_r, \ldots, I'_t\}) + C_{k+1,j}(\{I'_t, \ldots, I_s\}) + \right.
\]

\[
+ \sum_{u_t \leq k < l_{t+1}} \pi_{n,k}(C_{i,k-1}(\{I_r, \ldots, I_t\}) + C_{k+1,j}(\{I_t+1, \ldots, I_s\})) \right]
\]

\[
+ \sum_{i \leq k < l_r} \pi_{n,k}C_{k+1,j}(\{I_r, \ldots, I_s\})
\]

\[
+ \sum_{l_s \leq k \leq j} \pi_{n,k}C_{i,k-1}(\{I_r, \ldots, I_s\})
\]

\]

with \( I'_t = [l_t, k - 1] \) and \( I''_t = [k + 1, u_t] \).
How to solve the recurrence …

- We can solve this problem "iteratively", using generating functions
How to solve the recurrence ...

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- First we have $p = 1$ and $I_1 = [a, b]$ and we translate the recurrence for $C_{i,j}([a, b])$ into a functional equation for

$$C_{[a,b]}(u, v) = \sum_{1 \leq i \leq j} C_{i,j}([a, b]) u^i v^j,$$

which is actually a first-order linear differential equation.
How to solve the recurrence ...

Then you can do a similar thing for \( p = 2 \), by introducing

\[
C_{[a,b],[c,d]}(u,v) = \sum_{1 \leq i \leq j} C_{i,j}([a, b], [c, d]) u^i v^j,
\]

which satisfies a similar ODE but the independent term now involves \( C_{[a,b]}(u,v) \) and \( C_{[c,d]}(u,v) \).
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- A pattern emerges here, so that one can obtain a general form for the functional equation satisfied by \( C_{\{I_1, \ldots, I_p\}}(u, v) \).
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- Solve and extract \( [u^i v^j] C_{\cdots}(u, v) \)
...But how I actually did solve it

I guessed the solution from the known solutions to the algorithms which chunksort generalizes and I proved it by induction...
Theorem

The average number of element comparisons $C_n({I_1, \ldots, I_p}) \equiv C_{1,n}({I_1, \ldots, I_p})$ needed by chunksort given the intervals $\{I_1, \ldots, I_p\}$ is

$$C_n = 2n + u_p - \ell_1 + 2(n + 1)H_n - 7m - 2 + 15p - 2(\ell_1 + 2)H_{\ell_1} - 2(n + 3 - u_p)H_{n+1-u_p} \quad \text{for} \quad p-1$$

$$- 2 \sum_{k=1}^{p-1} (\overline{m}_k + 5)H_{\overline{m}_k+2},$$

where

- $\overline{m}_k = |\overline{I}_k| = \ell_{k+1} - u_k - 1$
- $m_k = |I_k| = u_k - \ell_k + 1$
- $m = m_1 + m_2 + \cdots + m_p$
Chunksort vs. quicksort+quickselect

- For small $p$ ($p = 1, 2$) it is perfectly reasonable to solve the problem using quickselect to find the beginning and end of each block, and then sort each block using quicksort.

- The order of magnitude of the average cost of chunksort and this alternative is similar; but there are significant differences for the second order terms.

- For example, if $p = 1$ and $I_1 = [\alpha \cdot n - f(n), \alpha \cdot n + f(n)]$ for some $\alpha < 1/2$ and $f(n) = o(n)$ then chunksort makes $2(1 - \alpha)n$ comparisons less.
Conclusions

- The formula for the average cost of chunksort generalizes the corresponding formulas for special cases: quicksort, quickselect, partial quicksort, multiple quickselect, …
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- The formula for the average cost of chunksort generalizes the corresponding formulas for special cases: quicksort, quickselect, partial quicksort, multiple quickselect, ...

- Despite being simple and "efficient", chunksort should not be used as a substitute for the specialized algorithms (maybe it could be used for the less frequent tasks of multiple selection or partial sorting)
Conclusions

- It is interesting to analyze the cost of the algorithm when taking into account the cost $\Theta(\log p)$ of locating the pivot’s position in the array of intervals.
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- I would like to know about possible applications for chunksort; e.g., partial quicksort has been used to improve significantly the practical performance of Kruskal’s algorithm for minimum spanning trees.
Thank you for your attention