Recursive Path Orderings can also be Incremental

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Abstract. In this paper the *Recursive Path Ordering* is adapted for proving termination of rewriting incrementally. The new ordering, called *Recursive Path Ordering with Modules*, has as ingredients not only a precedence but also an underlying ordering $\sqsupset_{\mathcal{B}}$. It can be used for incremental (innermost) termination proofs of hierarchical unions by defining $\sqsupset_{\mathcal{B}}$ as an extension of the termination proof obtained for the base system. Furthermore, there are practical situations in which such proofs can be done modularly.

1 Introduction

Term rewriting provides a simple (but Turing-complete) model for symbolic computation. A *term rewrite system* (TRS) is just a binary relation over the set of terms of a given signature. The pairs of the relation are used for computing by replacements until an irreducible term is eventually reached. Hence, the absence of infinite sequences of replacements, called *termination*, is a fundamental (though undecidable) property for most applications of rewriting in program verification and automated reasoning. For program verification, the termination of a particular rewriting strategy called innermost termination has special interest. In this strategy the replacements are performed inside-out, i.e. arguments are fully reduced before reducing the function. Therefore, it corresponds to the *"call by value"* computation rule of programming languages. This strategy is also important because for certain classes of TRSs, innermost termination and termination coincide [12].

Term rewrite systems are usually defined in hierarchies. This hierarchical structure is very important when reasoning about TRS properties in an incremental manner. Roughly, a property P is proved incrementally for a hierarchical TRS $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ if we can prove it by using information from the proof of P for the base system $\mathcal{R}_0$. The simplest form of incrementality is modularity, i.e. proving P for $\mathcal{R}$ just by proving P for $\mathcal{R}_0$ and $\mathcal{R}_1$ independently. However, termination is not a modular property even for disjoint unions of TRSs [21]. A stronger form of termination, called $\mathcal{C}_{\mathcal{E}}$-termination, and innermost termination, are indeed modular for a restricted class of hierarchical unions [17, 14], but not in general. Therefore, it is of great importance to tackle (innermost) termination of hierarchical systems using an incremental approach.

Regardless the previous facts, the problem of ensuring termination of a hierarchical union without finding (if possible) an alternate proof for the base system has received quite few attention. The first and important step was done by Urbain in [22]. He showed that from the knowledge that a base system is $\mathcal{C}_{\mathcal{E}}$-terminating, the conditions for the termination proof of a hierarchical union can be relaxed. In the context of the *Dependency Pair method* (DP) [1] (the most successful for termination of rewriting) this entails a significant reduction in the number and the strictness of the DP-constraints. Very recently, Urbain's contribution was used for improving the application of the *Size-Change Principle* (SCP) [15] to $\mathcal{C}_{\mathcal{E}}$-termination of rewriting [10]. In the latter paper it was shown that a termination measure for a base system $\mathcal{R}_0$ can be used for proving size-change termination of a hierarchical extension $\mathcal{R}_1$, and this guarantees $\mathcal{R}_0 \cup \mathcal{R}_1$ is $\mathcal{C}_{\mathcal{E}}$-terminating. Using this result, the next TRS is easily (and even modularly) proved simply terminating.

Example 1. The following hierarchical union ($\mathcal{R}_{plus}$ is taken from [19]) can be used for computing Sudan's function[3].

$$\mathcal{R}_{plus} = \begin{cases} plus(s(s(x)), y) & \to & s(plus(x, s(y))) \\ plus(x, s(s(y))) & \to & s(plus(s(x), y)) \\ plus(s(0), y) & \to & s(y) \\ plus(0, y) & \to & y \end{cases}$$

$$\mathcal{R}_F = \begin{cases} F(0, x, y) & \to & plus(x, y) \\ F(s(n), x, 0) & \to & x \\ F(s(n), x, s(y)) & \to & F(n, F(s(n), x, y), s(plus(F(s(n), x, y), y))) \end{cases}$$

In order to prove termination of $\mathcal{R} = \mathcal{R}_{plus} \cup \mathcal{R}_F$ (when using the DP-approach) the whole union must be included in some (quasi-) ordering. But $\mathcal{R}_{plus}$ requires semantical comparisons while $\mathcal{R}_F$ needs lexicographic ones. Therefore, no (quasi-) ordering traditionally used for automated proofs serves for this purpose. However, simple termination of $\mathcal{R}_{plus}$ is easy to prove e.g. using the *Knuth-Bendix Ordering* (KBO) [3]. Besides, every size-change graph of $\mathcal{R}_F$ decreases w.r.t. the lexicographic extension of KBO. Thus, $\mathcal{R}_F$ is size-change terminating w.r.t. KBO and we conclude $\mathcal{R}$ is simply terminating.

SCP provides a more general comparison than lexicographic and multiset ones. But it has as main drawback that it cannot compare *defined* function symbols (i.e. those appearing as root of left-hand sides) syntactically.

[3] Chronologically, Sudan's function [7] is the first example of a recursive but not primitive recursive function. Sudan's function $F(p, m, n)$ is greater than or equal to Ackermann's function $A(p, m, n)$ except at the single point $(2, 0, 0)$. The latter was used in [19] combined with $\mathcal{R}_{plus}$.

Example 2. Let $\mathcal{R}_{F'} = \mathcal{R}_F \cup \{F(s(n), F(s(n), x, y), z) \to F(s(n), x, F(n, y, z))\}$. The new rule [6][Lemma 6.7, page 47] can be used for computing an upper bound of the left-hand side while decreasing the size of the term. But now SCP fails in proving termination of $\mathcal{R}_{plus} \cup \mathcal{R}_{F'}$. This is due to the new rule which demands a lexicographic comparison determined by a subterm rooted with the defined symbol F.

When dealing with defined symbols, SCP cannot compete with classical syntactical orderings like the *Recursive Path Ordering* [8]. Therefore, it would be nice to adapt RPO in order to prove termination of $\mathcal{R}_{plus} \cup \mathcal{R}_{F'}$ and other hierarchical systems incrementally.

In this paper we present a new RPO-like ordering which can be used for these purposes, called the *Recursive Path Ordering with Modules* (RPOM). It has as ingredients not only a precedence, but also an *underlying ordering* $\sqsupset_{\mathcal{B}}$.

Actually RPOM defines a class of orderings that can be partitioned into three subclasses, RPOM-STAB, RPOM-MON and RPOM-IP-MON, where, under certain conditions, the first one contains stable orderings, the second one contains monotonic orderings (or a weak form of monotonocity related to $\sqsupset_{\mathcal{B}}$), and the third one contains IP-monotonic orderings.

We use these orderings for proving $\mathcal{C}_{\mathcal{E}}$-termination and innermost termination of a hierarchical union $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ incrementally as follows. The system $\mathcal{R}_0$ is known terminating, and perhaps an ordering $\succ_{\mathcal{B}}$ including the relation $\to_{\mathcal{R}_0}$ on terms of $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$ is given. An ordering $\sqsupset_{\mathcal{B}}$ is then constructed, perhaps as an extension of $\succ_{\mathcal{B}}$ to $\mathcal{T}(\mathcal{F}, \mathcal{X})$, or perhaps independently of the possible $\succ_{\mathcal{B}}$. Three orderings from RPOM-STAB, RPOM-MON and RPOM-IP-MON are then obtained from $\sqsupset_{\mathcal{B}}$, satisfying that the one in RPOM-STAB is included into the one in RPOM-MON under some conditions on $\sqsupset_{\mathcal{B}}$ and $\mathcal{R}_0$, and into the one in RPOM-IP-MON under weaker conditions. Including $\mathcal{R}_1$ in RPOM-STAB will then allow to prove $\mathcal{C}_{\mathcal{E}}$-termination or innermost termination of $\mathcal{R}$ depending on the original properties of $\sqsupset_{\mathcal{B}}$ and $\mathcal{R}_0$. Note that, in the case of innermost termination, no condition at all is imposed on $\sqsupset_{\mathcal{B}}$ and $\mathcal{R}_0$.

Our results are a first step towards the definition of a general framework for combining and extending different termination proof methods (this idea of combining ordering methods was early considered in [18]), and thus obtain termination proofs of hierarchical unions of TRS's whose modules have been proved using different techniques. As a first step, since based on RPO, these results are still weak to compete with the recent refinements of the DP method in [16, 20]. However, we believe that the extension of these results to more powerful path orderings, like the *Monotonic Semantic Path Ordering* in [5], will provide fairer comparison.

The remainder of the paper is organized as follows. In Section 2 we review basic notation, terminology and results. In Section 3 we define RPOM and prove its properties, and the ones corresponding to every subclass RPOM-STAB, RPOM-MON and RPOM-IP-MON. In Section 4 (resp. Section 5) we show how to use RPOM for proving $\mathcal{C}_{\mathcal{E}}$-termination (resp. innermost termination) incrementally. We present some concluding remarks in Section 6.

2 Preliminaries

We assume familiarity with the basics of term rewriting (see e.g. [2]).

The set of terms over a signature $\mathcal{F}$ is denoted as $\mathcal{T}(\mathcal{F}, \mathcal{X})$, where $\mathcal{X}$ represents a set of variables. The symbol labelling the root of a term t is denoted as $root(t)$. The root position is denoted by λ. The set of positions of t is denoted by $\mathcal{P}os(t)$. The *subterm* of t at position p is denoted as $t|_p$ and $t \rhd t|_p$ denotes the *proper subterm relation*. A *context*, i.e. a term with a hole, is denoted as $t[\ ]$. The term t with the hole replaced by s is denoted as $t[s]$, and the term $t[s]_p$ obtained by replacing $t|_p$ by s is defined in the standard way. For example, if t is $f(a, g(b, h(c)), d)$, then $t|_{2.2.1} = c$, and $t[d]_{2.2} = f(a, g(b, d), d)$. We denote $t[s_1]_{p_1}[s_2]_{p_2} \ldots [s_n]_{p_n}$ by $t[s_1, s_2, \ldots, s_n]_{p_1,p_2,\ldots,p_n}$. We write $p_1 > p_2$ (or, $p_2 < p_1$) if p_2 is a proper prefix of p_1. In this case we say that p_2 is *above* p_1, or that p_1 is *below* p_2. We will usually denote a term $f(t_1, \ldots, t_n)$ by the simplified form $ft_1 \ldots t_n$.

The notation $\bar{t}$ is ambiguously used to denote either the tuple $(t_1, \ldots, t_n)$ or the multiset $\{t_1, \ldots, t_n\}$, even in case of $t = f(t_1, \ldots, t_n)$. The number of symbols of t is denoted as $|t|$ while $|\bar{t}|$ denotes the number of elements in $\bar{t}$. Substitutions are denoted with the letter σ. A substitution application is written in postfix notation.

We say that a binary relation $\sqsupset$ on terms is *variable preserving* if $s \sqsupset t$ implies that every variable in t occurs in s. It is said that $\sqsupset$ is *non-duplicating* if $s \sqsupset t$ implies that every variable in t occurs at most as often as in s. If $s \sqsupset t$ implies $s\sigma \sqsupset t\sigma$ then $\sqsupset$ is *stable*. If for every function symbol f, $s \sqsupset t$ implies $f(\ldots s \ldots) \sqsupset f(\ldots t \ldots)$ then $\sqsupset$ is *monotonic*. It is said that a relation $\sqsupset$ is *well-founded* if there is no infinite sequence $s_1 \sqsupset s_2 \sqsupset s_3 \sqsupset \ldots$. The transitive and the reflexive-transitive closure of $\sqsupset$ are denoted as $\sqsupset^+$ and $\sqsupset^*$ resp. The union of $\sqsupset$ and the syntactical equality $\equiv$ is denoted as $\sqsupseteq$. We say that $\sqsupset$ is *compatible* with $\sqsupset'$ if $\sqsupset \circ \sqsupset' \subseteq \sqsupset$ and $\sqsupset' \circ \sqsupset \subseteq \sqsupset$.

A (strict partial) *ordering* on terms is an irreflexive transitive relation. A *reduction ordering* is a monotonic, stable and well-founded ordering. A *simplification ordering* is a reduction ordering including the strict subterm relation. A *precedence* $\succsim_{\mathcal{F}}$ over $\mathcal{F}$ is the union of a well-founded ordering $\succ_{\mathcal{F}}$ and a compatible equivalence relation $\approx_{\mathcal{F}}$. We say that a precedence $\succsim_{\mathcal{F}}$ is compatible with a partition of $\mathcal{F}$ if $f \approx_{\mathcal{F}} g$ implies that f and g belongs to the same part of $\mathcal{F}$.

The *multiset extension* of an ordering $\sqsupset$ on terms to multisets, denoted as $\sqsupset^{mul}$, is defined as $\bar{s} \sqsupset^{mul} \bar{t}$ iff there exists $\bar{u} \subset \bar{s}$ such that $\bar{u} \subseteq \bar{t}$ and for all $t' \in \bar{t} - \bar{u}$ there is some $s' \in \bar{s} - \bar{u}$ s.t. $s' \sqsupset t'$. The *lexicographic extension* of $\sqsupset$ to tuples, denoted as $\sqsupset^{lex}$, is defined as $\bar{s} \sqsupset^{lex} \bar{t}$ iff $s_i \sqsupset t_i$ for some $1 \leq i \leq |\bar{s}|$ and $s_j \equiv t_j$ for all $1 \leq j < i$. These extensions preserve irreflexivity, transitivity, stability and well-foundedness.

If $\sqsupset$ is defined on $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$ and $\mathcal{F}_0 \subset \mathcal{F}$ then $\sqsupset^{\mathcal{F}} = \{(s\sigma, t\sigma) \mid s \sqsupset t, \forall x \in \mathcal{X}, x\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{X})\}$ is called the *stable extension* of $\sqsupset$ to $\mathcal{F}$. The stable extension of a stable (and well-founded) ordering is also a stable (and well-founded) ordering [19].

A *term rewrite system* over $\mathcal{F}$ is denoted as $\mathcal{R}$. Here, we deal with variable preserving TRSs. Regarding termination, this restriction is not a severe one. A *rewriting step* with $\mathcal{R}$ is written as $s \to_{\mathcal{R}} t$. The notation $s \to_{\lambda,\mathcal{R}} t$ is used for a rewriting step at position λ.

A TRS $\mathcal{R}$ is *terminating* iff $\to_{\mathcal{R}}^{+}$ is well-founded. It is said that $\mathcal{R}$ is *simply terminating* iff $\mathcal{R} \cup \mathcal{E}mb_{\mathcal{F}}$ is terminating where $\mathcal{E}mb_{\mathcal{F}} = (\mathcal{F}, \{f(x_1, \ldots, x_n) \to x_k \mid f \in \mathcal{F}, 1 \leq k \leq n\})$ and $x_1, \ldots, x_n$ are pairwise distinct variables. It is said that $\mathcal{R}$ is *$\mathcal{C}_{\mathcal{E}}$-terminating* iff $\mathcal{R}_{\mathcal{E}} = \mathcal{R} \cup \mathcal{C}_{\mathcal{E}}$ is terminating, where $\mathcal{C}_{\mathcal{E}} = (\mathcal{G}, \{G(x, y) \to x, G(x, y) \to y\})$ and $\mathcal{G} = \mathcal{F} \uplus \{G\}$.

Given a TRS $\mathcal{R}$, $f(t_1, \ldots, t_n)$ is said to be *argument normalized* if for all $1 \leq k \leq n$, t_k is a normal form. A substitution σ is said to be *normalized* if $x\sigma$ is a normal form for all $x \in \mathcal{X}$. An *innermost redex* is an argument normalized redex. A term s *rewrites innermost* to t w.r.t. $\mathcal{R}$, written $s \to_i t$, iff $s \to t$ at position p and $s|_p$ is an innermost redex. A term s *rewrites innermost in parallel* to t w.r.t. $\mathcal{R}$, written $s \mathrel{+\!\!\!\!\!\!\!\!\;\;+\!\!\!\!\to}_{i,\mathcal{R}} t$, iff $s \to_{i,\mathcal{R}}^{+} t$ and either $s \to_{i,\mathcal{R}} t$ at position λ (denoted as $s \to_{i,\lambda,\mathcal{R}}$) or $s = f(\bar{s})$, $t = f(\bar{t})$ and for all $1 \leq k \leq |\bar{s}|$ either $s_k \mathrel{+\!\!\!\!\!\!\!\!\;\;+\!\!\!\!\to}_{i,\mathcal{R}} t_k$ or $s_k = t_k$ is a normal form (denoted as $\bar{s} \mathrel{+\!\!\!\!\!\!\!\!\;\;+\!\!\!\!\to}_{i,\mathcal{R}} \bar{t}$). A binary relation $\sqsupset$ is *IP-monotonic* w.r.t. $\mathcal{R}$ iff $\mathrel{+\!\!\!\!\!\!\!\!\;\;+\!\!\!\!\to}_{i,\mathcal{R}} \subseteq \sqsupset$ [11].

A TRS $\mathcal{R}$ is *innermost terminating* iff $\to_{i,\mathcal{R}}^{+}$ is well-founded. Alternatively, we have the following characterization for innermost termination.

Theorem 1. *[11] A TRS $\mathcal{R}$ is innermost terminating iff there exists a well-founded relation which is IP-monotonic w.r.t. $\mathcal{R}$.*

The *defined* symbols of a TRS $\mathcal{R}$ are $\mathcal{D} = \{root(l) \mid l \to r \in \mathcal{R}\}$ and the *constructors* are $\mathcal{C} = \mathcal{F} - \mathcal{D}$. The union $\mathcal{R}_0 \cup \mathcal{R}_1$ is said to be *hierarchical* if $\mathcal{F}_0 \cap \mathcal{D}_1 = \emptyset$.

3 RPOM

In this section we define RPOM in terms of an underlying ordering $\sqsupset_{\mathcal{B}}$ and show that it is an ordering. Moreover, we prove that well-foundedness of $\sqsupset_{\mathcal{B}}$ implies well-foundedness of RPOM.

Actually RPOM defines a class of orderings that depends on three parameters. These are the underlying ordering $\sqsupset_{\mathcal{B}}$, the (usual in RPO and other orderings) statusses of the symbols in the signature, and a last parameter $mc \in \{rmul, mul, set\}$. Due to mc, this class of orderings can be partitioned into three subclasses, RPOM-STAB, RPOM-MON and RPOM-IP-MON, where, under certain conditions, the first one contains stable orderings, the second one contains monotonic orderings (or a weak form of monotonocity related to $\sqsupset_{\mathcal{B}}$), and the third one contains IP-monotonic orderings. At the end of this section we prove the corresponding properties to every subclass.

Before going into the definition of RPOM, we need some additional notation. Apart from the multiset extension $\sqsupset^{mul}$ of an ordering $\sqsupset$ defined in the preliminaries we need two other extensions of orderings to multisets: the *set* extension and the *rmul* extension.

Definition 1. *Let $\sqsupset$ be an arbitrary ordering. Given two multisets S and T, $S \sqsupset^{set} T$ if $S' \sqsupset^{mul} T'$, where S' and T' are obtained from S and T, respectively, by removing repetitions. $S \sqsupset^{rmul} T$ if $S \neq \emptyset$ and for all $t \in T$ there is some $s \in S$ such that $s \sqsupset t$.*

It is easy to see that the relation $\sqsupset^{rmul}$ is included into $\sqsupset^{mul}$ and $\sqsupset^{set}$, and that it preserves irreflexivity, transitivity, stability and well-foundedness, whereas $\sqsupset^{set}$ preserves all these properties except for stability.

We will use the notation $\supseteq_{set}$ and $\supseteq_{mul}$ for denoting the inclusion in the sense of sets and multisets, respectively, in the cases where $\supseteq$ alone is not clear by the context. For facility of notations, we identify $\supseteq_{rmul}$ with $\supseteq_{set}$.

The ordering $RPOM$ is defined as the union of the underlying ordering $\sqsupset_{\mathcal{B}}$, and a RPO-like ordering $\succ$. Hence, we need a definition of $\succ$ not in contradiction (or even more, compatible) with $\sqsupset_{\mathcal{B}}$. Since $\sqsupset_{\mathcal{B}}$ will be generally obtained as an extension of an ordering $\succ_{\mathcal{B}}$ on the base signature $\mathcal{B} = \mathcal{F}_0$, it seems natural to demand this ordering to relate pairs of terms where at least one is rooted by a base symbol (i.e. a symbol in $\mathcal{B}$), but as we see as follows, a more strict condition is needed for $\sqsupset_{\mathcal{B}}$.

The definition of $s \succ t$ differs depending on if the roots of s and t are or not in $\mathcal{B}$. If no root is in $\mathcal{B}$, then we use a classical RPO-like recursive definition. If some root is in $\mathcal{B}$, we eliminate all the context containing symbols of $\mathcal{B}$, resulting in two multisets, and compare them with the corresponding extension $\succ^{rmul}$, $\succ^{mul}$ or $\succ^{set}$.

Definition 2. *Given a signature $\mathcal{B}$, we say that p is a* frontier position *and $t|_p$ is a* frontier term occurrence *of t if $root(t|_p) \notin \mathcal{B}$ and $root(t|_{p'}) \in \mathcal{B}$, for all $p' < p$. The multiset of all frontier subterm occurrences of t is denoted as $frt_{\mathcal{B}}(t)$*[4]*.*

For example, if $\mathcal{B} = \{f\}$, then $frt_{\mathcal{B}}(f(g(a), f(g(f(g(a), g(b))), g(a)))$ is $\{g(a), g(f(g(a), g(b))), g(a)\}$.

If we want $frt_{\mathcal{B}}(s) \succ^{rmul} frt_{\mathcal{B}}(t)$ or $frt_{\mathcal{B}}(s) \succ^{mul} frt_{\mathcal{B}}(t)$ or $frt_{\mathcal{B}}(s) \succ^{set} frt_{\mathcal{B}}(t)$ to be not in contradiction with $\sqsupset_{\mathcal{B}}$, it is necessary to demand that $s \sqsupset_{\mathcal{B}} t$ implies $frt_{\mathcal{B}}(s) \supseteq_{rmul} frt_{\mathcal{B}}(t)$ or $frt_{\mathcal{B}}(t) \supseteq_{mul} frt_{\mathcal{B}}(t)$ or $frt_{\mathcal{B}}(s) \supseteq_{set} frt_{\mathcal{B}}(t)$, depending on the case. We call frontier preserving (w.r.t. $rmul$, mul or set) to this property.

Definition 3. *Let $mc \in \{mul, rmul, set\}$, $\mathcal{B} \subset \mathcal{F}$ and $\succsim_{\mathcal{F}}$ be a precedence over $\mathcal{F} - \mathcal{B}$ compatible with the partition of $\mathcal{F} - \mathcal{B}$, $\mathcal{F}_{\mathcal{M}ul} \uplus \mathcal{F}_{\mathcal{L}ex}$. Moreover, let $\sqsupset_{\mathcal{B}}$ be an ordering on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ s.t. $s \sqsupset_{\mathcal{B}} t$ implies $frt_{\mathcal{B}}(s) \supseteq_{mc} frt_{\mathcal{B}}(t)$. Then, the* Recursive Path Ordering with Modules (RPOM) *is defined as $\succ_{rpom} = \sqsupset_{\mathcal{B}} \cup \succ$ where $s = f(\bar{s}) \succ t$ iff one of the following conditions holds:*

1. *$f, root(t) \notin \mathcal{B}$ and $s' \succeq t$ for some $s \rhd s'$.*
2. *$t = g(\bar{t})$, $f \succ_{\mathcal{F}} g$ and $s \succ t'$, for all $t' \in \bar{t}$.*

[4] Note that these multisets include not only maximal subterms of t rooted by non-base function symbols, but also variables.

3. $t = g(\bar{t})$, $f \approx_{\mathcal{F}} g$, $f \in \mathcal{F}_{\mathcal{M}ul}$ *and* $\bar{s} \succ^{mul}_{rpom} \bar{t}$.
4. $t = g(\bar{t})$, $f \approx_{\mathcal{F}} g$, $f \in \mathcal{F}_{\mathcal{L}ex}$, $\bar{s} \succ^{lex}_{rpom} \bar{t}$ *and* $s \succ t'$, *for all* $t' \in \bar{t}$.
5. $f \in \mathcal{B}$ *or* $root(t) \in \mathcal{B}$, $s \notin \mathcal{T}(\mathcal{B}, \mathcal{X})$, *and* $frt_{\mathcal{B}}(s) \succ^{mc} frt_{\mathcal{B}}(t)$.

We define $\succ_{rpom-stab}$, $\succ_{rpom-mon}$ *and* $\succ_{rpom-IP-mon}$ *to be* $\succ_{rpom}$ *in the cases where mc is rmul, mul and set, respectively. Analogously,* $\succ_{stab}$, $\succ_{mon}$ *and* $\succ_{IP-mon}$ *refer to* $\succ$.

It is not difficult to show (using induction on the size of s and t) that RPOM is well-defined. In order to prove that RPOM is an ordering, first we show that $\succ$ is compatible with $\sqsupset_{\mathcal{B}}$, and then, it suffices to show that $\succ$ is transitive and irreflexive.

Lemma 1. $s \succ t$ *iff* $s \notin \mathcal{T}(\mathcal{B}, \mathcal{X})$ *and* $frt_{\mathcal{B}}(s) \succ^{mc} frt_{\mathcal{B}}(t)$.

Proof. The result holds by definition if $root(s) \in \mathcal{B}$ or $root(t) \in \mathcal{B}$. Otherwise, $root(s), root(t) \notin \mathcal{B}$ and $\{s\} = frt_{\mathcal{B}}(s) \succ^{mc} frt_{\mathcal{B}}(t) = \{t\}$ iff $s \succ t$. □

Lemma 2. $\succ$ *is compatible with* $\sqsupset_{\mathcal{B}}$.

Proof. It has to be shown that $u \sqsupset_{\mathcal{B}} s \succ t \sqsupset_{\mathcal{B}} v$ implies $u \succ v$. By the frontier preserving condition of $\sqsupset_{\mathcal{B}}$ and Lemma 1 we have $frt_{\mathcal{B}}(u) \supseteq_{mc} frt_{\mathcal{B}}(s) \succ^{mc} frt_{\mathcal{B}}(t) \supseteq_{mc} frt_{\mathcal{B}}(v)$. This implies $u \notin \mathcal{T}(\mathcal{B}, \mathcal{X})$ and $frt_{\mathcal{B}}(u) \succ^{mc} frt_{\mathcal{B}}(v)$ by definition of $\succ^{mc}$. Therefore, using again Lemma 1, $u \succ v$ holds. □

Lemma 3. *If* $root(s) \notin \mathcal{B}$ *and* $s \rhd t \succeq_{rpom} u$ *then* $s \succ u$.

Proof. Either $t \sqsupseteq_{\mathcal{B}} u$ and hence $frt_{\mathcal{B}}(t) \supseteq_{mc} frt_{\mathcal{B}}(u)$, or $t \succ u$ and hence, by Lemma 1, $frt_{\mathcal{B}}(t) \succ^{mc} frt_{\mathcal{B}}(u)$. In both cases, for all $u' \in frt_{\mathcal{B}}(u)$, there exists $t' \in frt_{\mathcal{B}}(t)$ s.t. $s \rhd t' \succeq u'$ holds, and we obtain $s \succ u'$ by case 1. Thereby, $frt_{\mathcal{B}}(s) = \{s\} \succ^{mc} frt_{\mathcal{B}}(u)$ and the required result follows by Lemma 1. □

Lemma 4. $\succ$ *is transitive. More generally,* $s \succ t(\succ \cup \rhd)u$ *implies* $s \succ u$.

Proof. Assuming that $s_1 \succ s_2(\succ \cup \rhd)s_3$ we prove that $s_1 \succ s_3$, and we do it by induction on the multiset $\{|s_1|, |s_2|, |s_3|\}$ and the multiset extension of the usual ordering on naturals.

First, note that if $s_2' \rhd s_3$ for some $s_2' \in frt_{\mathcal{B}}(s_2)$, then, $s_2' \succ s_3$, and hence, by Lemma 1 $frt_{\mathcal{B}}(s_2) \supseteq_{mc} frt_{\mathcal{B}}(s_2') \succ^{mc} frt_{\mathcal{B}}(s_3)$, which implies $frt_{\mathcal{B}}(s_2) \succ^{mc} frt_{\mathcal{B}}(s_3)$, and $s_2 \succ s_3$ by Lemma 1 again. Therefore, in general we have that either $s_2 \succ s_3$ or $frt_{\mathcal{B}}(s_2) \supseteq_{mc} frt_{\mathcal{B}}(s_3)$.

Assume that some of the symbols $root(s_1)$, $root(s_2)$ or $root(s_3)$ are in $\mathcal{B}$. By Lemma 1 and previous observation, $frt_{\mathcal{B}}(s_1) \succ^{mc} frt_{\mathcal{B}}(s_2)(\succ^{mc} \cup \supseteq_{mc})frt_{\mathcal{B}}(s_3)$. By induction hypothesis, transitivity holds for smaller terms, and since the extension mc preserves transitivity and is compatible with $\supseteq_{mc}$, we can conclude that $frt_{\mathcal{B}}(s_1) \succ^{mc} frt_{\mathcal{B}}(s_3)$. Again by Lemma 1, $s_1 \succ s_3$.

Hence, from now on we can assume that all $root(s_1)$, $root(s_2)$ or $root(s_3)$ are not in $\mathcal{B}$, and therefore case 5 of the definition of RPOM does not apply any more and, moreover, by our first observation, $s_2 \succ s_3$.

If $s_1 \succ s_2$ by case 1, then there exists a proper subterm s'_1 of s_1 satisfying $s'_1 \succeq s_2$. Either because $s'_1 \equiv s_2$ or by induction hypothesis, $s'_1 \succ s_3$, and $s_1 \succ s_3$ holds by case 1. Hence, from now on assume that $s_1 \succ s_2$ is not due to case 1.

At this point it is easy to show that $s_1 \succ s'_2$ for any proper subterm s'_2 of s_2. Note that for such s'_2 there is some s''_2 in $\overline{s_2}$ that contains s'_2 as subterm. If $s_1 \succ s_2$ is due to case 2 or 4, then $s_1 \succ s''_2$. Otherwise, if it is due to case 3, for some s'_1 in $\overline{s_1}$, $s'_1 \succeq_{rpom} s''_2$, and by Lemma 3, we obtain $s_1 \succ s''_2$ again. In any case $s_1 \succ s''_2$, and either s''_2 is s'_2 and hence $s_1 \succ s'_2$ directly, or $s''_2 \rhd s'_2$ and by induction hypothesis on $s_1 \succ s''_2 \rhd s'_2$ we obtain $s_1 \succ s'_2$ again.

If $s_2 \succ s_3$ by case 1, then there exists a proper subterm s'_2 of s_2 satisfying $s'_2 \succeq s_3$. By the previous observation, $s_1 \succ s'_2$, and by induction hypothesis, $s_1 \succ s_3$. Hence, from now on we can assume that case 1 does not apply in $s_1 \succ s_2 \succ s_3$.

Reasoning analogously as before, it is easy to show that $s_2 \succ s'_3$ for any proper subterm s'_3 of s_3. Moreover, by induction hypothesis on $s_1 \succ s_2 \succ s'_3$, we obtain $s_1 \succ s'_3$ for any of such s'_3's. Hence, if $root(s_1) \succ_{\mathcal{F}} root(s_3)$, then $s_1 \succ s_3$ by case 2. On the other hand $root(s_3) \succ_{\mathcal{F}} root(s_1)$ can not happen since case 1 does not apply in $s_1 \succ s_2$ and $s_2 \succ s_3$. Therefore, from now on we can assume that $root(s_1) \approx_{\mathcal{F}} root(s_3)$. Again since case 1 does not apply, we have $root(s_1) \approx_{\mathcal{F}} root(s_2) \approx_{\mathcal{F}} root(s_3)$.

If such a root symbol is from $\mathcal{F}_{\mathcal{M}ul}$ ($\mathcal{F}_{\mathcal{L}ex}$) then, since the mul (lex) extension preserves transitivity, $\bar{s_1} \succ^{mul}_{rpom} \bar{s_3}$ ($\bar{s_1} \succ^{lex}_{rpom} \bar{s_3}$): note that $\succ_{rpom}$ is transitive on smaller subterms since, by induction hypothesis, $\succ$ is, and, moreover, it is compatible with $\sqsupset_{\mathcal{B}}$, which is transitive too. Hence (using that $s_1 \succ s'_3$ for any proper subterm s'_3 of s_3 in the case where the root symbol is from $\mathcal{F}_{\mathcal{L}ex}$) we conclude that $s_1 \succ s_3$. $\square$

Lemma 5. *$\succ$ is irreflexive.*

Proof. Obviously, $s \not\succ s$ for all $s \in \mathcal{X}$. Hence, we proceed by contradiction, using induction on the size of s. Depending on the case $s \succ s$ holds we consider 3 cases. If $s \succ s$ holds by case 1 then, $root(s) \in \mathcal{F} - \mathcal{B}$ and for some $s \rhd s'$, $s' \succ s$ holds. But by Lemma 3, $s \succ s'$, and by transitivity $s' \succ s \succ s'$ implies $s' \succ s'$ contradicting the induction hypothesis. The irreflexivity of $\succ_{\mathcal{F}}$ is contradicted if $s \succ s$ holds by case 2. Finally, $s \succ s$ holding by case 3, 4 or 5 implies either $\bar{s} \succ^{mul}_{rpom} \bar{s}$, $\bar{s} \succ^{lex}_{rpom} \bar{s}$ or $frt_{\mathcal{B}}(s) \succ^{mc} frt_{\mathcal{B}}(s)$. But $\sqsupset_{\mathcal{B}}$ is irreflexive and, by the induction hypothesis, $\succ$ is irreflexive for the subterms of s. Hence, since the multiset and lexicographic extensions preserve irreflexivity we obtain $\bar{s} \not\succ^{mul}_{rpom} \bar{s}$, $\bar{s} \not\succ^{lex}_{rpom} \bar{s}$ or $frt_{\mathcal{B}}(s) \not\succ^{mc} frt_{\mathcal{B}}(s)$ which is a contradiction. $\square$

Well-foundedness of RPO follows from the fact that it is a monotonic ordering which includes the subterm relation. This is not the case of $\succ$ when $mc \neq mul$: for example, even if $\mathcal{B} = \{f\}$ and $a \succ_{\mathcal{F}} b$, $faab \not\succ fabb$. Therefore, we prove its well-foundedness directly by contradiction.

Lemma 6. *If $\sqsupset_{\mathcal{B}}$ is well-founded then $\succ$ is well-founded.*

Proof. Proceeding by contradiction, suppose there is an infinite sequence with $\succ$. We choose a minimal one w.r.t. the size of the terms involved; that is, the infinite sequence $S = s_1, s_2, s_3, \ldots$ satisfies that for any other sequence $t_1, t_2, t_3, \ldots$ with different sequence of sizes, i.e. with $|s_1|, |s_2|, |s_3|, \ldots \neq |t_1|, |t_2|, |t_3|, \ldots$, there exists an $i > 0$ such that $|t_i| > |s_i|$ and $|t_j| = |s_j|$ for all $j < i$.

If there exists a step in S s.t. $s_i \succ s_{i+1}$ holds by case 1, then the minimality of S is contradicted. Note that if so, by definition of $\succ$ and Lemma 3 we have $s_i \succ s' \succeq s_{i+1}$ for some $s_i \rhd s'$. Hence, by transitivity we obtain the sequence $S' = s_1, s_2, \ldots, s_{i-1}, s', s_{i+2}, \ldots$, which is smaller than S. This also applies when $s_i \succ s_{i+1}$ holds by case 5 and $root(s_{i+1}) \notin \mathcal{B}$. In this case $s' \succeq s_{i+1}$ holds for some $s' \in frt_{\mathcal{B}}(s_i)$ and when $i > 1$ by Lemma 4 we have $s_{i-1} \succ s'$. Therefore, there is at most one step in S s.t. $root(s_i) \notin \mathcal{B}$ and $root(s_{i+1}) \in \mathcal{B}$. Thus, any other step in S holding by case 5 involves terms which are both rooted by a base symbol.

By the previous facts and since $\succsim_{\mathcal{F}}$ is a precedence, we conclude that there is some $i \geq 1$ satisfying that for all $j > i$, $s_j \succ s_{j+1}$ holds by the same case 3, 4 or 5. In cases 3 and 4, by definition of the multiset and lexicographic extensions, from the infinite sequence $\bar{s}_{i+1}, \bar{s}_{i+2}, \bar{s}_{i+3}, \ldots$ with $\succ_{rpom}^{mul}$ or $\succ_{rpom}^{lex}$ we extract another infinite sequence $t_1, t_2, t_3, \ldots$ with $\succ_{rpom}$ with $t_1 \in \bar{s}_{i+1}$. Since $\sqsupset_{\mathcal{B}}$ is well-founded and $\succ$ is compatible with $\sqsupset_{\mathcal{B}}$, from the latter sequence we construct another infinite sequence $s'_{i+1}, s'_{i+2}, s'_{i+3}, \ldots$ with $\succ$ and where $s'_{i+1} = t_1$. In case 5, from the infinite sequence $frt_{\mathcal{B}}(s_{i+1}), frt_{\mathcal{B}}(s_{i+2}), frt_{\mathcal{B}}(s_{i+3}), \ldots$ with $\succ^{mc}$ we construct another infinite sequence $s'_{i+1}, s'_{i+2}, s'_{i+3}, \ldots$ with $\succ$ and where $s_{i+1} \in frt_{\mathcal{B}}(s_{i+1})$. Thus, we have $s_{i+1} \rhd s'_{i+1}$ and $s_i \succ s'_{i+1}$ holds by Lemma 4. Therefore, we construct the infinite sequence $s_1, s_2, \ldots, s_i, s'_{i+1}, s'_{i+2}, s'_{i+3}, \ldots$ with $\succ$ which again contradicts the minimality of S. $\square$

Corollary 1. *$\succ_{rpom}$ is an ordering. If $\sqsupset_{\mathcal{B}}$ is well-founded then $\succ_{rpom}$ is well-founded.*

3.1 A stable subclass of RPOM

In this subsection we show that $\succ_{rpom-stab}$ preserves the stability of $\sqsupset_{\mathcal{B}}$.

Proposition 1. *If $\sqsupset_{\mathcal{B}}$ is stable, then $\succ_{rpom-stab}$ is stable.*

Proof. We just need to show that $s \succ_{stab} t$ implies $s\sigma \succ_{stab} t\sigma$ for every substitution σ. We use induction on the size of s and t.

If $s \succ_{stab} t$ holds by a case different from 5 then $s\sigma \succ_{stab} t\sigma$ is easily obtained by the same case using the induction hypothesis and the stability of $\rhd$, $\sqsupset_{\mathcal{B}}$ and the multiset and lexicographic extensions. In the case where $s \succ_{stab} t$ holds by case 5, note that $s \notin \mathcal{F}(\mathcal{B}, \mathcal{X})$ implies $s\sigma \notin \mathcal{F}(\mathcal{B}, \mathcal{X})$, and hence $frt_{\mathcal{B}}(s\sigma)$ is not empty. Besides, every term in $frt_{\mathcal{B}}(t\sigma)$ is either at a position p such that $t|_p$ is in $frt_{\mathcal{B}}(t)$ and $root(t|_p) \notin \mathcal{B}$, or at a position of the form $p.p'$ such that $t|_p$ is a variable x, and $t\sigma|_{p.p'} \in frt_{\mathcal{B}}(x\sigma)$. In the first case, there is a term $s' \in frt_{\mathcal{B}}(s)$ such that $s' \succ t|_p$ and $root(s') \notin \mathcal{B}$. Hence, $s'\sigma \in frt_{\mathcal{B}}(s\sigma)$ and by induction hypothesis $s'\sigma \succ_{stab} t|_p\sigma$. In the second case, there is a term $s' \in frt_{\mathcal{B}}(s)$ with $root(s) \notin \mathcal{B}$ that has x as proper subterm, and hence $s'\sigma \in frt_{\mathcal{B}}(s)$ and $s'\sigma \succ_{stab} t|_{p.p'}\sigma$ by case 1. Altogether shows that $frt_{\mathcal{B}}(s\sigma) \succ_{stab}^{rmul} frt_{\mathcal{B}}(t\sigma)$, and hence $s\sigma \succ_{stab} t\sigma$ holds by case 5. □

The orderings $\succ_{rpom-mon}$ and $\succ_{rpom-IP-mon}$ are not stable. This is due to the terms rooted by base function symbols which are compared by using the frontier subterms and the multiset extension. Note that, after applying a substitution, some frontier positions (corresponding to variables) may disappear and thus a strict superset relation (which is included in the multiset extension) may become equality. For example, for $\mathcal{B} = \{g, a\}$, we have $s = g(a, h(x), y) \succ_{mon} g(a, a, h(x)) = t$ but $s\sigma \not\succ_{mon} t\sigma$ if $y\sigma$ is a ground term of $\mathcal{T}(\mathcal{B}, \mathcal{X})$. The same and more complex situations hold for $\succ_{IP-mon}$.

3.2 A monotonic subclass of RPOM

In this subsection we show that $\succ_{mon}$ is monotonic, and $\succ_{rpom-mon}$ preserves the monotonicity of $\sqsupset_{\mathcal{B}}$. Moreover, even if $\sqsupset_{\mathcal{B}}$ is not monotonic, there is a monotonic relation between $\succ_{mon}$ and $\sqsupset_{\mathcal{B}}$ that we define as follows.

Definition 4. *A relation $\sqsupset$ on terms is monotonic on an other relation $\sqsupset'$ w.r.t. a set of symbols $\mathcal{F}_1$ if for all $f \in \mathcal{F}_1$, $s \sqsupset t$ implies $f(\ldots, s, \ldots) \sqsupset' f(\ldots, t, \ldots)$.*

Proposition 2. *$\succ_{mon}$ is monotonic, and $\sqsupset_{\mathcal{B}}$ is monotonic on $\succ_{mon}$ w.r.t. $\mathcal{F} - \mathcal{B}$.*

Proof. Let $u = f(\ldots, s, \ldots)$, $v = f(\ldots, t, \ldots)$. If $f \in \mathcal{B}$ and $s \succ_{mon} t$, then $frt_{\mathcal{B}}(s) \succ_{mon}^{mul} frt_{\mathcal{B}}(t)$ by Lemma 1, and hence $frt_{\mathcal{B}}(u) \succ_{mon}^{mul} frt_{\mathcal{B}}(v)$, which implies $u \succ_{mon} v$ by case 5.

If $f \notin \mathcal{B}$ and either $s \succ_{mon} t$ or $s \sqsupset_{\mathcal{B}} t$, then $s \succ_{rpom-mon} t$. Hence $\bar{u} \succ_{rpom-mon}^{mul} \bar{v}$ and $\bar{u} \succ_{rpom-mon}^{lex} \bar{v}$. If $f \in \mathcal{F}_{\mathcal{M}ul}$ then $u \succ_{mon} v$ holds by case 3. If $f \in \mathcal{F}_{\mathcal{L}ex}$ then $u \succ_{mon} v$ holds by case 4 because by Lemma 3, $u \succ_{mon} v'$ holds for all $v' \in \bar{v}$. □

Corollary 2. *If $\sqsupset_{\mathcal{B}}$ is monotonic then $\succ_{rpom-mon}$ is monotonic.*

3.3 An IP-monotonic subclass of RPOM

In this subsection we show that, for a given hierarchical TRS $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ and under certain conditions, IP-monotonicity of $\sqsupset_{\mathcal{B}}$ w.r.t. $\mathcal{R}_0$ (on terms of $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$) implies IP-monotonicity of $\succ_{rpom-IP-mon}$ w.r.t. $\mathcal{R}$ (on terms of $\mathcal{T}(\mathcal{F}, \mathcal{X})$). Since $\sqsupset_{\mathcal{B}}$ will usually be an extension from an ordering orienting $\mathcal{R}_0$, it is not expectable to be IP-monotonic on terms on the extended signature $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$. Even more, including $\not\!\!\rightarrow_{i,\mathcal{R}_0}$ applied to terms on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ into $\sqsupset_{\mathcal{B}}$ is not possible because then the condition stating that $s \sqsupset_{\mathcal{B}} t$ implies $frt_{\mathcal{B}}(s) \supseteq_{set} frt_{\mathcal{B}}(t)$ is violated for terms rooted by $f \notin \mathcal{B}$. Instead of including the whole relation $\not\!\!\rightarrow_{i,\mathcal{R}_0}$ in $\sqsupset_{\mathcal{B}}$ we demand a weaker condition based on the following definition.

Definition 5. *Let s and t be terms in $\mathcal{T}(\mathcal{F},\mathcal{X})$. Then we write $s \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}_0,\mathcal{F}_0} t$ if $s \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}_0} t$ and all innermost redexes in s are at positions p such that for all $p' \leq p$, $root(s|_p) \in \mathcal{F}_0$.*

Proposition 3. *Let $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ be a hierarchical TRS, $\mathcal{B} = \mathcal{F}_0$ and $\sqsupset_{\mathcal{B}}$ be an ordering on $\mathcal{T}(\mathcal{F},\mathcal{X})$ s.t. $s \sqsupset_{\mathcal{B}} t$ implies $frt_{\mathcal{B}}(s) \sqsupseteq_{set} frt_{\mathcal{B}}(t)$, and $\mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}_0,\mathcal{B}} \subseteq \sqsupset_{\mathcal{B}}$. Let $\rightarrow_{i,\lambda,\mathcal{R}_1} \subseteq \succ_{IP-mon}$.*

Then $\succ_{rpom-IP-mon}$ is IP-monotonic w.r.t. $\mathcal{R}$.

For proving the previous lemma we need the following basic facts concerning the set extension of any ordering.

Proposition 4. *Let $\sqsupset$ be any ordering.*

- *$S \sqsupset^{set} T$, $S' \sqsupseteq^{set} T'$ and $S \cap S' = \emptyset$ imply $S \cup S' \sqsupset^{set} T \cup T'$.*
- *$\{s_1\} \sqsupset T_1, \ldots, \{s_n\} \sqsupset T_n$ implies $\{s_1, \ldots, s_n\} \sqsupset^{set} T_1 \cup \ldots \cup T_n$.*

Proof. (of Proposition 3) To prove that $s \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}} t$ implies $s \succ_{rpom-IP-mon} t$, we prove, by induction on term structure, a more general statement: $s \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}} t$ implies $s \succ_{rpom-IP-mon} t$ and if $root(s) \notin \mathcal{B}$ then $s \succ_{IP-mon} t$. We distinguish two cases depending on whether or not $root(s)$ is in $\mathcal{B}$.

Assume that $root(s) \notin \mathcal{B}$. If $s \rightarrow_{i,\lambda,\mathcal{R}_1} t$ then trivially $s \succ_{IP-mon} t$ by the assumptions of the lemma. Otherwise, s and t are of the form $f(s_1 \ldots s_m)$ and $f(t_1 \ldots t_m)$, respectively, every s_j is either an $\mathcal{R}$-normal form or $s_j \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}} t_j$, and for some $j \in \{1 \ldots m\}$, $s_j \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}} t_j$. By induction hypothesis, every s_j is either a normal form or $s_j \succ_{rpom-IP-mon} t_j$, and for some $j \in \{1 \ldots m\}$, $s_j \succ_{rpom-IP-mon} t_j$. If $f \in \mathcal{F}_{\mathcal{M}ul}$, $s \succ_{IP-mon} t$ by case 3. If $f \in \mathcal{F}_{\mathcal{L}ex}$, then $s \succ_{IP-mon} t$ holds by case 4 because by Lemma 3 we have $s \succ_{IP-mon} t_j$ for all $j \in \{1 \ldots m\}$.

Assume now that $root(s) \in \mathcal{B}$. We consider the set containing only the minimal positions from $\{p \mid s|_p \in frt_{\mathcal{B}}(s)$ or $s|_p$ is an innermost redex$\}$, i.e. the ones in this set such that no other is above them. This set is of the form $\{p_1, \ldots, p_n, p'_1, \ldots, p'_m\}$ where the p_j's are frontier positions without innermost redexes above them, and the p'_j's are redex positions satisfying $root(s|_{p'}) \in \mathcal{B}$ for every $p' \leq p'_j$. Hence, s and t can be written as $s[s_1, \ldots, s_n,\ s'_1, \ldots, s'_m]_{p_1,\ldots,p_n,p'_1,\ldots,p'_m}$ and $s[t_1, \ldots, t_n, t'_1, \ldots, t'_m]_{p_1,\ldots,p_n,p'_1,\ldots,p'_m}$, respectively, where every s_j satisfies $root(s_j) \notin \mathcal{B}$ and either $s_j \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}} t_j$ or s_j is a normal form and $s_j = t_j$, and every s'_j satisfies $s'_j \rightarrow_{i,\lambda,\mathcal{R}_0} t'_j$. Moreover, $frt_{\mathcal{B}}(s) = \{s_1, \ldots, s_n\} \cup frt_{\mathcal{B}}(s'_1) \cup \ldots \cup frt_{\mathcal{B}}(s'_m)$, and $frt_{\mathcal{B}}(t) = frt_{\mathcal{B}}(t_1) \cup \ldots \cup frt_{\mathcal{B}}(t_n) \cup frt_{\mathcal{B}}(t'_1) \cup \ldots \cup frt_{\mathcal{B}}(t'_m)$. If all the s_j's are normal forms, then $s \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}_0,\mathcal{B}} t$, and by our assumptions $s \sqsupset_{\mathcal{B}} t$, and hence $s \succ_{rpom-IP-mon} t$ holds. Hence, assume that for some $j \in \{1 \ldots n\}$, $s_j \mathrel{+\!\!\!\!\!\!\mapsto}_{i,\mathcal{R}} t_j$. By induction hypothesis $s_j \succ_{IP-mon} t_j$, and by Lemma 1, $\{s_j\} \succ^{set}_{IP-mon} frt_{\mathcal{B}}(t_j)$. Similarly, for the rest of $j \in \{1, \ldots, n\}$ we have that either $\{s_j\} \succ^{set}_{IP-mon} frt_{\mathcal{B}}(t_j)$ or $s_j = t_j$, depending on whether or not s_j is a normal form. Since every s'_j satisfies $s'_j \rightarrow_{i,\lambda,\mathcal{R}_0} t'_j$, $frt_{\mathcal{B}}(s'_j) \sqsupseteq_{set} frt_{\mathcal{B}}(t'_j)$, and hence $frt_{\mathcal{B}}(s'_j) \succeq^{set}_{IP-mon} frt_{\mathcal{B}}(t'_j)$. By propositon 4, $\{s_1, \ldots, s_n\} \cup frt_{\mathcal{B}}(s'_1) \cup \ldots \cup frt_{\mathcal{B}}(s'_m) \succ^{set}_{IP-mon} frt_{\mathcal{B}}(t_1) \cup \ldots \cup frt_{\mathcal{B}}(t_n) \cup frt_{\mathcal{B}}(t'_1) \cup \ldots \cup frt_{\mathcal{B}}(t'_m)$, and hence $s \succ_{IP-mon} t$ by case 5, and $s \succ_{rpom-IP-mon} t$. $\square$

4 Proving $\mathcal{C}_\mathcal{E}$-termination incrementally with RPOM

Assume we have a hierarchical system $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$, and we want to prove it terminating using RPOM. We will usually have a reduction ordering $\succ_\mathcal{B}$ defined on $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$ orienting $\mathcal{R}_0$, or more generally, a well founded ordering $\succ_\mathcal{B}$ including $\rightarrow_{\mathcal{R}_0}$ on $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$, and we will want to obtain from it an ordering $\succ_{rpom-stab}$ orienting $\mathcal{R}_1$. A first simple idea is to extend $\succ_\mathcal{B}$ to some $\sqsupset_\mathcal{B}$ for terms on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ including $\rightarrow_{\mathcal{R}_0}$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. But this will not be useful with RPOM since rewriting with $\mathcal{R}_0$ on a term rooted by a symbol $f \notin \mathcal{B}$ does not preserve the frontier. An alternative idea is then to restrict the extension of $\succ_\mathcal{B}$ to rewriting steps with $\mathcal{R}_0$ not below a symbol $f \notin \mathcal{B}$. Then, we can take $\sqsupset_\mathcal{B} = \succ_\mathcal{B}^\mathcal{F}$ to this end, which is not monotonic, but preserves well foundedness of $\succ_\mathcal{B}$ (recall that $\succ_\mathcal{B}^\mathcal{F}$ is the stable extension of $\succ_\mathcal{B}$ to $\mathcal{F}$).

Definition 6. *Let s and t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Then we write $s \rightarrow_{\mathcal{R}_0, \mathcal{F}_0} t$ if $s \rightarrow_{\mathcal{R}_0} t$ and the involved redex is at a position p such that for all $p' \leq p$, $root(s|_p) \in \mathcal{F}_0$.*

The following theorem combines the use of $\succ_{stab}$ and $\succ_{mon}$ constructed from $\sqsupset_\mathcal{B}$. The monotonicity of the second requires $\sqsupset_\mathcal{B}$ to be frontier preserving in the sense of multisets. Therefore, when $\sqsupset_\mathcal{B}$ is defined as $\succ_\mathcal{B}^\mathcal{F}$, we also need $\succ_\mathcal{B}$ to be non-duplicating.

Theorem 2. *Let $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ be a hierarchical union, $\mathcal{B} = \mathcal{F}_0$ and $\sqsupset_\mathcal{B}$ be a stable, well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, such that $s \sqsupset_\mathcal{B} t$ implies $frt_\mathcal{B}(s) \sqsupseteq_{mul} frt_\mathcal{B}(t)$, and $\rightarrow_{\mathcal{R}_0, \mathcal{B}} \subseteq \sqsupset_\mathcal{B}$.*

If $\mathcal{R}_1 \subseteq \succ_{stab}$ then $\mathcal{R}$ is $\mathcal{C}_\mathcal{E}$-terminating.

Proof. Recall that $\mathcal{R}_\mathcal{E} = \mathcal{R} \cup \mathcal{C}_\mathcal{E}$. We prove that $\rightarrow_{\mathcal{R}_\mathcal{E}}$ is included in $\succ_{rpom-mon}$ because then the well-foundedness of $\succ_{rpom-mon}$ implies termination of $\mathcal{R}_\mathcal{E}$. First note that $\mathcal{R}_1 \cup \mathcal{C}_\mathcal{E} \subset \succ_{stab}$, and since $\succ_{stab}$ is stable, $\succ_{stab} \subset \succ_{mon}$ and $\succ_{mon}$ is monotonic we conclude that $\rightarrow_{\mathcal{R}_1 \cup \mathcal{C}_\mathcal{E}}$ is included in $\succ_{mon}$. Now, let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ be s.t. $s \rightarrow_{\mathcal{R}_0} t$ at position p. If every position above p is rooted by a symbol in $\mathcal{F}_0$ then we have $s \sqsupset_\mathcal{B} t$ by the assumptions of the theorem. It remains to see the case where there exist a context $u[\ ]$, a symbol $f \notin \mathcal{F}_0$ and a position $q < p$ s.t. $s = u[f(\ldots, s', \ldots)]_q$, $t = u[f(\ldots, t', \ldots)]_q$ and $s' \rightarrow_{\mathcal{R}_0} t'$ with every position between q and p rooted by a symbol in $\mathcal{F}_0$. In this case $s' \sqsupset_\mathcal{B} t'$ holds by the assumptions of the theorem, and $s \succ_{mon} t$ is obtained by Proposition 2. $\Box$

Corollary 3. *A hierarchical TRS $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ is $\mathcal{C}_\mathcal{E}$-terminating if there is a non-duplicating reduction ordering $\succ_\mathcal{B}$ s.t. $\mathcal{R}_0 \subseteq \succ_\mathcal{B}$ and $\mathcal{R}_1 \subseteq \succ_{stab}$.*

In addition, if $\succ_\mathcal{B}$ is a simplification ordering then $\mathcal{R}$ is simply terminating.

Example 3. Simple termination of $\mathcal{R}_{plus} \cup \mathcal{R}_{F'}$ in Example 2 is easily obtained using RPOM. Since $\mathcal{R}_{plus}$ is simply terminating, $\succ_\mathcal{B}$ can be defined as the non-duplicating part of any simplification ordering including $\mathcal{R}_{plus}$. The rules of the extension $\mathcal{R}_{F'}$ (listed below) are oriented using $\succ_{stab}$ with $\mathcal{F}_{\mathcal{L}ex} = \{F\}$.

$$\begin{cases} F(0, x, y) & \rightarrow & plus(x, y) \\ F(s(n), x, 0) & \rightarrow & x \\ F(s(n), x, s(y)) & \rightarrow & F(n, F(s(n), x, y), s(plus(F(s(n), x, y), y))) \\ F(s(n), F(s(n), x, y), z) & \rightarrow & F(s(n), x, F(n, y, z)) \end{cases}$$

Note that the first rule, here denoted as $l_1 \rightarrow r_1$, holds by case 5 of the definition of RPOM. This is because $l_1 \succ_{stab} x$ and $l_1 \succ_{stab} y$ hold by case 1 and therefore we have $frt(l_1) = \{l_1\} \succ_{stab}^{rmul} \{x, y\} = frt(r_1)$. The second rule trivially holds by case 1. The last two hold by case 4. We detail the proof for the third one, denoted as $l_3 \rightarrow r_3$. First note that $s(t) \succ_{\mathcal{B}} t$ for every term t. Thereby, we have $\bar{l_3} \succ_{\mathcal{B}}^{lex} \bar{r_3}$. By the former fact and using case 1 we obtain $l_3 \succ_{stab} F(s(n), x, y)$ by case 4. Finally, $l_3 \succ_{stab} s(plus(F(s(n), x, y), y))$ holds by case 5 since $frt(l_3) = \{l_3\} \succ_{stab}^{rmul} \{F(s(n), x, y), y\} = frt(s(plus(F(s(n), x, y), y)))$.

Analogously to the case of SCP, there are situations where the proofs with RPOM can be done modularly.

Theorem 3. *Let $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ be a hierarchical TRS where $\mathcal{R}_0$ is non-duplicating and terminating. Let $\sqsupset_{\mathcal{B}}$ be $\rhd_0^{\mathcal{F}}$ where $\rhd_0$ is $\rhd$ on $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$, and let $\mathcal{R}_1 \subseteq \succ_{stab}$.*

Then $\mathcal{R}$ is $\mathcal{C}_{\mathcal{E}}$-terminating.

Proof. The result can be obtained by Theorem 2 if we use $\sqsupset'_{\mathcal{B}} = (\rightarrow_{\mathcal{R}_0, \mathcal{F}_0} \cup \sqsupset_{\mathcal{B}})^+$ and the corresponding $\succ'_{stab}$ instead of $\sqsupset_{\mathcal{B}}$ and $\succ_{stab}$. Trivially $\mathcal{R}_1 \subseteq \succ'_{stab}$ and $\rightarrow_{\mathcal{R}_0, \mathcal{F}_0} \subseteq \sqsupset'_{\mathcal{B}}$. We just need to show that $\sqsupset'_{\mathcal{B}}$ is frontier preserving, stable and well-founded. The first two properties follow from the fact that $\rightarrow_{\mathcal{R}_0, \mathcal{F}_0}$ and $\rhd_0$ are non-duplicating and stable, and the stable extension preserves these properties. Well-foundedness of $\sqsupset'_{\mathcal{B}}$ follows from the fact that $\mathcal{R}_0$ is terminating, and that any derivation with $(\rightarrow_{\mathcal{R}_0, \mathcal{F}_0} \cup \sqsupset_{\mathcal{B}})$ can be commuted to a derivation with $(\rightarrow_{\mathcal{R}_0, \mathcal{F}_0})$ followed by a derivation with $\rhd_0^{\mathcal{F}}$, preserving the number of rewrite steps. $\square$

Example 4. Actually, $\mathcal{R}_{F'}$ in Example 2 is included in $\succ_{stab}$ with $\sqsupset_{\mathcal{B}}$ defined as $\rhd_0^{\mathcal{F}}$. Hence, by Theorem 3, the hierarchical union of $\mathcal{R}_{F'}$ and any non-duplicating base system $\mathcal{R}_{plus}$ is $\mathcal{C}_{\mathcal{E}}$-terminating whenever $\mathcal{R}_{plus}$ is so.

Example 5. Consider the following system which describes some properties of the conditional operator.

$$\mathcal{R}_{if} = \begin{cases} if(0, y, z) & \rightarrow & z \\ if(s(x), y, z) & \rightarrow & y \\ if(x, y, y) & \rightarrow & y \\ if(if(x, y, z), x_1, x_2) & \rightarrow & if(x, if(y, x_1, x_2), if(z, x_1, x_2)) \\ if(x, if(x, y, x_1), z) & \rightarrow & if(x, y, z) \\ if(x, y, if(x, x_1, z)) & \rightarrow & if(x, y, z) \\ if(x, plus(y, x_1), plus(z, x_2)) & \rightarrow & plus(if(x, y, z), if(x, x_1, x_2)) \end{cases}$$

The rules of $\mathcal{R}_{if}$ are included in RPOM with $\mathcal{F}_{\mathcal{L}ex} = \{if\}$ and $\sqsupset_{\mathcal{B}}$ defined as $\rhd_0^{\mathcal{F}}$. The first three rules hold by case 1 and the three next by case 4. The last rule holds by case 5. Note that $plus(x, y) \sqsupset_{\mathcal{B}} x$ and $plus(x, y) \sqsupset_{\mathcal{B}} y$ hold. Hence, using case 4 we obtain $if(x, plus(y, x_1), plus(z, x_2)) \succ_{stab} if(x, y, z)$ and $if(x, plus(y, x_1), plus(z, x_2)) \succ_{stab} if(x, x_1, x_2)$. Therefore, by Theorem 3 we conclude that the hierarchical union of $\mathcal{R}_{if}$ and any base system $\mathcal{R}_{plus}$ is $\mathcal{C}_{\mathcal{E}}$-terminating whenever $\mathcal{R}_{plus}$ is non-duplicating and $\mathcal{C}_{\mathcal{E}}$-terminating.

We stress that $\mathcal{R}_{F'}$ and $\mathcal{R}_{if}$ are hierarchical extensions which are not proper and where SCP cannot be used. Hence, no previous modularity result can be applied to these examples.

5 Proving innermost termination incrementally with RPOM

This section proceeds analogously to the previous one. The main difference is that, for proving innermost termination, $\sqsupset_{\mathcal{B}}$ needs to be frontier preserving only in the sense of sets. Hence, if $\sqsupset_{\mathcal{B}}$ is constructed from $\succ_{\mathcal{B}}$, the non-duplicating requirement on $\succ_{\mathcal{B}}$ disappears.

Theorem 4. *Let $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ be a hierarchical union, $\mathcal{B} = \mathcal{F}_0$ and $\sqsupset_{\mathcal{B}}$ be a stable, well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, such that $s \sqsupset_{\mathcal{B}} t$ implies $frt_{\mathcal{B}}(s) \sqsupseteq_{set} frt_{\mathcal{B}}(t)$, and $\mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}_{i,\mathcal{R}_0,\mathcal{B}} \subseteq \sqsupset_{\mathcal{B}}$.*

If $\mathcal{R}_1 \subseteq \succ_{stab}$ then $\mathcal{R}$ is innermost terminating.

Proof. By the assumptions and Proposition 1, $\succ_{rpom-stab}$ is stable. Hence, it includes $\rightarrow_{i,\lambda,\mathcal{R}}$. Since $\succ_{rpom-stab} \subseteq \succ_{rpom-IP-mon}$, it follows that $\rightarrow_{i,\lambda,\mathcal{R}} \subseteq \succ_{rpom-IP-mon}$. By the assumptions and Proposition 3, $\succ_{rpom-IP-mon}$ is IP-monotonic w.r.t. $\mathcal{R}$, and by Lemma 6, it is well-founded. Altogether with Theorem 1 imply that $\mathcal{R}$ is innermost terminating. $\square$

Theorem 5. *Let $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ be a hierarchical TRS where $\mathcal{R}_0$ is innermost terminating. Let $\sqsupset_{\mathcal{B}}$ be $\rhd_0^{\mathcal{F}}$ where $\rhd_0$ is $\rhd$ on $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$, and let $\mathcal{R}_1 \subseteq \succ_{stab}$.*

Then $\mathcal{R}$ is innermost terminating.

Proof. We use $\sqsupset'_{\mathcal{B}} = (\mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}_{i,\mathcal{R}_0,\mathcal{F}_0} \cup \sqsupset_{\mathcal{B}})^+$ and the corresponding $\succ'_{IP-mon}$. Note that $\sqsupset'_{\mathcal{B}}$ and $\succ'_{stab}$ are not necessarily stable whereas $\sqsupset_{\mathcal{B}}$ and $\succ_{stab}$ are, the second one by Proposition 1, and hence, $\succ_{stab}$ includes $\rightarrow_{i,\lambda,\mathcal{R}_1}$. Since $\succ_{rpom-stab} \subseteq \succ_{rpom-IP-mon} \subseteq \succ'_{rpom-IP-mon}$, it follows that $\rightarrow_{i,\lambda,\mathcal{R}} \subseteq \succ'_{rpom-IP-mon}$.

By the definition of $\sqsupset'_{\mathcal{B}}$, it is IP-monotonic w.r.t. $\mathcal{R}_0$ in $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$. It is also well-founded since any derivation with $\mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}_{i,\mathcal{R}_0,\mathcal{F}_0} \cup \sqsupset_{\mathcal{B}}$ can be commuted to a derivation with $\mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}_{i,\mathcal{R}_0,\mathcal{F}_0}$ followed by a derivation with $\rhd_0^{\mathcal{F}}$, with the same number of rewrite steps, and the fact that $\mathcal{R}_0$ is innermost terminating.

By the assumptions and Proposition 3, $\succ'_{rpom-IP-mon}$ is IP-monotonic w.r.t. $\mathcal{R}$, and by Lemma 6, it is well-founded. Altogether with Theorem 1 imply that $\mathcal{R}$ is innermost terminating. $\square$

Example 6. Recall the systems $\mathcal{R}_{F'}$ in Example 2 and $\mathcal{R}_{if}$ in Example 5 are included in $\succ_{stab}$ with $\sqsupset_{\mathcal{B}}$ defined as $\rhd_0^{\mathcal{F}}$. Hence, by Theorem 5, the hierarchical union of $\mathcal{R}_{F'} \cup \mathcal{R}_{if}$ and any (possibly duplicating) base system $\mathcal{R}_{plus}$ is innermost terminating whenever $\mathcal{R}_{plus}$ is innermost terminating.

6 Conclusions

The stable subclass of the RPOM is suitable for proving termination automatically. It is more powerful than RPO since it allows the reuse of termination proofs. But at the same time it inherits from its predecessor the simplicity and all the techniques for the automated generation of the precedence. The two main differences between RPO and RPOM-STAB are the use of $\sqsupset_{\mathcal{B}}$ and the treatment of terms rooted by base function symbols. But these difference can be easily handled: frontier subterms can be computed in linear time and the decision between applying case 5 or $\sqsupset_{\mathcal{B}}$ is deterministic. Besides, if $\sqsupset_{\mathcal{B}}$ is defined as $\succ_{\mathcal{B}}^{\mathcal{F}}$, we can prove $s \sqsupset_{\mathcal{B}} t$ just by proving $s_r \succ_{\mathcal{B}} t_r$, where s_r and t_r are obtained by replacing each occurrence of a frontier subterm of s by the same fresh variable.

As future work we plan to investigate more deeply the use of RPOM for proving innermost termination incrementally, since for this particular case no condition need to be imposed on the base TRS. In particular, we will consider the combination of RPOM with the ideas from [9, 11]. Furthermore, we are interested in extending the given results to the monotonic semantic path ordering [5, 4] which will provide a much more powerful framework for combining orderings and prove termination incrementally.

Finally we are also interested in extending these results to the higher-order recursive path ordering [13], which will provide necessary results for hierarchical unions for the higher-order case.

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