# Random Variables and Expectation

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RA-MIRI 2023–2024

# Random variables

Flip 100 times a fair coin, each time if the outcome is $\mathrm{H}$ we give 1€, if it is $\mathrm{T}$ we get 1€. At the end, how much did we win or loose?. Notice $\Omega = \{\mathrm{T}, \mathrm{H}\}^{100}$

Given $\Omega$, a random variable is a function $X : \Omega \to \mathbb{R}$.
$X$ can be interpreted as a quantity, whose value depends on the outcome of the experiment.

**Example**

In the previous example, our total gain (or loss) is a random variable $X$,

$$X = \text{number of H's minus the number of T's.}$$

The number of heads $W$ and the number of tails $L$ are also random variables (and $X = W - L$).

# Events and random variables

Given a random variable $X$ on $\Omega$ and $a \in \mathbb{R}$ the event $X \geqslant a$ represents the set $\{\omega \in \Omega | X(\omega) \geqslant a\}$.

$$\mathbb{P}[X \geqslant a] = \sum_{\omega \in \Omega : X(\omega) \geqslant a} \mathbb{P}[\omega]$$

**Example**

In the previous example of 100 coin flips, for the event $W = 50$ we have $\mathbb{P}[W = 50] = \frac{\binom{100}{50}}{2^{100}}$ (*)

Given an event $A$ define the indicator r.v. $\mathbb{I}_A$:

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ true} \\ 0 & \text{otherwise} \end{cases}$$

**Example**

If $A$ = exactly 50 wins, $\mathbb{P}[A] = \mathbb{P}[\mathbb{I}_A = 1] = \mathbb{P}[W = 50]$, which is exactly (*)

# Expectation

The expectation $\mathbb{E}[X]$ of a r.v. $X : \Omega \to \mathbb{R}$ is defined as

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot \mathbb{P}[X = x] \,.$$

Expectation (mean, average) is just the weighted sum over all values of the r.v.

Notice: If $X$ is a r.v. then $\mathbb{E}[X] \in \mathbb{R}$.

**Example**

Let $X$ be an integer generated u.a.r. between 1 and 6. Then $\mathbb{E}[X] = \sum_{x=1}^{6} x \cdot \mathbb{P}[X = x] = \sum_{x=1}^{6} \frac{x}{6} = 3.5$, which is not a possible value for $X$.

# Linearity of expectation

*Theorem*

1. *Given r.v.* $X, Y$, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
2. *Given any constant* $c$, *and a rv* $X$, *then* $\mathbb{E}[cX] = c\,\mathbb{E}[X]$.
3. *More generally, given r.v.* $\{X_i\}_{i=1}^n$ *and* $n$ *real numbers* $\{a_i\}_{i=1}^n$, $\mathbb{E}[\sum_{i=1}^n a_i X_i] = \sum_{i=1}^n a_i\,\mathbb{E}[X_i]$.

*The proof is standard and relies on the fact that the sum of r.v. is a r.v.*

# Independent r.v.

Two random variables $X$ and $Y$ are said to be independent if

$$\forall x, y \in \mathbb{R}, \mathbb{P}[(X = x) \cap (Y = y)] = \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y] \, .$$

Two r.v. which are not independent are said to be dependent or correlated.

**Example**

Rolling two dice, let $X_1$ be a r.v. counting the pips in die 1, and let $X_2$ be a r.v. counting the pips in die 2. Then $X_1$ and $X_2$ are independent rv.

**Example**

Rolling two dice, let $X_1$ be a r.v. counting the pips in die 1, and let $X_3$ count the sum of pips in the two rollings, then $X_1$ and $X_3$ are correlated.

# Inversions in Permutations

Given and array $A[1, \dots, n]$ containing $n$ different keys, chosen u.a.r. from one permutation of the set of $n$ keys, let $a_i$, $1 \leqslant i \leqslant n$, be the key contained in $A[i]$. We say $a_i$ and $a_j$ are inverted if $i < j$ but $a_i > a_j$. Compute the expected number of inversions in $A$.
Let $X$ count the number of inversions in $A$.
For every pair $1 \leqslant i < j \leqslant n$ of positions in $A$ define an indicator r.v.:

$$X_{i,j} = \begin{cases} 1 & \text{if } a_i > a_j \\ 0 & \text{otherwise} \end{cases}$$

$X = \sum_{i<j} X_{i,j} \Rightarrow \mathbb{E}[X] = \sum_{i<j} \mathbb{E}\left[X_{i,j}\right] = \sum_{i<j} 1 \cdot \underbrace{\mathbb{P}\left[a_i > a_j\right]}_{=1/2}$

Notice $|\{(i,j)|1 \leqslant i < j \leqslant n\}| = (n-1) + (n-2) + \cdots + 2 + 1$

therefore, $\mathbb{E}[X] = \frac{1}{2} \sum_{i=1}^{n} (n-i) = \frac{1}{2} \sum_{i=1}^{n-1} i = \frac{n(n-1)}{4}$

# Records in Permutations

We have $n$ students $\{1, \ldots, n\}$, we want to hire the best one to help us. The $i$-th interviewed student has score/rank $\sigma(i)$; each time we find one that is more suitable that the previous ones (a record), we preselect that candidate. At the end, we hire the last one pre-selected, but we indemnify with $S > 0$ € each of the pre-selected candiadtes who are not hired. How much will we be paying?

**procedure** HIRING($n$)<br>
    $best := 0$<br>
    **for** $i := 1$ **to** $n$ **do**<br>
        interview $i$-th candidate<br>
        **if** $\sigma(i)$ is better than $\sigma(best)$ **then**<br>
            $best := i$ and pre-select $i$<br>
        **end if**<br>
    **end for**<br>
**end procedure**

# Records in Permutations


- In the worst-case the list of students is given in increasing order of score, $\sigma(i) = i$, and we will be pre-selecting everyone $\implies$ we pay $S \cdot (n-1)$ €.
- In the best-case, the first candidate is the one with best rank, $\sigma(1) = n$, and the only one to be preselected. We have no indemnizations to pay.

# Average analysis of the hiring algorithm

There are $n!$ possible orders of the students; we assume any of them has identical probability $\frac{1}{n!}$.

*Lemma*

*The expected number of pre-selected candidates is* $H_n = \sum_{1 \leqslant i \leqslant n} 1/i = \ln n + \mathcal{O}(1)$.

Proof

Let $X$ be a r.v. counting the number of pre-selected students. For each $1 \leqslant i \leqslant n$ define an indicator r.v. $X_i = \mathbb{I}_{i\text{-th is preselected}}$. Then, $X = \sum_{i=1}^{n} X_i$ and

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} 1 \cdot \underbrace{\frac{1}{i}}_{\text{why?}} = \ln n + \mathcal{O}(1).$$

□

# Randomized algorithm for the hiring problem

To fool the input given by an adversary: Permute the input

```
procedure RAND-HIRE-STUDENT(n)
    Randomly permute the list [1, ..., n]
    best := 0
    for i := 1 to n do
        interview i-th candidate
        if i-th candidate is better than best then
            best := i and pre-select i-th candidate
        end if
    end for
end procedure
```

Let $X(n)$ a r.v. counting the number of pre-selections, on an input of $n$ students. Then $\mathbb{E}[X(n)] = \ln n + \mathcal{O}(1)$, with the expectation taken over our random choices (the initial permutation of the input) and not on any assumption on the probability of the possible inputs.