Modeling stationary data by a class of generalized Ornstein-Uhlenbeck processes.

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Abstract

We analyze in this work the effect of the iterated application of the linear operator that maps a Wiener process onto an Ornstein-Uhlenbeck process. The processes obtained after p iterations are called *Ornstein-Uhlenbeck processes of order p* (denoted OU(p)).

Technically our composition of operators is easy to manipulate and its parameters can be computed efficiently because, as we show, in most cases the result of the iteration is a linear combination of the same operators, and exceptionally it involves simple generalizations of them. This provides a straightforward computation of covariances.

We also give a state space model representation of OU(p) and from this setup show that the discrete process resulting from sampling the linear combination of Ornstein-Uhlenbeck processes, at equally spaced periods of time, is a parsimonious ARMA process. Experiments on real data show that the empirical autocorrelation for large lags can be fairly modeled with OU(p) processes with approximately half the number of parameters than ARMA processes.

Key words and phrases: *Ornstein-Uhlenbeck process, models for stationary processes, iterated processes, empirical covariances*

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1 Introduction

The Ornstein-Uhlenbeck process (from now on OU) was introduced by L. S. Ornstein and E. G. Uhlenbeck [Uhlenbeck and Ornstein, 1930] as a model for the velocities of a particle subject to the collisions with surrounding molecules. It improves Einstein's model (a Wiener process) because it also applies to fluids with finite viscosity, and since the 1950's is a well studied and accepted model for thermodynamics, chemical and other various stochastic processes found in physics and the natural sciences [Gardiner, 2004]. Moreover, the OU process is the unique non-trivial stochastic process that is stationary, Markovian and Gaussian [Breiman, 1992]. Additionally it is mean-reverting, and for all these properties it has found its way into financial engineering, first as a model for the term structure of interest rates in a form due to Vasicek [1977], and then under other variants or generalizations (e.g. where the underlying random noise is a Lévy process) as a model of financial time series with applications to option pricing, portfolio optimization and risk theory, among others [Nicolato and Venardos, 2003, Barndorff-Nielsen and Shephard, 2001a,b, Maller et al., 2009, and references therein].

The OU process can be thought of as continuous time interpolation of an autoregressive process of order one (i.e. an AR(1) process), a link that we shall make evident in Section 2. Beginning with this relation to the autoregressive model, one can seek to define and analyze the result of iterating the application of the operator that maps a Wiener process onto an OU process, in order to obtain a higher order OU process. This operator is defined in Section 3 and denoted $\mathcal{OU}$, with subscripts denoting the parameters involved. The p iterations of $\mathcal{OU}$, for each positive integer p, give rise to a new family of processes, the *Ornstein-Uhlenbeck processes of order* p, denoted OU(p), proposed as models for either stationary continuous time processes or the series obtained by observing these continuous processes at equally spaced instants. We show that these higher order OU process can be expressed as a linear combination of ordinary OU processes, or generalized OU processes as defined below by Eq. (9). This result resembles the aggregations of Gaussian (and non-Gaussian) processes so much studied in connection with deconstructing a complicated economic model into simpler constituents; long memory models; flexible models of dependence structures, and others [Granger and Morris, 1976, Granger, 1980, Barndorff-Nielsen and Shephard, 2001a].

A consequence of writing the OU(p) process as aggregation of simpler ones is the derivation of a closed formula for its covariance. This has important practical implications since it allows to easily estimate the parameters of a OU(p) process by maximum likelihood or, as an alternative, by matching correlations, the latter being a procedure resembling the method of moments.

A distinctive point of our $\mathcal{OU}$ operator from others contained in the vast literature on aggregations (or superpositions) of stochastic processes (see, e.g., [Granger, 1980, Barndorff-Nielsen, 2001, Eliazar and Klafter, 2009], among many others) is that while the common assumption of these already known aggregated models is such that the processes involved are driven by *independent* Wiener processes, in our situation of the stochastic processes obtained by convolution of the $\mathcal{OU}$ operator, the linear combination expressing the result of the convolution of the $\mathcal{OU}$ operator is comprised of processes driven by the *same* Wiener process.

To explore the characteristics of this homogeneous aggregation, in Section 4, we show how to write the discrete version of a OU(p) as state space model, and from this representation show in Section 5 that for $p > 1$, a OU(p) behaves like aggregation of AR processes (in the manner considered by [Granger and Morris, 1976]), that turns out to be an ARMA(p, q), with $q \leq p-1$. Notwithstanding this structural similarity, the family of discretized OU(p) processes is more parsimonious than the family of ARMA$(p, p-1)$ processes, and we shall see empirically that it is able to fit well the auto covariances for large lags. Hence, OU processes of higher order appear as a new continuos model, competitive in a discrete time setting with higher order autoregressive processes (AR or ARMA). The estimation of the parameters of OU(p) processes is attempted in Section 6, and examples showing the comparison of the proposed methods for that estimation and the application of OU(p) models to real data are provided in Section 7. Section 8 contains our concluding remarks.

2 Preliminaries

Let us call w a standard Wiener process, that is, a Gaussian, centered process with independent increments with variance $\mathbf{E}(w(t) - w(s))^2 = |t - s|$. We impose further (as usual) that $w(0) = 0$, but shall not limit the domain of the parameter to $\mathbf{R}^+$ and assume that $w(t)$ is defined for t in $\mathbf{R}$. Then, an

Ornstein-Uhlenbeck process with parameters $\lambda > 0, \sigma > 0$ can be written as

$$\xi_{\lambda,\sigma}(t) = \sigma \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} dw(s) \tag{1}$$

or, in differential form,

$$d\xi_{\lambda,\sigma}(t) = -\lambda \xi_{\lambda,\sigma}\, dt + \sigma dw(t) \tag{2}$$

We may think of $\xi_{\lambda,\sigma}$ as the result of accumulating a random noise, with reversion to the mean (that we assume to be 0) of exponential decay with rate λ. The magnitude of the noise is given by σ.

When the Ornstein-Uhlenbeck process x is sampled at equally spaced times $\{i\tau : i = 0, 1, 2, \ldots, n\}$, $\tau > 0$, the series $X_i = x(i\tau)$ obeys an autoregressive model of order 1, AR(1), since

$$X_{i+1} = \sigma \int_{-\infty}^{(i+1)\tau} \mathrm{e}^{-\lambda((i+1)\tau - s)} dw(s)$$

$$= \sigma \mathrm{e}^{-\lambda\tau} \int_{-\infty}^{i\tau} \mathrm{e}^{-\lambda(i\tau - s)} dw(s) + \sigma \int_{i\tau}^{(i+1)\tau} \mathrm{e}^{-\lambda((i+1)\tau - s)} dw(s) = \mathrm{e}^{-\lambda\tau} X_i + Z_{i+1},$$

where $Z_{i+1} = \sigma \int_{i\tau}^{(i+1)\tau} \mathrm{e}^{-\lambda((i+1)\tau - s)} dw(s)$ is a Gaussian innovation (independent of $\{w(t) : t \leq i\tau\}$ and $\{x(t) : t \leq i\tau\}$) with variance

$$\sigma^2 \int_{i\tau}^{(i+1)\tau} \mathrm{e}^{-2\lambda((i+1)\tau - s)} ds = \sigma^2 \int_{-\tau}^{0} \mathrm{e}^{2\lambda s} ds = \frac{\sigma^2}{2\lambda}(1 - \mathrm{e}^{-2\lambda\tau}).$$

Hence, we can consider the OU process as continuous time interpolation of an AR(1) process. Notice that both models are stationary. As we show in §5, the result of iterating p times the operator that carries Wiener process into Ornstein-Uhlenbeck process is an ARMA process with at most $2p - 1$ parameters plus the variance. Let us recall that an ARMA or *autoregressive moving average* process has the following form

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \theta_0 \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

where $\phi_1, \ldots, \phi_p$ are the autoregressive parameters, $\theta_0, \ldots, \theta_q$ are the moving average parameters, and the white-noise process ϵ_t has variance one. Such a

processes is referred to as ARMA(p, q). Denote by B the backshift operator that carries x_t into x_{t-1}. By considering the polynomials in the backshift operator,

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \quad \text{and} \quad \theta(B) = \theta_0 + \theta_1 B + \cdots + \theta_q B^q$$

the ARMA(p, q) model can be written as

$$\phi(B)x_t = \theta(B)\epsilon_t \tag{3}$$

This compact expression of autoregressive models comes in handy for analyzing structural properties of time series and moreover for multivariate time series. It also links to another representation of ARMA processes useful for simplifying maximum likelihood estimation and forecasting; namely, the *state space model* representation. A state space model has the general form

$$\boldsymbol{Y}_t = \boldsymbol{A}\boldsymbol{Y}_{t-1} + \boldsymbol{\eta}_t \tag{4}$$

$$x_t = \boldsymbol{K}^{tr}\boldsymbol{Y}_t + \boldsymbol{N}_t \tag{5}$$

where (4) is the state equation and (5) is the observation equation, with $\boldsymbol{Y}_t$ the m-dimensional state vector, $\boldsymbol{A}$ and $\boldsymbol{K}^{tr}$ are $m \times m$ and $m \times k$ coefficient matrices, $\boldsymbol{K}^{tr}$ denotes the transpose of $\boldsymbol{K}$, $\boldsymbol{\eta}$ and $\boldsymbol{N}$ are m and k dimensional Gaussian white noises, where $\boldsymbol{N}$ would be present only if the process x_t is observed subject to additional noise (see Box et al. [1994] for further details). We present in §4 a state space model representation of our generalized OU process.

3 Ornstein-Uhlenbeck processes of order p

Let $\mathcal{OU}_\lambda$ be defined as the operator that maps σw onto $\xi_{\lambda,\sigma}(t)$, and also maps a process $y(t), t \in \mathbf{R}$ onto

$$\mathcal{OU}_\lambda y(t) = \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} dy(s), \tag{6}$$

whenever the integral can be defined. The definition is extended to include complex processes, by replacing λ by $\kappa = \lambda + \imath\mu$, $\lambda > 0$, $\mu \in \mathbf{R}$ in (6). The set of complex numbers with positive real part is denoted by $\mathbf{C}^+$.

For $p \geq 1$, the process

$$x = \mathcal{OU}_{\boldsymbol{\kappa}}(\sigma w) := \mathcal{OU}_{\kappa_1}\mathcal{OU}_{\kappa_2}\cdots\mathcal{OU}_{\kappa_p}(\sigma w) = \prod_{j=1}^{p}\mathcal{OU}_{\kappa_j}(\sigma w) \tag{7}$$

will be called *Ornstein-Uhlenbeck process of order p with parameters $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_p) \in (\mathbf{C}^+)^p$ and $\sigma > 0$*. The composition $\prod_{j=1}^{p}\mathcal{OU}_{\kappa_j}$ is unambiguously defined because the application of $\mathcal{OU}_{\kappa_j}$ operators is commutative as shown in Theorem 1(i) below.

For technical reasons, it is convenient to introduce the *Ornstein-Uhlenbeck operator $\mathcal{OU}_{\kappa}^{(h)}$ of degree h with parameter κ* that maps y onto

$$\mathcal{OU}_{\kappa}^{(h)}(t)y(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa(t-s)}\frac{(-\kappa(t-s))^h}{h!}dy(s) \tag{8}$$

and σw onto

$$\xi_{\kappa,\sigma}^{(h)}(t) = \sigma\int_{-\infty}^{t} \mathrm{e}^{-\kappa(t-s)}\frac{(-\kappa(t-s))^h}{h!}dw(s) \tag{9}$$

We call the process (9) *generalized Ornstein-Uhlenbeck process* of order 1 and degree h.

3.1 Properties

The following statements summarize some properties of products (compositions) of the operators defined by (7) and (8), and correspondingly, of the stationary centered Gaussian processes $\xi_{\kappa,\sigma}^{(h)}$, $h \geq 0$. In particular, the Ornstein-Uhlenbeck processes of order 1 and degree 0, $\xi_{\kappa,\sigma}^{(0)} = \xi_{\kappa,\sigma}$ are the ordinary Ornstein-Uhlenbeck processes (1).

Corollary 1 establishes that the operator $\mathcal{OU}_{\boldsymbol{\kappa}}$ with p-vector parameter $\boldsymbol{\kappa}$ can be written as a linear combination of p operators $\mathcal{OU}_{\kappa}$ or $\mathcal{OU}_{\kappa}^{(h)}$ for suitable scalar values κ and non negative integer h. Consequently, the process $x = \mathcal{OU}_{\boldsymbol{\kappa}}\sigma w$ can be written as a linear combination of OU processes driven by the same Wiener process, as stated in Corollary 1.

Theorem 1.

(i) When $\kappa_1 \neq \kappa_2$, the product $\mathcal{OU}_{\kappa_2}\mathcal{OU}_{\kappa_1}$ can be computed as

$$\frac{\kappa_1}{\kappa_1 - \kappa_2}\mathcal{OU}_{\kappa_1} + \frac{\kappa_2}{\kappa_2 - \kappa_1}\mathcal{OU}_{\kappa_2}$$

and is therefore commutative.

(ii) The composition $\prod_{j=1}^{p} \mathcal{OU}_{\kappa_j}$ *constructed with values of* $\kappa_1, \ldots, \kappa_p$ *pairwise different, is equal to the linear combination*

$$\prod_{j=1}^{p} \mathcal{OU}_{\kappa_j} = \sum_{j=1}^{p} K_j(\kappa_1, \ldots, \kappa_p) \mathcal{OU}_{\kappa_j}, \tag{10}$$

with coefficients

$$K_j(\kappa_1, \ldots, \kappa_p) = \frac{1}{\prod_{\kappa_l \neq \kappa_j}(1 - \kappa_l/\kappa_j)}. \tag{11}$$

(iii) For $i = 1, 2, \ldots$, $\mathcal{OU}_{\kappa} \mathcal{OU}_{\kappa}^{(i)} = \mathcal{OU}_{\kappa}^{(i)} - \kappa \mathcal{OU}_{\kappa}^{(i+1)}$.

(iv) For any positive integer p *the* p*-th power of the Ornstein-Uhlenbeck operator has the expansion*

$$\mathcal{OU}_{\kappa}^{p} = \sum_{j=0}^{p-1} \binom{p-1}{j} \mathcal{OU}_{\kappa}^{(j)}. \tag{12}$$

(v) Let $\kappa_1, \ldots, \kappa_q$ *be pairwise different complex numbers with positive real parts, and* $p_1, \ldots, p_q$ *positive integers, and let us denote by* $\boldsymbol{\kappa}$ *a complex vector in* $(\mathbf{C}^+)^p$ *with components* κ_h *repeated* p_h *times,* $p_h \geq 1$, $h = 1, \ldots, q$, $\sum_{h=1}^{q} p_h = p$. *Then, with* $K_h(\boldsymbol{\kappa})$ *defined by (11),*

$$\prod_{h=1}^{q} \mathcal{OU}_{\kappa_h}^{p_h} = \sum_{h=1}^{q} \frac{1}{\prod_{l \neq h}(1 - \kappa_l/\kappa_h)^{p_l}} \mathcal{OU}_{\kappa_h}^{p_h} = \sum_{h=1}^{q} K_h(\boldsymbol{\kappa}) \mathcal{OU}_{\kappa_h}^{p_h}.$$

Corollary 1. *i. The process*

$$x = \mathcal{OU}_{\boldsymbol{\kappa}}(\sigma w) = \prod_{h=1}^{q} \mathcal{OU}_{\kappa_h}^{p_h}(\sigma w)$$

can be expressed as the linear combination

$$x = \sum_{h=1}^{q} K_h(\boldsymbol{\kappa})(1 + \xi_{\kappa_h,\sigma})^{(p_h - 1)}, \quad (1 + \xi_{\kappa_h,\sigma})^{(p_h-1)} = \sum_{j=0}^{p_h - 1} \binom{p_h - 1}{j} \xi_{\kappa_h,\sigma}^{(j)} \tag{13}$$

of the p *processes* $\{\xi_{\kappa_h,\sigma}^{(j)} : h = 1, \ldots, q, j = 0 \ldots, p_h - 1\}$ *(see (9)).*

ii. Consequently,

$$x(t) = \sigma \sum_{h=1}^{q} K_h(\boldsymbol{\kappa}) \sum_{j=0}^{p_h-1} \binom{p_h-1}{j} \int_{-\infty}^{t} \mathrm{e}^{-\kappa_h(t-s)} \frac{(-\kappa_h(t-s))^j}{j!} dw(s)$$

Corollary 2. *For real λ, μ, with $\lambda > 0$, the product $\mathcal{OU}_{\lambda+\imath\mu}\mathcal{OU}_{\lambda-\imath\mu}$ is real, that is, applied to a real process produces a real image.*

Proof of the Theorem and its corollaries:

Parts (i) and (iii) are obtained by direct computation of the integrals, (ii) follows from (i) by finite induction, as well as (iv) from (iii).

From the continuity of the integrals with respect to the parameter κ, the power $\mathcal{OU}_\kappa^p$ satisfies

$$\mathcal{OU}_\kappa^p = \lim_{\delta \downarrow 0} \prod_{j=1}^{p} \mathcal{OU}_{\kappa+j\delta} = \lim_{\delta \downarrow 0} \sum_{j=1}^{p} K'_j(\delta, \kappa, p) \mathcal{OU}_{\kappa+j\delta} \tag{14}$$

with

$$K'_j(\delta, \kappa, p) = \frac{1}{\prod_{1 \le l \le p, l \ne j} \left(1 - \frac{\kappa + l\delta}{\kappa + j\delta}\right)}.$$

On the other hand, by (i),

$$\prod_{h=1}^{q} \mathcal{OU}_{\kappa_h}^{p_h} = \lim_{\boldsymbol{\delta} \downarrow 0} \prod_{h=1}^{q} \prod_{j=1}^{p_h} \mathcal{OU}_{\kappa_h + j\delta_h} = \lim_{\boldsymbol{\delta} \downarrow 0} \sum_{h=1}^{q} \sum_{j=1}^{p_h} K''_{h,j}(\boldsymbol{\delta}, \boldsymbol{\kappa}) \mathcal{OU}_{\kappa_h + j\delta_h} \tag{15}$$

where $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_q)$,

$$K''_{h,j}(\boldsymbol{\delta}, \boldsymbol{\kappa}) = \frac{1}{\prod_{\substack{1 \le h' \le q, 1 \le j' \le p_h, \\ (h', j') \ne (h, j)}} \left(1 - \frac{\kappa_{h'} + j'\delta_{h'}}{\kappa_h + j\delta_h}\right)} = K'''_{h,j}(\boldsymbol{\delta}, \boldsymbol{\kappa}) K'_j(\delta_h, \kappa_h, p_h),$$

and

$$K'''_{h,j}(\boldsymbol{\delta}, \boldsymbol{\kappa}) = \frac{1}{\prod_{\substack{1 \le h' \le q, \\ h' \ne h}} \prod_{j'=1}^{p_{h'}} (1 - (\kappa_{h'} + j'\delta_{h'})/(\kappa_h + j\delta_h))} \to K_h(\boldsymbol{\kappa}) \text{ as } \boldsymbol{\delta} \downarrow 0$$

For the h-th term in the right-hand side of (15), we compute

$$\lim_{\boldsymbol{\delta} \downarrow 0} \sum_{j=1}^{p_h} K''_{h,j}(\boldsymbol{\delta}, \boldsymbol{\kappa}) \mathcal{OU}_{\kappa_h + j\delta_h} = \lim_{\boldsymbol{\delta} \downarrow 0} \sum_{j=1}^{p_h} K'''_{h,j}(\boldsymbol{\delta}, \boldsymbol{\kappa}) K'_j(\delta_h, \kappa_h, p_h) \mathcal{OU}_{\kappa_h + j\delta_h}$$

$$= \lim_{\boldsymbol{\delta}\downarrow 0} \sum_{j=1}^{p_h} (K'''_{h,j}(\boldsymbol{\delta}, \boldsymbol{\kappa}) - K_h(\boldsymbol{\kappa})) K'_j(\delta_h, \kappa_h, p_h) \mathcal{OU}_{\kappa_h + j\delta_h}$$

$$+ K_h(\boldsymbol{\kappa}) \lim_{\boldsymbol{\delta}\downarrow 0} \sum_{j=1}^{p_h} K'_j(\delta_h, \kappa_h, p_h) \mathcal{OU}_{\kappa_h + j\delta_h} = K_h(\boldsymbol{\kappa}) \mathcal{OU}^{p_h}_{\kappa_h}$$

by Eq. (14) since, in addition, each term in the first sum tends to zero. This ends the verification of (v).

Corollary 1 is an immediate consequence of (iv) and (v), and Corollary 2 follows by applying (i) to compute

$$\mathcal{OU}_{\lambda+\imath\mu}\mathcal{OU}_{\lambda-\imath\mu} = \frac{\lambda+\imath\mu}{2\imath\mu}\mathcal{OU}_{\lambda+\imath\mu} - \frac{\lambda-\imath\mu}{2\imath\mu}\mathcal{OU}_{\lambda-\imath\mu}$$

$$= \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} \Big[\tfrac{\lambda+\imath\mu}{2\imath\mu}(\cos(\mu(t-s)) + \imath \sin(\mu(t-s))) - \tfrac{\lambda-\imath\mu}{2\imath\mu}(\cos(\mu(t-s)) - \imath \sin(\mu(t-s))) \Big] dw(s)$$

$$= \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} (\cos(\mu(t-s)) + \tfrac{\lambda}{\mu} \sin(\mu(t-s))) dw(s).$$

□

3.2 Computing the covariances

The representation

$$x = \mathcal{OU}_{\boldsymbol{\kappa}}(\sigma w) = \sum_{h=1}^{q} K_h \sum_{j=1}^{p_h} \binom{p_h - 1}{j-1} \mathcal{OU}^{(j-1)}_{\kappa_h}(\sigma w)$$

of x as a linear combination of the processes $\xi^{(i)}_{\kappa_h,\sigma} = \mathcal{OU}^{(i)}_{\kappa_h}(\sigma w)$ allows a direct computation of the covariances $\gamma(t) = \mathbf{E}x(t)\bar{x}(0)$ through a closed formula, in terms of the covariances $\gamma^{(i_1,i_2)}_{\kappa_1,\kappa_2,\sigma}(t) = \mathbf{E}\xi^{(i_1)}_{\kappa_1,\sigma}(t)\bar{\xi}^{(i_2)}_{\kappa_2,\sigma}(0)$:

$$\gamma(t) = \sum_{h'=1}^{q} \sum_{i'=0}^{p_{h'}-1} \sum_{h''=1}^{q} \sum_{i''=0}^{p_{h''}-1} K_{h'}(\boldsymbol{\kappa}) \bar{K}_{h''}(\boldsymbol{\kappa}) \binom{p_{h'}-1}{i'} \binom{p_{h''}-1}{i''} \gamma^{(i',i'')}_{\kappa_{h'},\kappa_{h''},\sigma}(t) \quad (16)$$

with

$$\gamma^{(i_1,i_2)}_{\kappa_1,\kappa_2,\sigma}(t) = \sigma^2(-\kappa_1)^{i_1}(-\bar{\kappa}_2)^{i_2}\int_{-\infty}^{0} \mathrm{e}^{-\kappa_1(t-s)}\frac{(t-s)^{i_1}}{i_1!}\mathrm{e}^{-\bar{\kappa}_2(-s)}\frac{(-s)^{i_2}}{i_2!}ds$$

$$= \sigma^2(-\kappa_1)^{i_1}(-\bar{\kappa}_2)^{i_2}\mathrm{e}^{-\kappa_1 t}\sum_{j=0}^{i_1}\binom{i_1}{j}\frac{t^j}{i_1!i_2!}\int_{-\infty}^{0}\mathrm{e}^{(\kappa_1+\bar{\kappa}_2)s}(-s)^{i_1+i_2-j}ds$$

$$= \frac{\sigma^2(-\kappa_1)^{i_1}(-\bar{\kappa}_2)^{i_2}\mathrm{e}^{-\kappa_1 t}}{i_2!}\sum_{j=0}^{i_1}\frac{t^j(i_1+i_2-j)!}{j!(i_1-j)!(\kappa_1+\bar{\kappa}_2)^{(i_1+i_2-j+1)}} \tag{17}$$

A real expression for the covariance when the imaginary parameters appear as conjugate pairs is much more involved than this one, that contains complex terms.

4 The OU(p)-process as a state space model

Theorem 1 and its corollaries lead to express the OU(p) models by means of linear state space models. The state space modeling provides a unified methodology for the analysis of time series [Durbin and Koopman, 2001].

In the simplest case, where the elements of $\boldsymbol{\kappa}$ are all different, the process $\boldsymbol{x}(t) = \mathcal{OU}_{\boldsymbol{\kappa}}\sigma w(t)$ is a linear combination of the state vector $\boldsymbol{\xi}_{\boldsymbol{\kappa}}(t) = (\xi_{\kappa_1}(t), \xi_{\kappa_2}(t), \ldots, \xi_{\kappa_p}(t))^{\mathrm{tr}}$, where $\xi_{\kappa_j} = \mathcal{OU}_{\kappa_j}(\sigma w)$.

More precisely, the vectorial process

$$\boldsymbol{\xi}_{\boldsymbol{\kappa}}(t) = (\xi_{\kappa_1}(t), \xi_{\kappa_2}(t), \ldots, \xi_{\kappa_p}(t))^{\mathrm{tr}},\ \xi_{\kappa_j} = \mathcal{OU}_{\kappa_j}(\sigma w)$$

and $\boldsymbol{x}(t) = \mathcal{OU}_{\boldsymbol{\kappa}}\sigma w(t)$ satisfy the linear equations

$$\boldsymbol{\xi}_{\boldsymbol{\kappa}}(t) = \mathrm{diag}(\mathrm{e}^{-\kappa_1\tau}, \mathrm{e}^{-\kappa_2\tau}, \ldots, \mathrm{e}^{-\kappa_p\tau})\boldsymbol{\xi}_{\boldsymbol{\kappa}}(t-\tau) + \boldsymbol{\eta}_{\boldsymbol{\kappa},\boldsymbol{\tau}}(t) \tag{18}$$

and

$$\boldsymbol{x}(t) = \boldsymbol{K}^{\mathrm{tr}}(\boldsymbol{\kappa})\boldsymbol{\xi}(t), \tag{19}$$

$$\boldsymbol{\eta}_{\boldsymbol{\kappa},\boldsymbol{\tau}}(t) = (\eta_{\kappa_1,\tau}(t), \eta_{\kappa_2,\tau}(t), \ldots, \eta_{\kappa_p,\tau}(t))^{\mathrm{tr}},\ \ \eta_{\kappa_j,\tau}(t) = \sigma\int_{t-\tau}^{t}\mathrm{e}^{-\kappa_j(t-s)}dw(s),$$

$$\mathbf{Var}(\boldsymbol{\eta}_{\boldsymbol{\kappa},\boldsymbol{\tau}}(t)) = \sigma^2((v_{j,l})), v_{j,l} = \mathbf{E}\int_{t-\tau}^{t}\mathrm{e}^{-(\kappa_j+\bar{\kappa}_l)(t-s)}ds = \frac{1-\mathrm{e}^{-(\kappa_j+\bar{\kappa}_l)\tau}}{\kappa_j+\bar{\kappa}_l} \tag{20}$$

and the coefficients from (11)

$$\boldsymbol{K}^{\mathrm{tr}}(\boldsymbol{\kappa}) = (K_1(\boldsymbol{\kappa}), K_2(\boldsymbol{\kappa}), \ldots, K_p(\boldsymbol{\kappa})).$$

The initial value $\boldsymbol{\xi}(0)$ is estimated by means of its conditional expectation $\hat{\boldsymbol{\xi}}(0) = \mathbf{E}(\xi(0)|x(0)) = \frac{\boldsymbol{K}^{\mathrm{tr}}(\boldsymbol{\kappa})Vx(0)}{\boldsymbol{K}^{\mathrm{tr}}(\boldsymbol{\kappa})V\bar{K}}$, with $V = \mathbf{Var}(\xi(0)) = ((\frac{1}{\kappa_j+\bar{\kappa}_l}))$.

An application of Kalman filter to this state space model leads to compute the likelihood of $\boldsymbol{x} = (x(0), x(\tau), \ldots, x(n\tau))$. Some Kalman filter programs included in software packages require the processes in the state space to be real. That condition is not fulfilled by the model described by equations (18,19). An equivalent description by means of real processes can be obtained by ordering the parameters $\boldsymbol{\kappa}$ with the imaginary components paired with their conjugates in such a way that $\kappa_{2h} = \bar{\kappa}_{2h-1}$, $h = 1, 2, \ldots, c$ and $\Im(\kappa_j) = 0$ if and only if $2c < j \leq p$.

Then the matrix $M = ((M_{j,k}))$ with all elements equal to zero except

$$M_{2h-1,2h-1} = M_{2h-1,2h} = 1, \quad -M_{2h,2h-1} = M_{2h,2h} = \imath, \ h = 1, 2, \ldots, c$$

and

$$M_{j,j} = 1, \ 2c < j \leq p$$

induces the linear transformation $\boldsymbol{\xi} \mapsto M\boldsymbol{\xi}$ that leads to the new state space description

$$M\boldsymbol{\xi}(t) = M\mathrm{diag}(\mathrm{e}^{-\kappa_1\tau}, \mathrm{e}^{-\kappa_2\tau}, \ldots, \mathrm{e}^{-\kappa_p\tau})M^{-1}M\boldsymbol{\xi}(t-1) + M\boldsymbol{\eta}(t), \qquad (21)$$

$$\boldsymbol{x}(t) = \boldsymbol{K}^{\mathrm{tr}}M^{-1}M\boldsymbol{\xi}(t), \qquad (22)$$

where the processes $M\boldsymbol{\xi}$ are real.

Observe that there is no loss of generality in choosing the spacing τ between observations as unity, for the derivation of the state space equations. Hence, we set $\tau = 1$ in the sequel and, in addition, τ will be omitted from the notation.

When $\kappa_1, \ldots, \kappa_q$ are all different, $p_1, \ldots, p_q$ are positive integers, $\sum_{h=1}^{q} p_h = p$ and $\boldsymbol{\kappa}$ is a p-vector with p_h repeated components equal to κ_h, the OU(p) process $x_{\boldsymbol{\kappa}}$ is a linear function of the state space vector

$$\left(\xi_{\kappa_1}^{(0)}, \xi_{\kappa_1}^{(1)}, \ldots, \xi_{\kappa_1}^{(p_1-1)}, \ldots, \xi_{\kappa_q}^{(0)}, \xi_{\kappa_q}^{(1)}, \ldots, \xi_{\kappa_q}^{(p_q-1)}\right)$$

where the components are given by (9), and the transition equation is no longer expressed by a diagonal matrix. In this case the state space model has the following form

$$\begin{aligned} \boldsymbol{\xi}(t) &= A\boldsymbol{\xi}(t-1) + \boldsymbol{\eta}(t) \\ x(t) &= \boldsymbol{K}^{\mathrm{tr}}\boldsymbol{\xi}(t) \end{aligned} \tag{23}$$

We leave the technical details of this derivation to an Appendix. The terms $\boldsymbol{\xi}(t)$, A, $\boldsymbol{\eta}(t)$ and $\boldsymbol{K}$ are precisely defined in (42). The real version of (23), when the process $\boldsymbol{\xi}$ has imaginary components is obtained by multiplying both equations by a block-diagonal matrix C (which is defined precisely in the aforementioned Appendix), given us the real state space model

$$C\boldsymbol{\xi}(t) = (CAC^{-1})(C\boldsymbol{\xi}(t-1)) + C\boldsymbol{\eta}(t), \tag{24}$$

$$x(t) = (\boldsymbol{K}^{\mathrm{tr}}C^{-1})(C\boldsymbol{\xi}(t)). \tag{25}$$

5 The OU(p) as an ARMA($p, p-1$)

The studies of properties of linear transformations and aggregations of similar processes have produced a great amount of work stemming from the seminal paper of Granger and Morris [1976] on the invariance of MA and ARMA processes under these operations. These results and extensions to Vector Autoregressive Moving Average (VARMA) processes are compiled in the textbook by Lütkepohl [2005].

The description of the OU(p) process $x = \mathcal{OU}_{\boldsymbol{\kappa}}(\sigma w)$ with parameters $\boldsymbol{\kappa}, \sigma$ as a linear state space model, given in the previous section, will allow us to show that the series $x(0), x(1), \ldots, x(n)$ satisfies an ARMA(p, q) model with q smaller than p. We refer the reader to Ch. 11 of Lütkepohl [2005] for a presentation on VARMA processes and, in particular, to the following result on the invariance property of VARMA processes under linear transformations contained in §11.6 which we quote with a minor change of notation:

Theorem 2 (Lütkepohl [2005] Cor. 11.1.2)**.** *Let y_t be a d-dimensional, stable, invertible VARMA($\tilde{p}$,$\tilde{q}$) process and let F be an $(m \times d)$ matrix of rank m. Then the process $z_t = Fy_t$ has a VARMA($\check{p}, \check{q}$) representation with $\check{p} \leq (d-m+1)\tilde{p}$ and $\check{q} \leq (d-m)\tilde{p} + \tilde{q}$.* □

Equation (24) shows that $C\boldsymbol{\xi}(t)$ is a p-dimensional autoregressive vector (a p-dimensional VARMA(1,0) process) and Eq. (25) expresses $x(t)$ as a

linear transformation of $C\boldsymbol{\xi}(t)$ by the $(1 \times p)$ matrix $F = \boldsymbol{K}^{\mathrm{tr}} C^{-1}$. Using Theorem 2 (with $d = p$, $\tilde{p} = 1$, $\tilde{q} = 0$, $m = 1$) we conclude that $(x(t) : t = 0, 1, \ldots, n)$ is an ARMA($\check{p}$,$\check{q}$) process with $\check{p} \leq p$ and $\check{q} \leq p - 1$:

$$x(i) = \sum_{j=1}^{p} \phi_j x(i-j) + \sum_{l=0}^{p-1} \theta_l \epsilon_{i-l} \tag{26}$$

where ϵ is a Gaussian white noise with variance one and the parameters $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^{\mathrm{tr}}$, $\boldsymbol{\theta} = (\theta_0, \ldots, \theta_{p-1})^{\mathrm{tr}}$ of the ARMA process are functions of the parameters $\boldsymbol{\kappa}$ and σ of the OU.

By using the backshift operator B, and the polynomials $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$, $\theta(z) = \sum_{l=0}^{p-1} \theta_l z^l$, (26) is written as

$$\phi(B)x = \theta(B)\epsilon. \tag{27}$$

We proceed now to identify the coefficients $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ of the ARMA model.
Step 1. Let us consider first the simplest case, analyzed at the beginning of §4. For each j, the series $\xi_{\kappa_j} = (\xi_{\kappa_j}(i))_{i \in \boldsymbol{Z}}$ satisfies the AR(1) model

$$(1 - \mathrm{e}^{-\kappa_j} B)\xi_{\kappa_j} = \eta_{\kappa_j}$$

(see (18)), and from (19) the series $x = (x(i))_{i \in \boldsymbol{Z}}$ given by

$$x = \sum_{j=1}^{p} K_j(\boldsymbol{\kappa}) \xi_{\kappa_j}$$

follows the ARMA model

$$\prod_{j=1}^{p} (1 - \mathrm{e}^{-\kappa_j} B)x = \sum_{j=1}^{p} K_j(\boldsymbol{\kappa}) \prod_{l \neq j} (1 - \mathrm{e}^{-\kappa_l} B)\eta_{\kappa_j}.$$

The sum of moving averages in the right-hand term is distributed as the moving average

$$\zeta = \sum_{h=0}^{p-1} \theta_h B^h \epsilon$$

where ϵ is a white noise with variance one and the coefficients θ_h are suitably chosen. It is readily verified that the autocovariances $c_l = \mathbf{E}\zeta(i)\bar{\zeta}(i-l)$ of this MA are the coefficients in the sum of powers of z

$$\left(\sum_{h=0}^{p-1} \theta_h z^h \right) \left(\sum_{k=0}^{p-1} \bar{\theta}_k z^{-h} \right) = \sum_{l=-p+1}^{p-1} c_l z^l. \tag{28}$$

A similar formula that takes into account the correlations (20) between the noises η_{κ_k}indicates that the same autocovariances are given by the identity

$$J(z) := \sum_{j=1}^{p}\sum_{l=1}^{p} K_j\bar{K}_l G_j(z)\bar{G}_l(1/z)v_{j,l} = \sum_{l=-p+1}^{p-1} c_l z^l \tag{29}$$

where $G_j(z) = \prod_{l\neq j}(1-\mathrm{e}^{-\kappa_l}z) = \sum_{l=0}^{p-1} g_{j,l}z^l$.

The coefficients $g_{j,l}$ and the function J are completely determined from the parameters of the OU process. In order to express the parameters of the ARMA(p,$p-1$) process in terms of $\boldsymbol{\kappa}, \sigma$ it remains to obtain the coefficients θ_h in the factorization (28).

The roots ρ_j $(j = 1, 2, \ldots, p-1)$ of

$$\theta(z) = \sum_{h=0}^{p-1}\theta_h z^h = \theta_0\prod_{j=1}^{p-1}(1 - z/\rho_j) \tag{30}$$

are obtained by choosing the roots of the polynomial $z^{p-1}\theta(z)\bar{\theta}(1/z) = z^{p-1}J(z)$ with modules greater than one (the remaining roots are their inverses). Then all θ_h are written in terms of the ρ_h and the size factor θ_0 by applying (30).

The value of θ_0 follows by using an additional equation, namely, the equality of the terms of degree zero in $J(z)$ and $\theta(z)\bar{\theta}(1/z)$, thus obtaining

$$\sum_{l=0}^{p-1}|\theta_l|^2 = \sum_{j=1}^{p}\sum_{l=1}^{p} K_j\bar{K}_l v_{j,l}\sum_{h=0}^{p-1} g_{j,h}\bar{g}_{l,h}.$$

Step 2. As a second step, let us approach the OU(p) process x with parameter equal to the p-vector with equal components $\boldsymbol{\kappa} = (\kappa, \kappa, \ldots, \kappa)^{\mathrm{tr}}$ as the limit of $x_\delta = \mathcal{OU}_{\boldsymbol{\kappa}(\delta)}\sigma w$, $\boldsymbol{\kappa}(\delta) = (\kappa(1+\delta), \kappa(1+2\delta), \ldots, \kappa(1+p\delta))^{\mathrm{tr}}$ when δ tends to zero.

From the previous step we use the representation

$$x_\delta = \sum_{j=1}^{p} K_j\xi_j, \quad K_j = \frac{(1+j\delta)^{p-1}}{\delta^{p-1}}\prod_{l\neq j}\frac{1}{j-l} \tag{31}$$

in terms of the vector

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_p)^{\mathrm{tr}}, \quad \xi_j(t) = \int_{-\infty}^{t}\mathrm{e}^{-\kappa(1+j\delta)(t-s)}dw(s)$$

that satisfies

$$\boldsymbol{\xi} = \mathrm{diag}(\mathrm{e}^{-\kappa(1+j\delta)})B\boldsymbol{\xi} + \boldsymbol{\eta}$$

with

$$\eta_j(t) = \int_{t-1}^{t} \mathrm{e}^{-\kappa(1+j\delta)(t-s)} dw(s)$$

and introduce the power expansions

$$\xi_j(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa(t-s)} \sum_{h=0}^{\infty} \frac{(j\delta)^h(-\kappa(t-s))^h}{h!} dw(s) = \sum_{h=0}^{\infty} (j\delta)^h \xi_\kappa^{(h)}(t)$$

with $\xi_\kappa^{(h)}(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa(t-s)} \frac{(-\kappa(t-s))^h}{h!} dw(s)$ and the similar expansion for the innovations

$$\eta_j(t) = \sum_{h=0}^{\infty} (j\delta)^h \eta_\kappa^{(h)}(t) \text{ with } \eta_\kappa^{(h)}(t) = \int_{t-1}^{t} \mathrm{e}^{-\kappa(t-s)} \frac{(-\kappa(t-s))^h}{h!} dw(s). \quad (32)$$

We write now the ARMA model

$$\prod_{j=1}^{p} (1 - \mathrm{e}^{-\kappa(1+j\delta)}B) x_\delta = \sum_{j=1}^{p} \prod_{l \neq j} (1 - \mathrm{e}^{-\kappa(1+l\delta)}B) K_j \eta_j$$

and notice that the limit when $\delta \to 0$ of the left-hand side is $(1 - \mathrm{e}^{-\kappa}B)^p x$.

In order to take limits at the right-hand side, we replace K_j for its expression in (31), expand $\prod_{l \neq j}(1 - \mathrm{e}^{-\kappa(1+l\delta)}B)$ as the product of the series

$$\prod_{l=1}^{p} (1 - \mathrm{e}^{-\kappa(1+l\delta)}B) = \sum_{\nu=0}^{\infty} a_\nu \delta^\nu \quad (33)$$

independent of j and

$$(1 - \mathrm{e}^{-\kappa(1+j\delta)}B)^{-1} = \sum_{h=0}^{\infty} (\mathrm{e}^{-\kappa(1+j\delta)}B)^h = \sum_{\mu=0}^{\infty} b_\mu (j\delta)^\mu \quad (34)$$

with coefficients independent of j and substitute the expansion (32) for η_j thus obtaining the series

$$\sum_{j=1}^{p} \left(\sum_{\nu=0}^{\infty} a_\nu \delta^\nu \times \sum_{\mu=0}^{\infty} b_\mu (j\delta)^\mu \times (1 + j\delta)^{p-1} \prod_{l \neq j} \frac{1}{j-l} \times \sum_{h=0}^{\infty} (j\delta)^h \eta_\kappa^{(h)} \right)$$

divided by δ^{p-1}.

After ordering this series by increasing powers of δ, it may be noticed that the terms in δ raised to a power smaller than $p-1$ vanish, because their coefficient include a factor $\sum_{j=1}^{p} j^h \prod_{l\neq j} \frac{1}{j-l}$ with $h \in \{0, 1, \ldots, p-2\}$ that is equal to zero as established in Lemma 1 below. Therefore, the limit when $\delta \to 0$ of the series divided by δ^{p-1} is the coefficient of δ^{p-1} in the series.

Unless the term a_0 of the first factor is taken, the power of j appearing in the coefficient of δ^{p-1} will be smaller than $p-1$ and again Lemma 1 leads to conclude that the coefficient vanishes. Therefore, since the same lemma establishes that $\sum_{j=1}^{p} j^{p-1} \prod_{l\neq j} \frac{1}{j-l} = 1$, the required limit is the linear combination of moving averages

$$a_0 \sum_{\mu+\imath+h=p-1} \binom{p-1}{i} b_\mu \eta_\kappa^{(h)} \tag{35}$$

where it remains to make explicit the dependence with respect to the back-shift operator B.

From (33) it follows immediately that $a_0 = (1-\mathrm{e}^{-\kappa}B)^p$, while from (34) we get $b_\mu j^\mu \mu! = \left[\frac{\partial^\mu}{\partial\delta^\mu} \sum_{h=0}^{\infty} \mathrm{e}^{-\kappa h} B^h \mathrm{e}^{-hj\delta}\right]_{\delta=0} = (-j)^\mu \sum_{h=0}^{\infty} (\mathrm{e}^{-\kappa}B)^h h^\mu$ and hence

$$b_\mu = \frac{(-1)^\mu}{\mu!} \sum_{\nu=0}^{\infty} (\mathrm{e}^{-\kappa}B)^\nu \nu^\mu.$$

Now we apply Lemma 2 (stated at the end of this section) such that, with the coefficients $\alpha_{\mu,\nu}$ there defined, leads us to write

$$\sum_{\nu=0}^{\infty} (\mathrm{e}^{-\kappa}B)^\nu \nu^\mu = \sum_{\nu=0}^{\mu} \alpha_{\mu,\nu} (1-\mathrm{e}^{-\kappa}B)^{-\nu-1}$$

and therefore (35) is equal to the moving average of order at most $p-1$

$$\sum_{\mu+\imath+h=p-1} \binom{p-1}{i} \frac{(-1)^\mu}{\mu!} \sum_{\nu=0}^{\mu} \alpha_{\mu,\nu} (1-\mathrm{e}^{-\kappa}B)^{p-\nu-1} \eta_\kappa^{(h)}. \tag{36}$$

Let us observe finally that the order of the moving average is actually $p-1$. The term in B^{p-1} corresponds to $\nu = 0$ and reduces to

$$\sum_{\mu+\imath+h=p-1} \binom{p-1}{i} \frac{(-1)^\mu}{\mu!} \alpha_{\mu,0} (-1)^{p-1} e^{-(p-1)\kappa} B^{p-1} \eta_\kappa^{(h)}.$$

At least the term in $B^{p-1}\eta_\kappa^{(p-1)}$ with coefficient $(-1)^{p-1}e^{-(p-1)\kappa}$ does not vanish. On the other hand, neither the term with lag zero in $\eta_\kappa^{(p-1)}$ vanishes, because its coefficient is $\alpha_{0,0}=1$.

Step 3. Our third and final step is to join the previous results for the general case with parameter $\boldsymbol{\kappa}$, a p-vector with p_j components equal to κ_j, $j=1,2,\ldots,q$, with $\kappa_1,\ldots,\kappa_q$ all different of each other and $\sum_{j=1}^q p_j = p$. We use the result of Theorem 1(v) and conclude that $x=\mathcal{OU}_{\boldsymbol{\kappa}}(\sigma w)$, satisfies the ARMA$(p,p-1)$ model

$$\prod_{j=1}^{q}(1-\mathrm{e}^{-\kappa_j}B)^{p_j}x=\sum_{j=1}^{q}K_j\prod_{l\neq j}(1-\mathrm{e}^{-\kappa_l}B)^{p_l}\mathrm{MA}_j \tag{37}$$

with MA_j the moving average of order p_j-1 given by Eq. (36).

We do not intend to express the coefficients θ_h of the moving average model $\sum_{h=0}^{p-1}\theta_h B^h\epsilon$ based on a white noise ϵ of variance one with the same distribution as $\sum_{j=1}^q K_j\prod_{l\neq j}(1-\mathrm{e}^{-\kappa_l}B)^{p_l}\mathrm{MA}_j$ for the general model, but will do it in particular cases treated below.

Lemma 1. *For each positive integer p, $\sum_{j=1}^p j^{p-1}\prod_{l\neq j}\frac{1}{j-l}=1$ and for $h=0,1,\ldots,p-2$, $\sum_{j=1}^p j^h\prod_{l\neq j}\frac{1}{j-l}=0$.*

Proof: The polynomial $G(z)=\sum_{j=1}^p\left(\frac{1}{j}\right)^{p-1-h}\prod_{l\neq j}\frac{1-lz}{1-l/j}$ has degree $p-1$ and coincides for p different values of the variable, namely $z=1/j$, $j=1,2,\ldots,p$, with the polynomial z^{p-1-h}, also of degree not greater than $p-1$ for $h=0,1,\ldots,p-1$. Therefore, both polynomials are identical, and hence $G(0)=0$ for $h<p-1$ and $G(0)=1$ for $h=p-1$. □

Lemma 2. *The power series $g(z,n)=\sum_{h=0}^\infty z^h h^n$, $|z|<1, n=0,1,2,\ldots$ has the sum*

$$\sum_{h=0}^{n}\alpha_{n,h}(1-z)^{-h-1}$$

with coefficients determined by $\alpha_{0,0}=1$ and the recurrence relations

$$\alpha_{n+1,h}=h\alpha_{n,h-1}-(h+1)\alpha_{n,h}, h=0,1,\ldots,n+1, n=0,1,2,\ldots,\alpha_{n,n+1}=0.$$

In particular, $\alpha_{n,0}=(-1)^n$. □

6 Estimation of the parameters of OU(p)

6.1 Reparameterization by means of real parameters

Since we wish to consider real processes x and the process itself and its covariance $\gamma(t)$ depend only on the unordered set of the components of $\boldsymbol{\kappa}$, we shall reparameterize the process by means of the real vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$ given by the polynomial identity

$$g(z) = \prod_{j=1}^{p}(1 + \kappa_j z) = 1 - \sum_{j=1}^{p} \beta_j z^j. \tag{38}$$

The resulting process is real, because of Corollary 2.

6.2 Maximum likelihood estimation (MLE)

We shall assume that the process x is observed at times $0, \tau, 2\tau, \ldots, n\tau$. By choosing τ the time unit of measure, as in §4, we assume without loss of generality that our observations are $\boldsymbol{x} = (x(0), x(1), \ldots, x(n))^{\mathrm{tr}}$.

The likelihood L of the vector $\boldsymbol{x}$ is given by

$$\log L(\boldsymbol{x}; \boldsymbol{\beta}, \sigma) = -\tfrac{n}{2}\log(2\pi) - \tfrac{1}{2}\log(\det(\Gamma(\boldsymbol{\beta}, \sigma)) - \tfrac{1}{2}\boldsymbol{x}^{\mathrm{tr}}(\Gamma(\boldsymbol{\beta}, \sigma))^{-1}\boldsymbol{x}$$

where Γ has components $\Gamma_{h,i} = \gamma(|h - i|)$ $(h, i = 0, 1, \ldots, n)$. The Kalman filter associated to the dinamical state space model in §4 provides an efficient alternative to compute the likelihood.

From these elements, a numerical optimization leads to obtain the maximum likelihood estimators $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ and $\hat{\sigma}^2$ of σ^2. If required, the estimations $\hat{\boldsymbol{\kappa}}$ follow by solving the analogue of the polynomial equation (38) written in terms of the estimators:

$$\prod_{j=1}^{p}(1 + \hat{\kappa}_j z) = 1 - \sum_{j=1}^{p} \hat{\beta}_j z^j.$$

The optimization for large n and the solution of the algebraic equation for large p require a considerable computation effort, but there are efficient programs to perform both operations, as `optim` and `polyroot` in `R` (R Development Core Team [2011]).

An alternative when the observed process is not assumed to be centered, is to maximize the log-likelihood of $\Delta\boldsymbol{x} = (x(1)-x(0), x(2)-x(1), \ldots, x(n)-x(n-1))$ given by

$$\log L(\boldsymbol{x};\boldsymbol{\beta},\sigma) = -\tfrac{n}{2}\log(2\pi) - \tfrac{1}{2}\log(\det(V(\boldsymbol{\beta},\sigma)) - \tfrac{1}{2}\Delta\boldsymbol{x}^{\mathrm{tr}}(V(\boldsymbol{\beta},\sigma))^{-1}\Delta\boldsymbol{x}$$

with $V(\boldsymbol{\beta},\sigma)$ equal to the $n \times n$ matrix with components

$$V_{h,i} = 2\gamma(|h-i|) - \gamma(|h-i|+1) - \gamma(|h-i|-1)$$

that reduce to $2(\gamma(0)-\gamma(1))$ at the diagonal $h = i$.

The optimization procedures require an initial guess about the value of the parameter to be estimated. The estimators obtained by *matching correlations* described in the next section can be used for that purpose.

6.3 Matching correlations estimation (MCE)

From the closed formula for the covariance γ (eq. (16)) and the relationship between $\boldsymbol{\kappa}$ and $\boldsymbol{\beta}$ (eq. (38)), we have a mapping $(\boldsymbol{\beta},\sigma^2) \mapsto \gamma(t)$, for each t. Since $\boldsymbol{\rho}^{(T)} := (\rho(1),\rho(2),\ldots,\rho(T))^{\mathrm{tr}} = (\gamma(1),\gamma(2),\ldots,\gamma(T))^{\mathrm{tr}}/\gamma(0)$ does not depend on σ^2, these equations determine a map $\mathcal{C} : (\boldsymbol{\beta},T) \mapsto \boldsymbol{\rho}^{(T)} = \mathcal{C}(\boldsymbol{\beta},T)$ for each T. After choosing a value of T and obtaining an estimate $\boldsymbol{\rho}_e^{(T)}$ of $\boldsymbol{\rho}^{(T)}$ based on the empirical covariances of x, we propose as a first estimate of $\boldsymbol{\beta}$, the vector $\check{\boldsymbol{\beta}}_T$ such that all the components of the corresponding $\boldsymbol{\kappa}$ have positive real parts, and such that the euclidean norm $\|\boldsymbol{\rho}_e^{(T)} - \mathcal{C}(\check{\boldsymbol{\beta}}_T, T)\|$ reaches its minimum. The procedure resembles the estimation by the *method of moments*. The components of $\boldsymbol{\rho}_e^{(T)}$for the series $(x_i)_{i=1,2,\ldots,n}$ are computed as $\rho_{e,h} = \gamma_{e,h}/\gamma_{e,0}$, $\gamma_{e,h} = \frac{1}{n}\sum_{i=1}^{n-h} x_i x_{i+h}$.

6.4 Some simulations

We have simulated the series $x(i), i = 0,1,2,\ldots,n$ obtained from an OU process x for $n = 300$ and three different values of the parameters and computed the MC and ML estimators $\check{\boldsymbol{\beta}}_T$, and $\hat{\boldsymbol{\beta}}$. The value of T for the MC estimation has been arbitrarily set equal to the integral part of $0.9 \cdot n$, but the graphs of $\check{\boldsymbol{\beta}}_T$ for several values of T show in each case that after T exceeds a moderate threshold, the estimates remain practically constant. One of such graphs is included below (see Figure 3).

The simulations show that the correlations of the series with the estimated parameters are fairly adapted to each other and to the empirical covariances. The departure from the theoretical covariances of x can be ascribed to the simulation intrinsic randomness.

Our first two examples describe OU(3) processes with arbitrarily (and randomly) chosen parameters and the third one imitates the behavior of Series A that appears in §7.

Example 1. A series $(x_i)_{i=0,1,\dots,n}$ of $n = 300$ observations of the $\mathrm{OU}_{\boldsymbol{\kappa}}$ process x ($p = 3$, $\boldsymbol{\kappa} = (0.9, 0.2 + 0.4\imath, 0.2 - 0.4\imath)$, $\sigma^2 = 1$) was simulated, and the parameters $\boldsymbol{\beta} = (-1.30,\ -0.56,\ -0.18)$ and $\sigma^2 = 1$ were estimated by means of $\check{\boldsymbol{\beta}}_T = (-1.9245,\ -0.6678,\ -0.3221)$, $T = 270$, $\hat{\boldsymbol{\beta}} = (-1.3546,\ -0.6707,\ -0.2355)$ and $\hat{\sigma}^2 = 0.8958$. The corresponding estimators for $\boldsymbol{\kappa}$ are $\check{\boldsymbol{\kappa}} = (1.6368,\ 0.1439\ +0.4196\imath,\ 0.14389\ -0.4196\imath)$ and $\hat{\boldsymbol{\kappa}} = (0.9001,\ 0.2273 + 0.4582\imath,\ 0.2273 - 0.4582\imath)$.

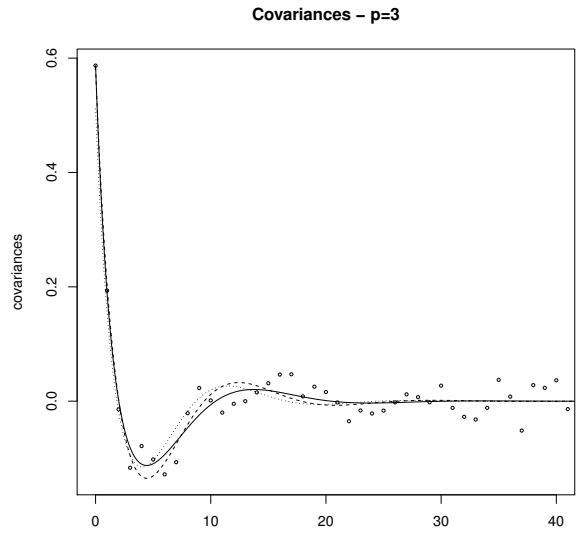

Figure 1: Empirical covariances (∘) and covariances of the MC (—) and ML (- - -) fitted OU models, for $p = 3$ corresponding to Example 1. The covariances of $\mathrm{OU}_{\boldsymbol{\kappa}}$ are indicated with a dotted line.

Figure 1 describes the theoretical, empirical and estimated covariances of x under the assumption $p = 3$, that is, the actual order of x. The results

obtained when the estimation is performed for $p = 2$ and $p = 4$ are shown in Figure 2. Finally, Figure 3 shows that the MC estimates of $\boldsymbol{\beta}$ become stable for T moderately large, and close to the already indicated estimations for $T = 270$ (the horizontal lines).

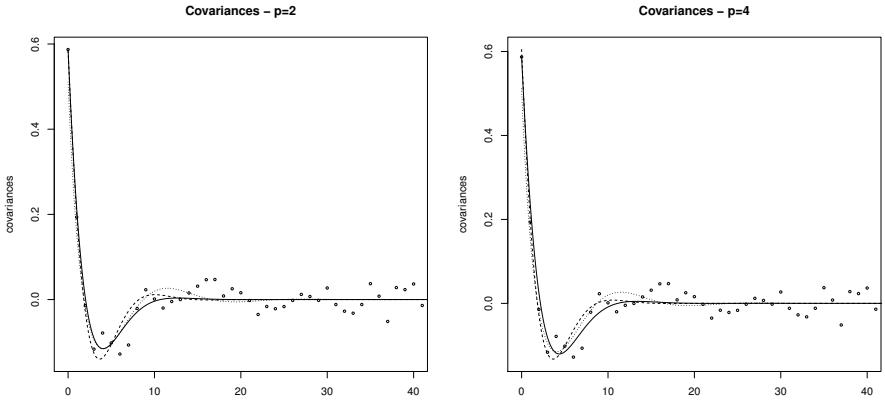

Figure 2: Empirical covariances ($\circ$) and covariances of the MC (—) and ML (- - -) fitted OU models, for $p = 2, 4$ corresponding to Example 1. The covariances of $\mathrm{OU}_{\boldsymbol{\kappa}}$ are indicated with a dotted line.

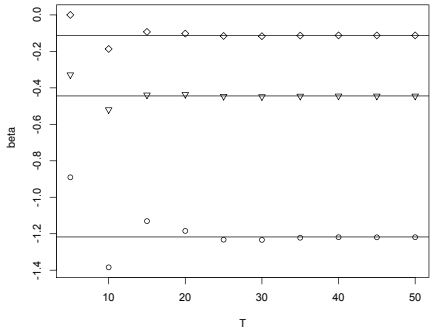

Figure 3: The MC estimations $\check{\beta}_1(\circ)$, $\check{\beta}_2(\triangledown)$ and $\check{\beta}_3(\diamond)$ for different values of T, corresponding to Example 1. The horizontal lines indicate the estimations for $T = 270$.

The coefficients ϕ_1, ϕ_2, ϕ_3 of the ARMA(3,2) model (27) satisfied by the series $(x(i))_{i=0,1,\ldots,300}$ are obtained by computing the product $\prod_{j=1}^{3}(1-\mathrm{e}^{-\kappa_j}B)$ $= 1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3 = 1 - 1.9148B + 1.2835B^2 - 0.2725B^3$.

As for the coefficients $\theta_0, \theta_1, \theta_2$, the first step is to compute the function

$$J(z) = 0.2995z^{-2} - 1.1943z^{-1} + 1.7904 - 1.1943z + 0.2995z^2,$$

then obtain the roots $\rho_1 = 1.1443\ -0.1944\imath$, $\rho_2 = 1.1443\ +0.1944\imath$, $\rho_3 = 0.8494\ -0.1443\imath$, $\rho_4 = 0.8494\ +0.1443\imath$ of the equation $z^2J(z) = 0$, ordered by decreasing moduli, discard the last two, and write the function $\theta(z) = \theta_0 + \theta_1 z + \theta_2 z^2$ defined in (30):

$$\theta_0 \prod_{j=1}^{2}(1 - B/\rho_1)(1 - B/\rho_2) = \theta_0(1 - 1.6988z + 0.7423z^2).$$

Solve $\theta_0^2(1 + (-1.6988)^2 + 0.74229^2) = 1.7904$ to have $\theta_0 = 0.6352$, and hence $\theta(B) = 0.6352 - 1.0791B + 0.4715B^2$.

Example 2. The process $x = \mathrm{OU}_{(0.04,0.21,1.87)}$ is analyzed as in Example 1. The resulting estimators are $\check{\boldsymbol{\beta}}_T = (-2.0611,\ -0.7459,\ -0.0553)$, $T = 270$, $\check{\boldsymbol{\kappa}} = (1.6224, 0.3378, 0.1009)$, $\hat{\boldsymbol{\beta}} = (-1.8253, -0.7340, -0.0680)$, $\hat{\sigma}^2 = 0.7842$, $\hat{\boldsymbol{\kappa}} = (1.3015,\ 0.3897,\ 0.1342)$, and the resulting covariances are shown in Figure 4 . The associated ARMA(3,2) model is

$$(1 - 1.9255B + 1.05185B^2 - 0.1200B^3)x = (0.4831 - 0.9044B + 0.4230B^2)\epsilon.$$

Example 3. The parameter $\boldsymbol{\kappa} = (0.83, 0.0041, 0.0009)$ used in the simulation of the OU process x treated in the present example is approximately equal to the parameter $\hat{\boldsymbol{\kappa}}$ obtained by ML estimation with $p = 3$ for Series A in §7.1. As in previous examples, a graphical presentation of the estimated covariances is given in Figure 5.

The associated ARMA(3,2) model is

$$(1 - 2.4311B + 1.8649B^2 - 0.4339B^3)x = (0.6973 - 1.3935B + 0.6962B^2)\epsilon$$

The description of the performance of the model is complemented by comparing in Figure 6 the simulated values of the process in 400 equally spaced points filling the interval (199,201) with the predicted values for the same interval, based on the OU(3) model and the assumed observed data $x(0), x(2), x(3), \ldots, x(200)$. Also a 2σ confidence band is included in the graph, in order to describe the precision of the predicted values.

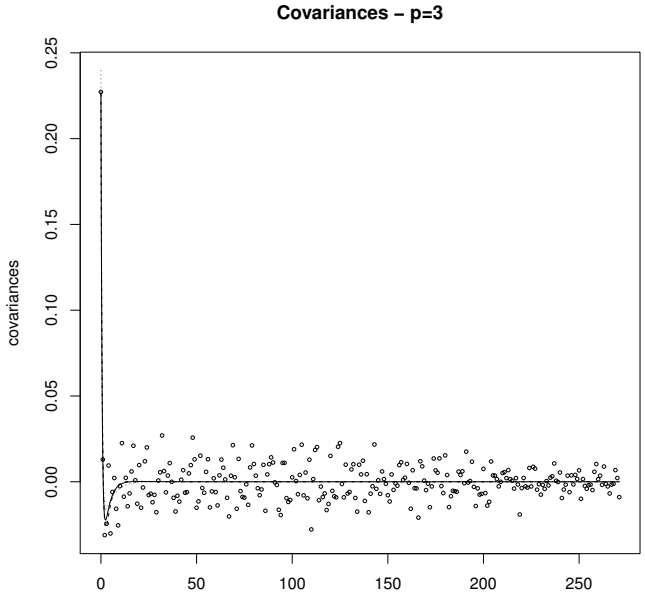

Figure 4: Empirical covariances (∘) and covariances of the MC (—) and ML (- - -) fitted OU models, for $p = 3$ corresponding to Example 2. The covariances of $\mathrm{OU}_{\boldsymbol{\kappa}}$ are indicated with a dotted line.

7 Applications to real data

In this section we present experimental results on three real data sets. We fit OU(p) processes for small values of p and also some ARMA processes. In each case we have observed that we can find an adequate value of p for which the empirical covariances are well approximated by the covariances of the adjusted OU(p) model. This is not the case for the ARMA models adjusted by maximum likelihood, in all three examples. We present a detailed comparison of both methodologies for the first example.

The first two data sets are taken from Box et al. [1994], and correspond to equally spaced observations of continuous time processes that might be assumed to be stationary. The third one is a series obtained by choosing one in every 100 terms of a high frequency recording of oxygen saturation in blood of a newborn child. The data were obtained by a team of researchers of Pereira Rossell Children Hospital in Montevideo, Uruguay, integrated by

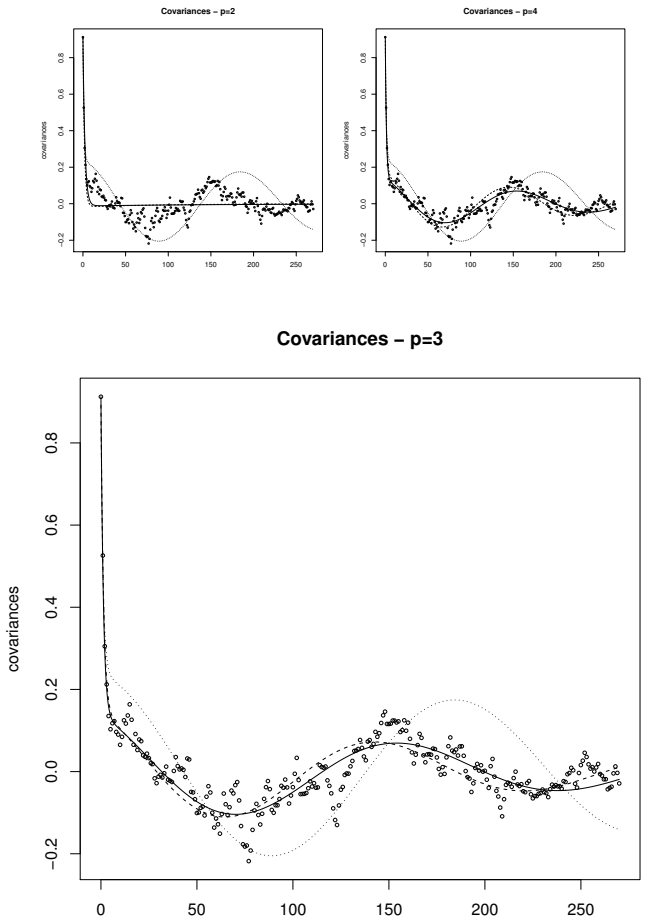

Figure 5: Empirical covariances (∘) and covariances of the MC (—) and ML (- - -) fitted OU models, for $p = 2, p = 4$ and $p = 3$, the actual value of the parameter, corresponding to Example 3. The covariances of $\mathrm{OU}_{\boldsymbol{\kappa}}$ are indicated with a dotted line.

L. Chiapella, A. Criado and C. Scavone. Their permission to analyze the data is gratefully acknowledged by the authors.

7.1 Box, Jenkins and Reinsel "Series A"

The Series A is a record of $n = 197$ chemical process concentration readings, taken every two hours, introduced with that name and analyzed in Chapter 4 of Box et al. [1994] (see also http://rgm2.lab.nig.ac.jp/RGM2/

Figure 6: Estimated interpolation and prediction of $x(t)$ for $199 < t < 200$ and $200 < t < 201$, respectively (- - -), 2σ confidence bands based on $(x(i))_{i=0,1,\ldots,200}$ $(\cdots)$, and a refinement of the simulation of $x(t)$ on $199 < t < 200$.

tfunc.php?rd_id=FitAR:SeriesA). The original data are plotted in Figure 7.

The diagrams in Figure 8 compare the empirical covariances of the series with the covariances of the estimated ARMA(p, q) process fitted by means

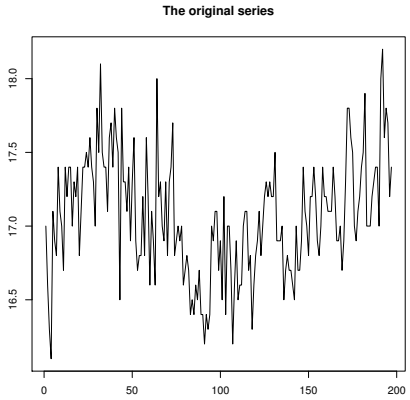

Figure 7: Series A

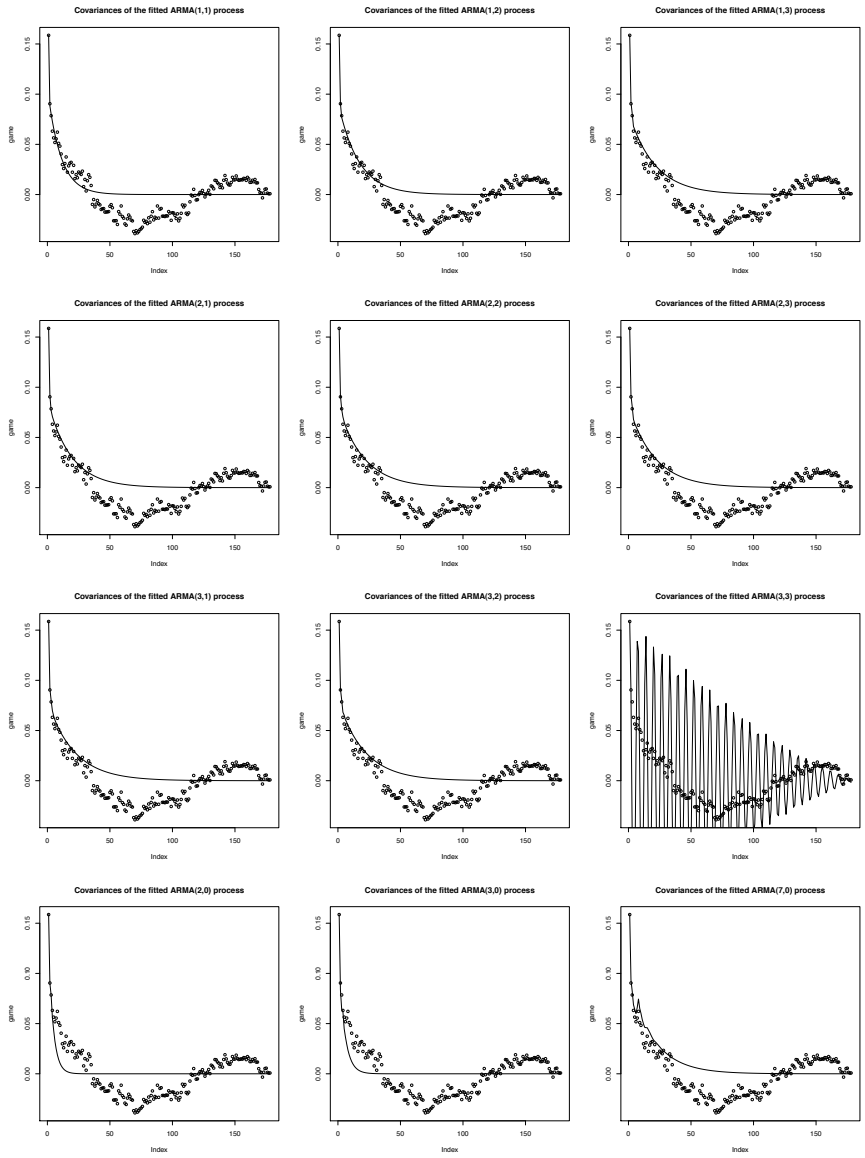

Figure 8: Empirical and ARMA-fitted covariances for Series A

of the `R` function `arima` for several values of p and q. In particular, the ARMA(1,1) is suggested as a model for this data in Box et al. [1994], and

subsets of AR(7) are proposed in Cleveland [1971] and McLeod and Zhang [2006] for the same purpose.

The ARMA(1,1) and the AR(7) fit fairly well the autocovariances for small lags, but fail to capture the structure of autocorrelations for large lags present in the series. However, the approximations obtained with the OU(3) process reflects both the short and long dependences, as shown in Figure 9.

It is interesting to consider jointly the ARMA(3,2) model (39) fitted to the original data by maximum likelihood (computed also with the `arima` function) and the ARMA(3,2) model (40) obtained by the procedure described in §5, corresponding to the OU(3) process also fitted to the data by maximum likelihood. The estimated parameters of this OU process are

$$\hat{\boldsymbol{\kappa}} = (0.8293, 0.0018 + 0.0330\imath, 0.0018 - 0.0330\imath) \quad \text{and} \quad \hat{\boldsymbol{\sigma}} = 0.4401$$

and the ARMA(3,2) processes are respectively

$$(1-0.7945B-0.3145B^2+0.1553B^3)x=0.3101(1-0.4269B-0.2959B^2)\epsilon \quad (39)$$

and

$$(1-2.4316B+1.8670B^2-0.4348B^3)x=0.4401(1-1.9675B+0.9685B^2)\epsilon. \quad (40)$$

The autocorrelations of both ARMA models, shown in Figure 10 together with the empirical correlations of the series were computed by means of the `R` function `ARMAacf`, although the ones corresponding to (40) could have been obtained as the restrictions to integer lags of the covariance function for continuous time described in §3.2. It is worth to notice that the autocorrelations of (39) do not approach the empirical correlations, indicated by circles, as much as the correlations of (40). The logarithms of the likelihoods of (39) and (40) are $\ell' = -49.23$, and $\ell'' = -50.95$, respectively. But since the number of parameters of the second model (which is four) is smaller than the number of parameters of the complete family of ARMA(3,2) processes (six), the Akaike Information Criterion (AIC) of the parsimonious OU model is $8 - 2\ell'' = 109.90$, slightly better than the AIC of the unrestricted ARMA model, equal to $12 - 2\ell' = 110.46$.

Finally we show in Figure 11 the predicted values of the continuous parameter process $x(t)$, for t between $n - 7$ and $n + 4$ (190-201), obtained as the best linear predictions based on the last 90 observed values, and on the correlations given by the fitted OU(3) model. The upper and lower lines are 2σ-confidence limits for each value of the process.

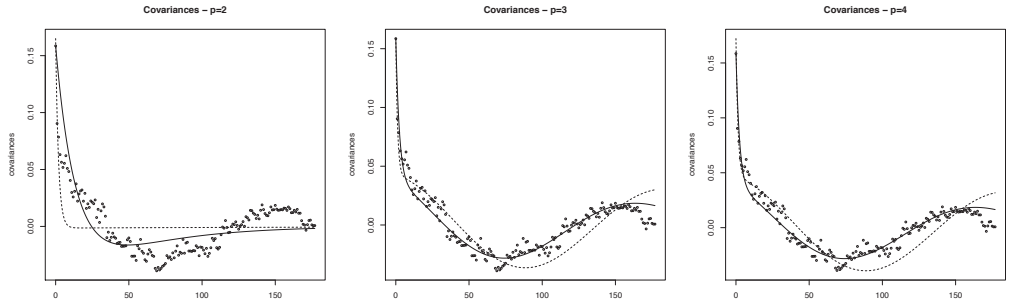

Figure 9: Empirical covariances (∘) and covariances of the MC (—) and ML (- - -) fitted OU(p) models, for $p = 2, 3, 4$ corresponding to Series A.

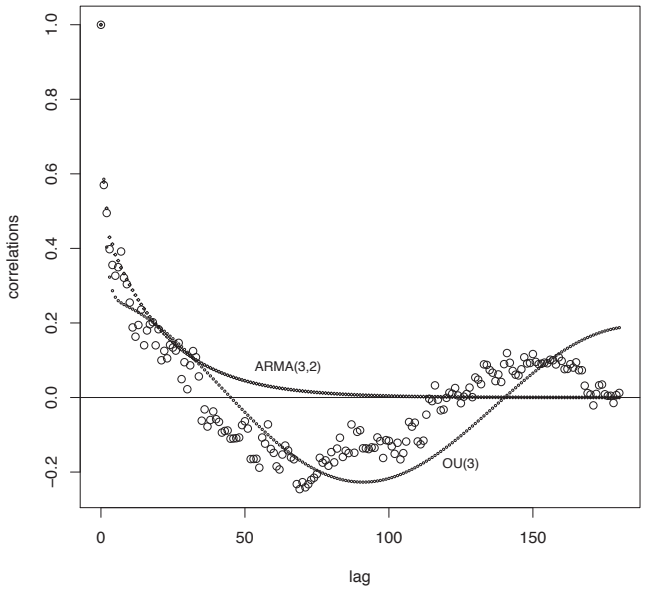

Figure 10: Empirical correlations (∘) of Series A, and autocorrelations of models (39) and (40) fitted by maximum likelihood from the family of all ARMA(3,2) and the restricted family of ARMA(3,2) derived from OU(3).

7.2 Box, Jenkins and Reinsel Series C

The Series C is a record of $n = 226$ chemical process temperature readings, taken every minute, introduced with that name in Box et al. [1994], p. 544.

As in the previous example, the fitted ARMA(p, q) and ARIMA$(p, 1, q)$ models for moderate values of p and q fail to capture the autocorrelations that might be present in the series. Figure 12 shows the empirical covariances of the series and the covariances of the MC (—) and ML (- - -) fitted OU(p) models for $p = 2$, $p = 3$ and $p = 4$. It is not surprising that the MC estimated covariances fit better than the ML ones the empirical covariances, since they have been obtained by optimizing that fit. The poor performance of the ML estimation is presumably due to the fact that the series does not obey an OU model. The corresponding graphs for the first differences of Series C are included in Figure 13.

7.3 Oxygen saturation in blood

The oxygen saturation in blood of a newborn child has been monitored during seventeen hours, and measures taken every two seconds. We assume that a series $x_0, x_1, \ldots, x_{304}$ of measures taken at intervals of 200 seconds is observed, and fit OU processes of orders $p = 2, 3, 4$ to that series.

Again the empirical covariances of the series and the covariances of the

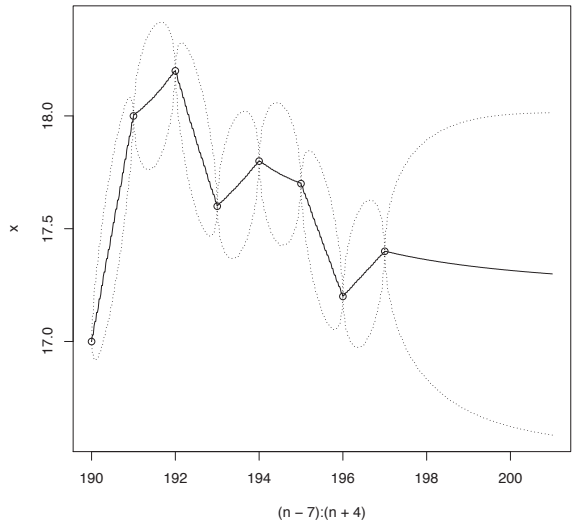

Figure 11: Confidence bands for interpolated and extrapolated values of Series A for continuous domain.

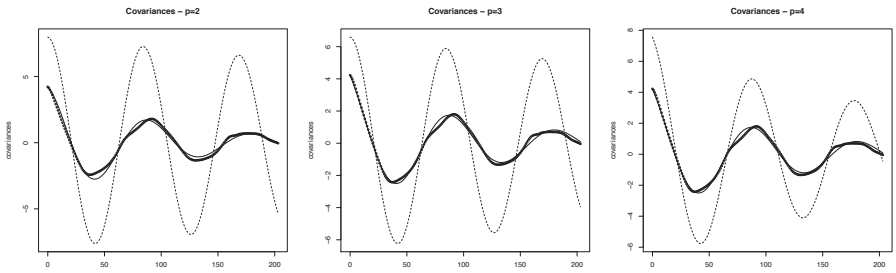

Figure 12: Empirical covariances (∘) and covariances of the MC (—) and ML (- - -) fitted OU(p) models for $p = 2, 3, 4$ corresponding to Series C.

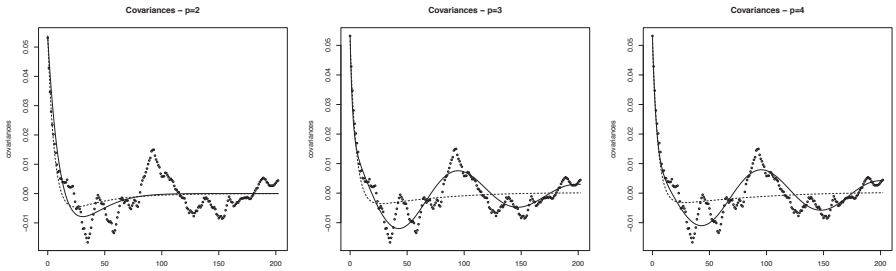

Figure 13: Empirical covariances (∘) and covariances of the MC (—) and ML (- - -) fitted OU(p) models for $p = 2, 3, 4$ corresponding to the first differences of Series C.

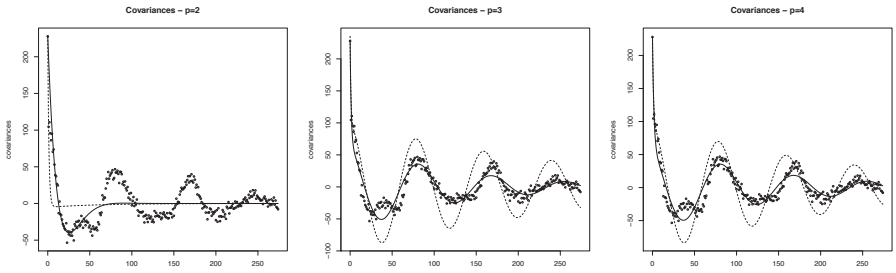

Figure 14: Empirical covariances (∘) and covariances of the MC (—) and ML (- - -) fitted OU(p) models for $p = 2, 3, 4$ corresponding to the series of O_2 saturation in blood.

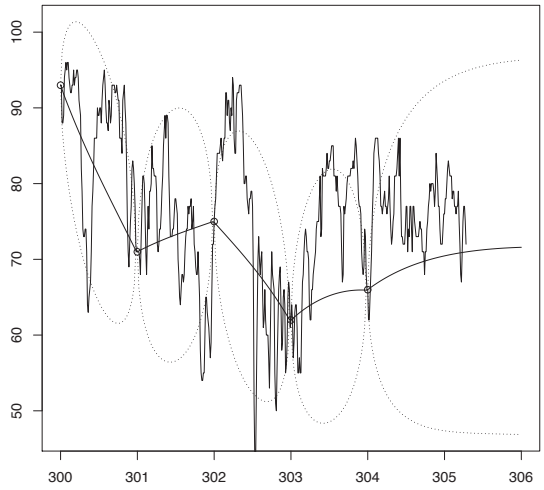

Figure 15: Partial graph showing the five last values of the series of O_2 saturation in blood at integer multiples of the 200 seconds unit of time ($\circ$), interpolated and extrapolated predictions (—), 2σ confidence bands (- - -), and actual values of the series.

fitted OU(p) models for $p = 2$, $p = 3$ and $p = 4$ are plotted (see Figure 14) and the estimated interpolation and extrapolation are shown in Figure 15. In the present case, the actual values of the series for integer multiples of 1/100 of the unit measure of 200 seconds are known, and plotted in the same figure.

8 Conclusions

We have proposed a family of continuous time stationary processes, OU(p), for each positive integer p, based on p iterations of the linear operator that maps a Wiener process onto an Ornstein-Uhlenbeck process. These operators have some nice properties, such as being commutative, and their p-compositions decompose as a linear combination of simple operators of the same kind. We remark that this last result, stated in Theorem 1, is independent of the process onto which the operators $\mathcal{OU}_{\boldsymbol{\kappa}}$ act on. We have preferred

to reduce the present scope of the applications envisaged by applying the operators only to Wiener processes, but other choices deserve consideration, for example, the results of applying the same operators to fractional Brownian motions.

An OU(p) process depends on $p+1$ parameters that can be easily estimated by either maximum likelihood (ML) or matching correlations (MC) procedures. Matching correlation estimators provide a fair estimation of the covariances of the data, even if the model is not well specified.

When sampled on equally spaced instants, the OU(p) family can be written as a discrete time state space model, namely, a VARMA model in a space of dimension p. As a consequence, the families of OU(p) models are a parsimonious subfamily of the ARMA($p, p-1$) processes. Furthermore, the coefficients of the ARMA can be deduced from those of the corresponding OU(p).

We have shown examples for which the ML-estimated OU model is able to capture a long term dependence that the ML-estimated ARMA model does not show. This leads to recommend the inclusion of OU models as candidates to represent stationary series to the users interested in such kind of dependence.

Appendix : Derivation of the state space model

The form of the equations for the state space representation of the OU(p) equations in the general case can be derived by considering three special cases:

1. When the components of $\boldsymbol{\kappa}$ are all different. This case is treated in §4.

2. When he components of $\boldsymbol{\kappa}$ are all equal.

   Let κ denote the common value of the components of $\boldsymbol{\kappa}$. The state of the system is described by the vector

   $$\boldsymbol{\xi}_{\kappa,p} = (\xi_\kappa^{(0)}, \xi_\kappa^{(1)}, \ldots, \xi_\kappa^{(p-1)})^{\mathrm{tr}},$$

   with components $\xi_\kappa^{(h)}(t) = \sigma \displaystyle\int_{-\infty}^{t} \mathrm{e}^{-\kappa(t-s)} \frac{(-\kappa(t-s))^h}{h!} dw(s).$

Each of these terms can be written as the sum

$$\xi_\kappa^{(h)}(t) = \sigma \mathrm{e}^{-\kappa} \int_{-\infty}^{t-1} \mathrm{e}^{-\kappa(t-1-s)} \frac{(-\kappa(t-1-s+1))^h}{h!} dw(s) + \eta_{\kappa,h}(t) \tag{41}$$

where $\eta_{\kappa,h}(t) = \sigma \int_{t-1}^{t} \mathrm{e}^{-\kappa(t-s)} \frac{(-\kappa(t-s))^h}{h!} dw(s)$.

The first term in the right-hand side of (41) is equal to

$$\sigma \mathrm{e}^{-\kappa} \sum_{j=0}^{h} \frac{(-\kappa)^{h-j}}{(h-j)!} \int_{-\infty}^{t-1} \mathrm{e}^{-\kappa(t-1-s)} \frac{(-\kappa(t-1-s))^j}{j!} dw(s)$$

$$= \mathrm{e}^{-\kappa} \sum_{j=0}^{h} \frac{(-\kappa)^{h-j}}{(h-j)!} \xi_\kappa^{(j)}(t-1)$$

and therefore, by introducing the matrix

$$A_{\kappa,p} = \mathrm{e}^{-\kappa} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{(-\kappa)}{1!} & 1 & 0 & \dots & 0 & 0 \\ \frac{(-\kappa)^2}{2!} & \frac{(-\kappa)}{1!} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{(-\kappa)^{p-2}}{(p-2)!} & \frac{(-\kappa)^{p-3}}{(p-3)!} & \frac{(-\kappa)^{p-4}}{(p-4)!} & \dots & 1 & 0 \\ \frac{(-\kappa)^{p-1}}{(p-1)!} & \frac{(-\kappa)^{p-2}}{(p-2)!} & \frac{(-\kappa)^{p-3}}{(p-3)!} & \dots & \frac{(-\kappa)}{1!} & 1 \end{pmatrix}$$

we may write

$$\boldsymbol{\xi}_{\kappa,p}(t) = A_{\kappa,p} \boldsymbol{\xi}_{\kappa,p}(t-1) + \boldsymbol{\eta}_{\kappa,p}$$

where

$$\boldsymbol{\eta}_{\kappa,p}(t) = (\eta_{\kappa,0}(t), \eta_{\kappa,1}(t), \dots, \eta_{\kappa,p-1}(t))^{\mathrm{tr}}$$

is a vector of centered Gaussian innovations (independent of the σ-algebra generated by $\{w(s) : s \leq t-1\}$) with covariance matrix $B_{\kappa,\kappa,p}$ obtained with $\kappa_1 = \kappa_2$ and $p_1 = p_2$ from the general expression of the $p_1 \times p_2$ matrix $B_{\kappa_1,\kappa_2,p_1,p_2} = ((b_{\kappa_1,\kappa_2,h_1,h_2}))_{1 \leq h_1 \leq p_1, 1 \leq h_2 \leq p_2}$, where

$$\begin{aligned} b_{\kappa_1,\kappa_2,h_1,h_2} &= \mathbf{E} \eta_{\kappa_1,h_1}(t) \bar{\eta}_{\kappa_2,h_2}(t) \\ &= \sigma^2 \int_{t-1}^{t} \mathrm{e}^{-(\kappa_1+\bar{\kappa}_2)(t-s)} (-\kappa_1)^{h_1} (-\bar{\kappa}_2)^{h_2} (t-s)^{h_1+h_2} ds \\ &= \sigma^2 \int_0^1 \mathrm{e}^{-(\kappa_1+\bar{\kappa}_2)y} (-\kappa)^{h_1} (-\bar{\kappa})^{h_2} y^{h_1+h_2} dy. \end{aligned}$$

The equation $x(t) = \boldsymbol{K}_p^{\mathrm{tr}} \boldsymbol{\xi}_{\boldsymbol{\kappa},p}(t)$, with $\boldsymbol{K}_p^{\mathrm{tr}} = (\binom{p-1}{0}, \binom{p-1}{1}, \ldots, \binom{p-1}{p-1})$ completes the description of the system state dynamics.

3. The vector κ has components $\kappa_1 = \lambda + \mu\imath$ and $\kappa_2 = \lambda - \mu\imath$, $\mu \neq 0$, each repeated p_1 times.

   A description involving imaginary processes is immediate from the previous case. The equations

$$\begin{pmatrix} \boldsymbol{\xi}_{\kappa_1,p_1}(t) \\ \boldsymbol{\xi}_{\kappa_2,p_1}(t) \end{pmatrix} = \begin{pmatrix} A_{\kappa_1,p_1} & 0 \\ 0 & A_{\kappa_2,p_1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_{\kappa_1,p_1}(t-1) \\ \boldsymbol{\xi}_{\kappa_2,p_1}(t-1) \end{pmatrix} + \begin{pmatrix} \boldsymbol{\eta}_{\kappa_1,p_1} \\ \boldsymbol{\eta}_{\kappa_2,p_1} \end{pmatrix}$$

$$x(t) = (\boldsymbol{K}_{p_1}^{\mathrm{tr}}, \boldsymbol{K}_{p_1}^{\mathrm{tr}}) \begin{pmatrix} \boldsymbol{\xi}_{\kappa_1,p_1}(t) \\ \boldsymbol{\xi}_{\kappa_2,p_1}(t) \end{pmatrix}$$

   hold, and $\mathbf{Var} \begin{pmatrix} \boldsymbol{\eta}_{\kappa_1,p_1} \\ \boldsymbol{\eta}_{\kappa_2,p_1} \end{pmatrix} = \begin{pmatrix} B_{\kappa_1,\kappa_1,p_1,p_1} & B_{\kappa_1,\kappa_2,p_1,p_1} \\ B_{\kappa_2,\kappa_1,p_1,p_1} & B_{\kappa_1,\kappa_1,p_1,p_1} \end{pmatrix}$.

   A description in terms of real processes is obtained by multiplying the first equation by the matrix

$$C_{p_1} = \begin{pmatrix} I_{p_1} & I_{p_1} \\ -\imath I_{p_1} & \imath I_{p_1} \end{pmatrix}$$

   (where I_p denotes the $p \times p$ identity matrix), because the vectorial process $C_{p_1} \begin{pmatrix} \boldsymbol{\xi}_{\kappa_1,p_1}(t) \\ \boldsymbol{\xi}_{\kappa_2,p_1}(t) \end{pmatrix}$ has real components. The new equations are

$$C_{p_1} \begin{pmatrix} \boldsymbol{\xi}_{\kappa_1,p_1}(t) \\ \boldsymbol{\xi}_{\kappa_2,p_1}(t) \end{pmatrix} = \left(C_{p_1} \begin{pmatrix} A_{\kappa_1,p_1} & 0 \\ 0 & A_{\kappa_2,p_1} \end{pmatrix} C_{p_1}^{-1} \right) \times \left(C_{p_1} \begin{pmatrix} \boldsymbol{\xi}_{\kappa_1,p_1}(t-1) \\ \boldsymbol{\xi}_{\kappa_2,p_1}(t-1) \end{pmatrix} \right)$$

$$+ C_{p_1} \begin{pmatrix} \boldsymbol{\eta}_{\kappa_1,p_1} \\ \boldsymbol{\eta}_{\kappa_2,p_1} \end{pmatrix}$$

   and

$$x(t) = \left((\boldsymbol{K}_{p_1}^{\mathrm{tr}}, \boldsymbol{K}_{p_1}^{\mathrm{tr}}) C_{p_1}^{-1} \right) \times \left(C_{p_1} \begin{pmatrix} \boldsymbol{\xi}_{\kappa_1,p_1}(t) \\ \boldsymbol{\xi}_{\kappa_2,p_1}(t) \end{pmatrix} \right)$$

General case, real processes

Let us assume that $\kappa_1, \ldots, \kappa_q$ are distinct components of $\boldsymbol{\kappa}$, each repeated $p_1, \ldots, p_q$ times. We assume in addition that the imaginary components are

$\kappa_1, \kappa_2 = \bar{\kappa}_1, \ldots, \kappa_{2c-1}, \kappa_{2c} = \bar{\kappa}_{2c-1}$ and the remaining $\kappa_{2c+1}, \ldots, \kappa_q$ are real. With this notation, $p_{2h-1} = p_{2h}$ for $h = 1, 2, \ldots, c$.

We make intensive use of the notations introduced in previous cases to write

$$\begin{aligned} \boldsymbol{\xi}(t) &= A\boldsymbol{\xi}(t-1) + \boldsymbol{\eta}(t), \\ x(t) &= K^{\mathrm{tr}}\boldsymbol{\xi}(t) \end{aligned} \tag{42}$$

with

$$\boldsymbol{\xi}(t) = \begin{pmatrix} \boldsymbol{\xi}_{\kappa_1,p_2}(t) \\ \boldsymbol{\xi}_{\kappa_2,p_2}(t) \\ \boldsymbol{\xi}_{\kappa_3,p_4}(t) \\ \boldsymbol{\xi}_{\kappa_4,p_4}(t) \\ \cdots \\ \boldsymbol{\xi}_{\kappa_{2c-1},p_{2c}}(t) \\ \boldsymbol{\xi}_{\kappa_{2c},p_{2c}}(t) \\ \boldsymbol{\xi}_{\kappa_{2c+1},p_{2c+1}}(t) \\ \boldsymbol{\xi}_{\kappa_{2c+2},p_{2c+2}}(t) \\ \cdots \\ \boldsymbol{\xi}_{\kappa_q,p_q}(t) \end{pmatrix}, \boldsymbol{\eta}(t) = \begin{pmatrix} \boldsymbol{\eta}_{\kappa_1,p_2}(t) \\ \boldsymbol{\eta}_{\kappa_2,p_2}(t) \\ \boldsymbol{\eta}_{\kappa_3,p_4}(t) \\ \boldsymbol{\eta}_{\kappa_4,p_4}(t) \\ \cdots \\ \boldsymbol{\eta}_{\kappa_{2c-1},p_{2c}}(t) \\ \boldsymbol{\eta}_{\kappa_{2c},p_{2c}}(t) \\ \boldsymbol{\eta}_{\kappa_{2c+1},p_{2c+1}}(t) \\ \boldsymbol{\eta}_{\kappa_{2c+2},p_{2c+2}}(t) \\ \cdots \\ \boldsymbol{\eta}_{\kappa_q,p_q}(t) \end{pmatrix},$$

$$A = \begin{pmatrix} A_{\kappa_1,p_2} & 0 & \ldots & 0 \\ 0 & A_{\kappa_2,p_2} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & A_{\kappa_q,p_q} \end{pmatrix}$$

and

$$\boldsymbol{K}^{\mathrm{tr}} = (K^{\mathrm{tr}}_{\kappa_1,p_1}, K^{\mathrm{tr}}_{\kappa_2,p_2}, \ldots, K^{\mathrm{tr}}_{\kappa_q,p_q}).$$

The real version, when the process $\boldsymbol{\xi}$ has imaginary components is obtained by multiplying (42) by the matrix

$$C = \begin{pmatrix} C_{p_2} & 0 & \ldots & 0 & 0 \\ 0 & C_{p_4} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & C_{p_{2c}} & 0 \\ 0 & 0 & \ldots & 0 & I_{p_{2c+1}+\cdots+p_q} \end{pmatrix} \tag{43}$$

thus obtaining

$$C\boldsymbol{\xi}(t) = (CAC^{-1}) \times (C\boldsymbol{\xi}(t-1)) + C\boldsymbol{\eta}(t), \tag{44}$$

$$x(t) = (\boldsymbol{K}^{\mathrm{tr}}C^{-1}) \times (C\boldsymbol{\xi}(t)). \tag{45}$$

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