Universitat Politècnica de Catalunya
Facultat d'Informàtica de Barcelona

Degree: Grau en Enginyeria Informàtica **Academic year:** 2023–2024 (Mid-term Exam)
Course: Randomized Algorithms (RA-MIRI) **Date:** October 27th, 2023
Time: 2h

1. **(2.5 points)** When the Dark Knight (not Batman) swiftly crossed the fields of Camelot on his horse, King Arthur, Lancelot, and Parsifal each shot an arrow at him. Arthur is a good archer and hits 3 out of 4 shots. Lancelot is mediocre and hits 1 out of 2 shots. Parsifal is the worst of the three and only hits 1 out of 4 times.

   (a) What is the probability that the Dark Knight received at least one arrow?

   (b) What is the average number of hits that he will receive?

---

2. **(2.5 points)** "Last Sunday, my wife and I went to a dinner with three other couples in a nice restaurant. We sat at random in the big round table that was assigned to us. Curiously enough, every one sat next (to the left or to the right) to his/her spouse".

   (a) How (un)likely was that to happen? Choose the correct option (i)–(iv) and explain.

   $$(i)\frac{1}{8} \qquad (ii)\frac{2}{105} \qquad (iii)\frac{1}{2520} \qquad (iv)\frac{1}{1260}$$

   (b) Give a formula for that probability as a function of the number n of couples.

   N.B. If you have produced an answer for the second question (b), you can use the formula to justify your answer to the first question (a).

---

3. **(2.5 points)** Consider a set $A \subseteq \{1, \ldots, n\}$ with size n_A which is a fraction of n. Therefore, drawing an integer u.a.r. from the range $[1..n]$ will produce an element $x \in A$ with probability $p_A = n_A/n > 0$. Our goal is to obtain a sample of k_A distinct elements of A, with $k_A \ll n_A$. Any subset of the $\binom{n_A}{k_A}$ subsets of A of size k_A must be generated with identical probability. You must design an algorithm that achieves this goal. Since $k_A \ll n_A$, the expected cost of the algorithm should be sublinear (for example, somewhere between $\Omega(k_A)$ —can't be less— and $\mathcal{O}(k_A \log n)$). Your algorithm can check if a given $x \in [1, \ldots, n]$ belongs to A or not in constant time. Hint: use the rejection method.

(a) How many integers in the range $[1, \ldots, n]$ must we draw on average to produce an element that belongs to A? The integers are independently drawn u.a.r. from $[1, \ldots, n]$.

(b) Suppose you know j elements of A, with $j < n_A$. How many integers in the range $[1, \ldots, n]$ must be drawn, on average, before we produce an element in A different from the j already known?

(c) Compute the expected cost (=expected number of random numbers generated in total) of your algorithm as a function of k_A, n_A and n.

Useful formulas:

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \sim \ln n + \gamma + \mathcal{O}(1/n), \qquad \gamma = 0.57721566\ldots \tag{1}$$

$$\ln(1 - x) \sim -x, \qquad \text{if } x \to 0 \tag{2}$$

---

4. **(2.5 points)** Suppose we have an unlimited amount of fair coins at our disposal. If we flip n of them, we expect $n/2$ to land heads and the other $n/2$ to land tails. We have been given the following (pseudo) code:

```
// 0 <= p <= 1
 int gen_number(double p) {
     int round = 1;
     while (rnd() > p) // rnd() returns a random real in [0,1]
         ++round;
     return round;
 }
 int flip_coins(double p) {
     int N = gen_number(p);
     flip N coins
     return number of heads in the N flips
 }
```

(a) Consider $X_1, X_2, \ldots$ independent and identically distributed random variables such that $\mathbb{E}[X_i] = \mu$ and another positive integer-value random variable $T \geq 0$. Let

$$Z = \sum_{1 \leq i \leq T} X_i$$

Prove that $\mathbb{E}[Z] = \mathbb{E}[T] \cdot \mu$.

(b) Prove that the expected number of heads returned by the call `flip_coins(p)` is $1/(2p)$.