A Soft Approach to Multi-objective Optimization*

Stefano Bistarelli[1,2], Fabio Gadducci[3], Javier Larrosa[4], and Emma Rollon[4]

[1] Department of Science, University "G. d'Annunzio" of Chieti-Pescara, Italy
[2] Institute of Informatics and Telematics (IIT), CNR Pisa, Italy
[3] Department of Informatics, University of Pisa, Polo "G. Marconi" (La Spezia), Italy
[4] Department of Software, Technical University of Catalonia, Barcelona, Spain

Abstract. Many combinatorial optimization problems require the assignment of a set of variables in such a way that an objective function is optimized. Often, the objective function involves different criteria, and it may happen that the requirements are in conflict: assignments that are good wrt. one objective may behave badly wrt. another. An optimal solution wrt. all criteria may not exist, and either the efficient frontier (the set of best incomparable solutions, all equally relevant in the absence of further information) or an approximation has to be looked after. The paper shows how the soft constraints formalism based on semirings, so far exploited for finding approximations, can embed also the computation of the efficient frontier in multi-objective optimization problems. The main result is the proof that the efficient frontier of a multi-objective problem can be obtained as the best level of consistency distilled from a suitable soft constraint problem.

1 Introduction

Many real world problems involve multiple measures of performance, or objectives, that should be optimized simultaneously: see e.g. [1] and the references therein. In such a situation a unique, perfect solution may not exist, while a set of solutions can be found that should be considered equivalent in the absence of information concerning the relevance of each objective wrt. the others. Hence, two solutions are equivalent if one of them is better than the other for some criteria, but worse for others; while one solution dominates the other if the former is better than the latter for all criteria.

The set of best solutions is the set of efficient (*pareto-optimal*) solutions. The task in a multi-objective problem is to compute the set of costs associated to efficient solutions (the *efficient frontier*) and, possibly, one efficient solution for any of its elements.

The main goal of the paper is to prove that the computation of the efficient frontier of multi-objective optimization problems can be modeled using soft CSP. More precisely, our main contribution is to show, given a (possibly partially ordered) semiring $\mathcal{K}$, how to compute a new semiring $\mathcal{L}(\mathcal{K})$ such that its elements corresponds to sets of elements of the original semiring; and such that the set of optimal costs of the original problem.

When applied to the multi-objective context, our work is summarized as follows: consider a problem where $\mathcal{K}_1 \ldots \mathcal{K}_p$ are the semirings associated to each objective. If we use their cartesian product $\mathcal{K}_C = \mathcal{K}_1 \times \ldots \times \mathcal{K}_p$ to model the multi-objective problem, the solution corresponds to the lowest vector dominating the efficient frontier; if we use $\mathcal{L}(\mathcal{K}_C)$ to model the problem, the solution coincides with the efficient frontier.

* Research partially supported by the EU IST-2004-16004 SENSORIA.

M. Garcia de la Banda and E. Pontelli (Eds.): ICLP 2008, LNCS 5366, pp. 764–768, 2008.

2 On Semiring-Based Frameworks

Semirings provide an algebraic framework for the specification of a general class of combinatorial optimization problems. Outcomes associated to variable instantiations are modeled as elements of a set A, equipped with a *sum* and a *product* operator. These operators are used for combining constraints: the intuition is that the sum operator induces a partial order $a \leq b$, meaning that b is a better outcome than a; whilst the product operator denotes the aggregation of outcomes coming from different soft constraints.

More in detail, a (commutative) semiring is a tuple $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$ such that A is a set, $\mathbf{1}, \mathbf{0} \in A$, and $+, \times : A \times A \to A$ are binary operators making the triples $\langle A, +, \mathbf{0}\rangle$ and $\langle A, \times, \mathbf{1}\rangle$ commutative monoids, satisfying distributiveness ($\forall a, b, c \in A.a \times (b + c) = (a \times b) + (a \times c)$) and absorptiveness wrt. $\times$ ($\forall a \in A.a \times \mathbf{0} = \mathbf{0}$). A semiring is *tropical* [2] if the sum operator $+$ is idempotent ($\forall a \in A.a + a = a$); it is *absorptive* if it satisfies absorptiveness wrt. $+$ ($\forall a \in A.a + \mathbf{1} = \mathbf{1}$).

Let $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$ be a tropical semiring. Then, the relation $\langle A, \leq\rangle$ such that $\forall a, b \in A.a \leq b$ iff $a + b = b$ is a partial order. Moreover, if $\mathcal{K}$ is absorptive, then $\mathbf{1}$ is the top element of the partial order. If additionally $\mathcal{K}$ is absorptive and *idempotent* (that is, the product operator $\times$ is idempotent: $\forall a \in A.a \times a = a$), then the partial order is actually a *lattice*, since $a \times b$ corresponds to the greatest lower bound of a and b.

2.1 Soft Constraints Based on Semirings

Let $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$ be an absorptive semiring; let V be a set of variables; and let D be a finite domain of interpretation for V. Then, a *constraint* $(V \to D) \to A$ is a function associating a value in A to each assignment $\eta : V \to D$ of the variables.[1]

Note that even if a constraint involves all the variables in V, it must depend on the assignment of a finite subset of them. For instance, a binary constraint $c_{x,y}$ over variables x, y is a function $c_{x,y} : (V \to D) \to A$ which depends only on the assignment of variables $\{x, y\} \subseteq V$. This subset is the *support* of the constraint [3] and correspond to the classical notion of scope of a constraint. Most often, whenever V is ordered, an assignment (over a support of cardinality k) is concisely presented by a tuple in D^k.

The *combination* operator $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is defined as $(c_1 \otimes c_2)\eta = c_1\eta \times c_2\eta$. Thus, combining two constraints means building a new constraint whose support involves all the variables of the original ones (i.e., $supp(c_1 \otimes c_2) \subseteq supp(c_1) \cup supp(c_2)$), and which associates to each tuple for such variables a semiring element, obtained by multiplying the elements associated by the original constraints to the appropriate subtuples.

Let $c \in \mathcal{C}$ be a constraint and $v \in V$ a variable. The *projection* of c over $V - \{v\}$ (denoted $c \Downarrow_{(V-\{v\})}$), is the constraint c' such that $c'\eta = \sum_{d \in D} c\eta[v := d]$. The projection operator is inductively extended to a set of variables $I \subseteq V$ by $c \Downarrow_{(V-I)} = c \Downarrow_{(V-\{v\})} \Downarrow_{(V-\{I-\{v\}\})}$. Informally, projecting eliminates variables from the support.

A *soft constraint satisfaction problem* is a pair $\langle C, con\rangle$, where C is a set of constraints over variables $con \subseteq V$. The set con is the set of variables of interest for the constraint set C, which may concern also variables not in con. The *solution* of a soft CSP $P = \langle C, con\rangle$ is the constraint $Sol(P) = (\bigotimes C) \Downarrow_{con}$.

[1] Alternatively, a constraint is a pair $\langle sc, def\rangle$: sc is the scope of a constraint, and def the function associating a value in A to each assignment of the variables in con.

The solution of a soft CSP plays the role of the objective function in optimization problems. Indeed, efficient solutions are referred to as *abstract solutions* in the soft CSP literature. The best approximation may be neatly characterized by so-called *best level*.

Proposition 1. *Let* $P = \langle C, con \rangle$ *be a soft CSP, and let* $blevel(P) = (\bigotimes C) \Downarrow_{\emptyset}$ *be denoted as the* best level of consistency *of* P*. Then,* $sup_{\eta}\{Sol(P)(\eta)\} = blevel(P)$*.*

The soft CSP framework may accomodate several soft constraint frameworks. For instance, the semiring $\mathcal{K}_{CSP} = \langle \{false, true\}, \vee, \wedge, false, true \rangle$ allows for recasting Classical CSPs; the semiring $\mathcal{K}_{WCSP} = \langle \mathcal{R}, min, +, \infty, 0 \rangle$ for Weighted CSPs.

When the set A is totally ordered, as the examples above, $blevel(P)$ is the optimum of P, and thus, it uniquely characterizes the efficient frontier $\mathcal{E}(P) = \{Sol(P)(\eta) \mid \forall \eta'.Sol(P)(\eta) \not\geq Sol(P)(\eta')\}$). That does not hold for partially ordered semirings, as those naturally arising in multi-objective optimization, obtained as the cartesian product of a family $\mathcal{K}_1, \ldots, \mathcal{K}_p$ of semirings, each one associated to an objective function.

Proposition 2. *Let* $\{K_i = \langle A_i, +_i, \times_i, \mathbf{0}_i, \mathbf{1}_i \rangle\}_{1 \leq i \leq p}$ *be semirings. Then, the tuple* $\mathcal{K}_C = \langle A_1 \times \ldots \times A_p, \overline{+}, \overline{\times}, \langle \mathbf{0}_1, \ldots, \mathbf{0}_p \rangle, \langle \mathbf{1}_1, \ldots, \mathbf{1}_p \rangle \rangle$ *is a semiring: its set of elements is the cartesian product of* A_i*s, and the operators are defined componentwise.*

Moreover, if each K_i *is tropical (absorptive, idempotent), then so is* K_C*.*

Proposition 1 tells us that the best level of consistency of a problem P over semiring $\mathcal{K}_C$ is the lowest vector dominating the efficient frontier $\mathcal{E}(P)$. In other words, calculating $blevel(P)$ gets only an approximation of $\mathcal{E}(P)$. Our solution is to consider basically the same problem, and choosing an alternative semiring wrt. $\mathcal{K}_C$ for solving it.

3 Semirings Based on Powersets

This section states the main theorem of the paper: for each soft CSP P over a semiring $\mathcal{K}$, a new semiring $\mathcal{L}(\mathcal{K})$ and a semiring morphism $l : \mathcal{K} \to \mathcal{L}(\mathcal{K})$ can be devised such that the best level of consistency for the problem $l(P)$ coincides with the efficient frontier of P. For the sake of readability, we fix a semiring $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$.

Definition 1 (downward closure). *Let* $\mathcal{K}$ *be tropical. Then, for a set* $S \subseteq A$ *we let* ΔS *denote its* downward closure*, i.e., the set* $\{a \in A \mid \exists s \in S.a \leq_{\mathcal{K}} s\}$*.*

A set S is *downward closed* if $S = \Delta S$ (and any downward closure is so, since $\Delta(\Delta S) = \Delta S$), and we denote by $\mathcal{L}(A)$ the family of downward closed subsets of A.

Proposition 3. *Let* $\mathcal{K}$ *be absorptive. Then, the tuple* $\mathcal{L}(\mathcal{K}) = \langle \mathcal{L}(A), \cup, \times, \{\mathbf{0}\}, A \rangle$ *is an absorptive semiring: its elements are the (not empty) downward-closed subsets of* A*,* $S \cup T$ *is set (of subsets) union, and* $S \times T = \Delta(\{s \times t \mid s \in S, t \in T\})$*.*

Note that the absorptiveness of $\mathcal{K}$ plays a pivotal role, since it means that $A = \Delta\{\mathbf{1}\}$. The ordering states that $\Delta S \leq_{\mathcal{L}(\mathcal{K})} \Delta T$ iff for each $s \in S$ there exists $t \in T$ such that $s \leq_{\mathcal{K}} t$. Our construction of $\mathcal{L}(\mathcal{K})$ is thus reminiscent of the *partial correctness* (or *Hoare*) power-domain, a well-known tool in denotational semantics.

Theorem 1. *Let $P = \langle C, con \rangle$ be a soft CSP over semiring $\mathcal{K}$; and let $\mathcal{L}(P) = \langle C_l, con \rangle$ be the soft CSP over semiring $\mathcal{L}(\mathcal{K})$ such that $C_l = \{l \circ c \mid c \in C\}$ (thus, $l \circ c(\eta) = \{c(\eta)\}$ for any assignment $\eta : V \to D$). Then, $\Delta(\mathcal{E}(P)) = blevel(\mathcal{L}(P))$.*

The closure $\Delta(\mathcal{E}(P))$ is necessary, since the sets in $\mathcal{L}(\mathcal{K})$ are downward-closed. However, note that each constraint of a soft CSP P is defined over a finite set of functions $V \to D$, since it is finitely supported and D is finite.Thus, the efficient frontier $\mathcal{E}(P)$ is a finite set: such a remark could be exploited to improve on the previous presentation, describing a downward-closed set S by the family of its *irreducible* elements, i.e., the set I_S such that for no pair s_1, s_2 the element s_1 dominates s_2, and $\Delta(I_S) = S$.

Proposition 4. *Let $\mathcal{K}$ be absorptive. If $\mathcal{K}$ is idempotent, then also $\mathcal{L}(\mathcal{K})$ is so.*

The result above ensures that the local consistency techniques [4] applied for the soft CSPs over an idempotent semiring $\mathcal{K}$ can still be applied for problems over $\mathcal{L}(\mathcal{K})$.

4 Recasting Multi-criteria CSP

A *multi-criteria* CSP (MC-CSP) is a soft CSP problem composed by a family of p soft CSPs. Each criterion can be defined over the semiring $\mathcal{K}_{CSP}$. Then, a MC-CSP problem is defined over semiring $\mathcal{L}(\mathcal{K}_{CSP_1} \times \ldots \times \mathcal{K}_{CSP_p})$.

Consider a problem with two variables $\{x, y\}$, two values in each domain $\{a, b\}$, and two criteria to be satisfied. For the first criteria, the assignments $(x = a, y = a)$, $(x = b, y = a)$, and $(x = a, y = b)$ are forbidden. For the second criteria, the assignments $(x = b, y = a)$, $(x = a, y = b)$, and $(x = b, y = b)$ are forbidden. Let $\mathcal{K}_{2-CSP} = \langle \{f, t\} \times \{f, t\}, \bar{\vee}, \bar{\wedge}, \langle f, f \rangle, \langle t, t \rangle \rangle$ be the cartesian product of two semirings $\mathcal{K}_{CSP}$ (one for each criterion), where f and t are short-hands for $false$ and $true$, respectively. $\bar{\vee}$ is the pairwise $\vee$ and $\bar{\wedge}$ is the pairwise $\wedge$. Then, the problem is represented as a soft CSP $P = \langle \mathcal{C}, \mathcal{X} \rangle$ over $\mathcal{K}_{2-CSP}$, where $\mathcal{C} = \{C_x, C_y, C_{xy}\}$ is defined as $C_x(a) = C_x(b) = C_y(a) = C_y(b) = \langle t, t \rangle$, $C_{xy}(a, a) = \langle f, t \rangle$, $C_{xy}(b, a) = \langle f, f \rangle$, $C_{xy}(a, b) = \langle f, f \rangle$, and $C_{xy}(b, b) = \langle t, f \rangle$.

The solution of P is the constraint $Sol(P)$ with support $\{x, y\}$ given by $C_x \bar{\wedge} C_y \bar{\wedge} C_{xy}$. Since the variables of the problem are the same as the ones in the support of the constraints, there is no need to project any variable out. Moreover, since for all η, $C_x(\eta) = C_y(\eta) = \langle t, t \rangle$ and $\langle t, t \rangle$ is the unit element with respect $\bar{\wedge}$, $Sol(P) = C_{xy}$. The best level of consistency of P is $blevel(P) = \underline{\vee}_\eta \{Sol(P)(\eta)\} = \langle t, t \rangle$.

However, we want to obtain as the best level of consistency the set of semiring values representing the efficient frontier $\mathcal{E}(P) = \{\langle f, t \rangle, \langle t, f \rangle\}$. To that end, we map the problem P to a new one, by changing the semiring $\mathcal{K}_{2-CSP}$ using the partial correctness transformation on finite representations. By applying the mapping, we obtain a problem $\mathcal{L}(P) = \langle \mathcal{C}', \mathcal{X} \rangle$ over semiring $\mathcal{L}(\mathcal{K}_{2-CSP})$, with the following constraint definition $C_x(a) = C_x(b) = C_y(a) = C_y(b) = \{\langle t, t \rangle\}$, $C_{xy}(a, a) = \{\langle f, t \rangle\}$, $C_{xy}(b, a) = \{\langle f, f \rangle\}$, $C_{xy}(a, b) = \{\langle f, f \rangle\}$, $C_{xy}(b, b) = \{\langle t, f \rangle\}$.

The solution of $\mathcal{L}(P)$ is the same as for P. However, its best level of consistency is $blevel(\mathcal{L}(P)) = \{\langle 0, 1 \rangle, \langle 1, 0 \rangle\}$, which is the efficient frontier of P. The corresponding pareto-optimal solutions are $(x = a, y = a)$ and $(x = b, y = b)$.

5 Conclusions, Related Works and Further Developments

Problems involving the optimization of more than one objective are ubiquitous in real world domains. They are probably the most relevant optimization problems with a partially ordered objective function. So far, nobody studied how to use the soft CSP framework to model multi-objective problems. The only attempt is [5], where the least upper bound is the used notion of solution, which is a relaxed one regarding pareto-optimality.

Our paper addresses exactly this issue. For the first time, we distill a semiring able to define problems such that their best level of consistency is the efficient frontier of a multi-objective problem. This formalization is important for two main reasons: we gain some understanding of the nature of multi-objective optimization problems; and we inherit some theoretical result from the soft CSP framework.

We are aware of few papers addressing the handling of preferences with structures that are reminiscent of downward closures. Among others, we cite the work on *preference queries* [6], where a *winnow* operator is introduced. The winnow operator selects from a given relation the set of most preferred tuples, according to a given preference relation. In the same context of relational databases a similar *Skyline* operator is investigated in [7]. The Skyline operator filters out a set of interesting points from a potentially large set of data points, where a point is interesting if it is not dominated by any other point. Moreover, Pareto preference constructors are defined in [8] in the framework of the so-called *Best-Matches-Only (BMO) query model*. In the paper, the author investigates how complex preference queries can be decomposed into simpler ones.

We are currently investigating the semiring $\mathcal{S}(\mathcal{K})$ of *saturated* closures, i.e., whose elements are both downward- and upward-closed sets. We are in general looking for suitable constructions resulting in a *division* semiring, if $\mathcal{K}$ is so. This would allow for the application of local consistency algorithm to a larger family of case studies [9].

References

1. Ehrgott, M., Gandibleux, X.: Multiple Criteria Optimization. State of the Art. Annotated Bibliographic Surveys. Kluwer, Dordrecht (2002)
2. Pin, J.E.: Tropical semirings. In: Idempotency, pp. 50–69. Cambridge University Press, Cambridge (1998)
3. Bistarelli, S., Montanari, U., Rossi, F.: Soft concurrent constraint programming. ACM Transactions in Compututational Logic 7(3), 563–589 (2006)
4. Bistarelli, S., Montanari, U., Rossi, F.: Semiring-based constraint solving and optimization. Journal of ACM 44(2), 201–236 (1997)
5. Bistarelli, S., Montanari, U., Rossi, F.: Soft constraint logic programming and generalized shortest path problems. Journal of Heuristics 8(1), 25–41 (2002)
6. Chomicki, J.: Semantic optimization techniques for preference queries. Information Systems 32(5), 670–684 (2007)
7. Börzsönyi, S., Kossmann, D., Stocker, K.: The skyline operator. In: Proc. ICDE 2001, pp. 421–430. IEEE Computer Society, Los Alamitos (2001)
8. Kießling, W.: Foundations of preferences in database systems. In: Proc. VLDB 2002, pp. 311–322. Morgan Kaufmann, San Francisco (2003)
9. Bistarelli, S., Gadducci, F.: Enhancing constraints manipulation in semiring-based formalisms. In: Brewka, G., Coradeschi, S., Perini, A., Traverso, P. (eds.) Proc. ECAI 2006, pp. 63–67. IOS Press, Amsterdam (2006)