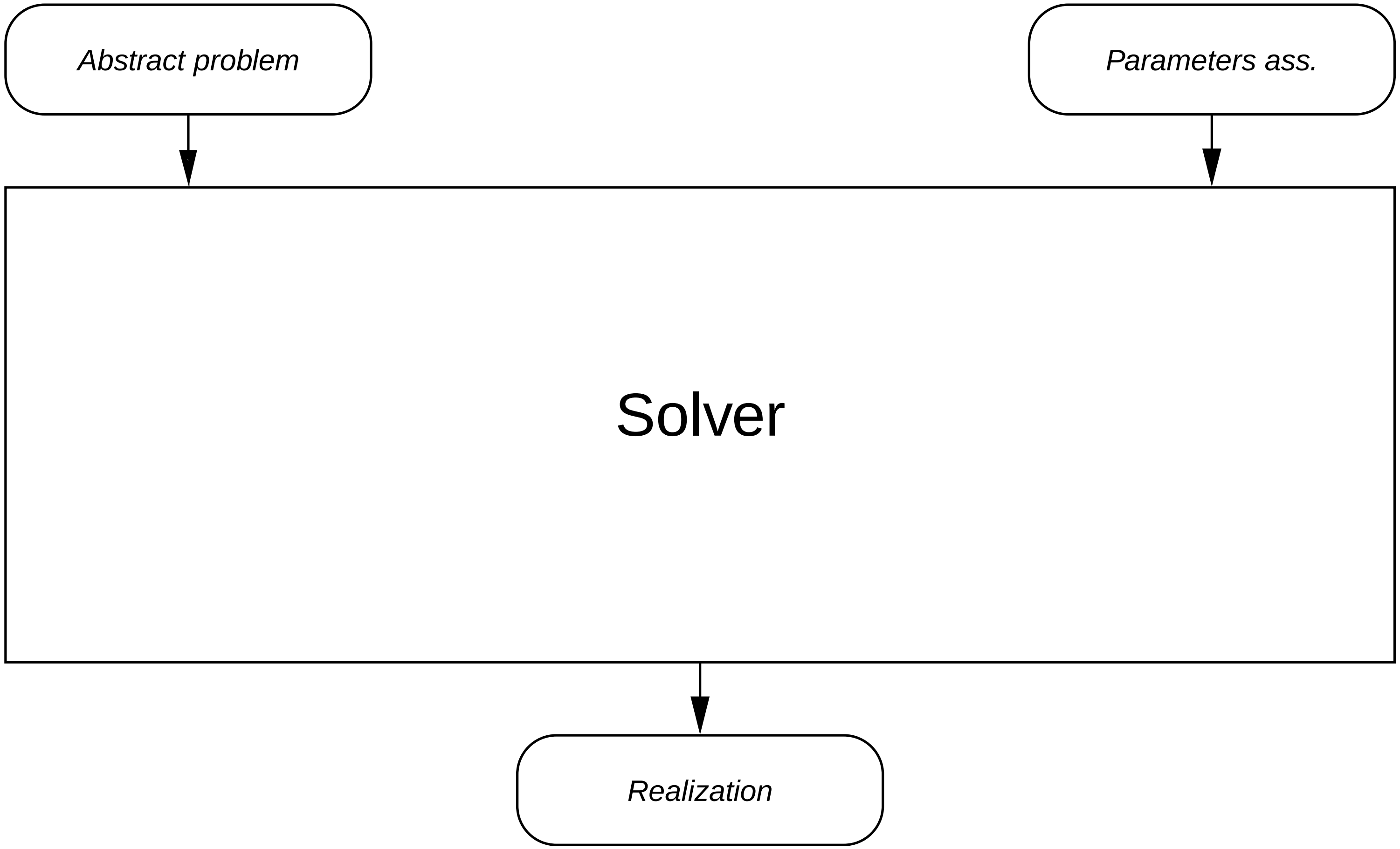
Geometric Constraint Solving



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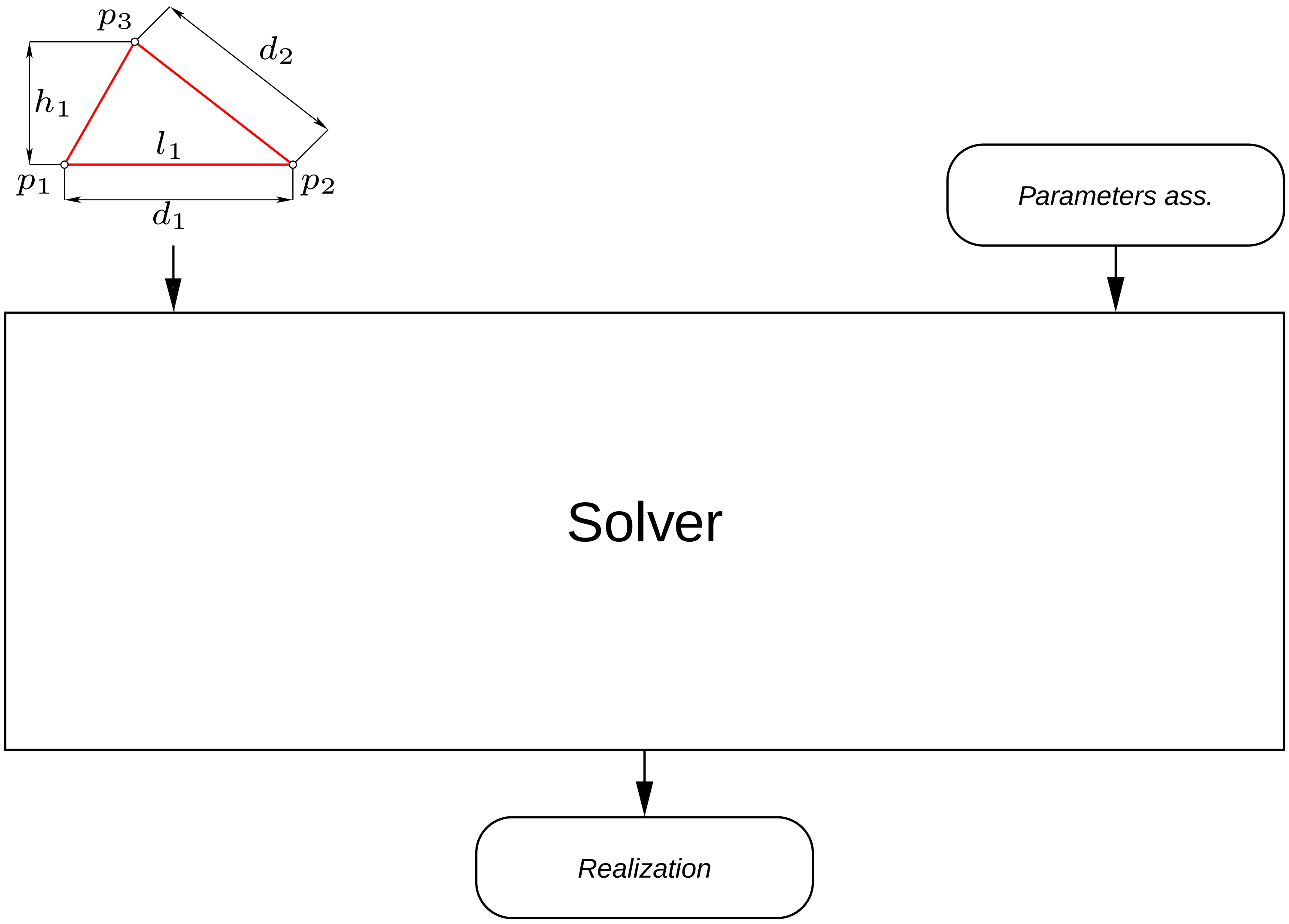

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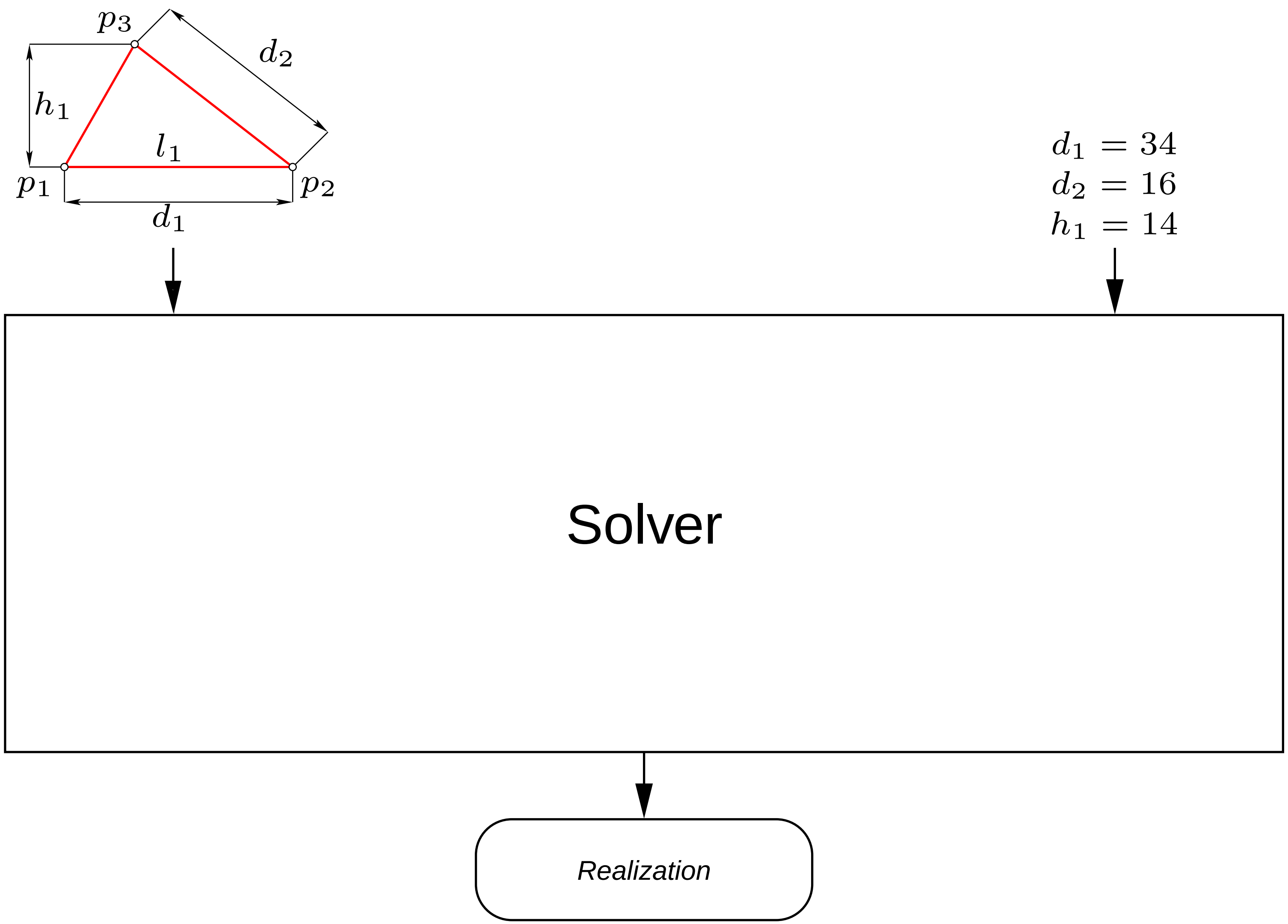

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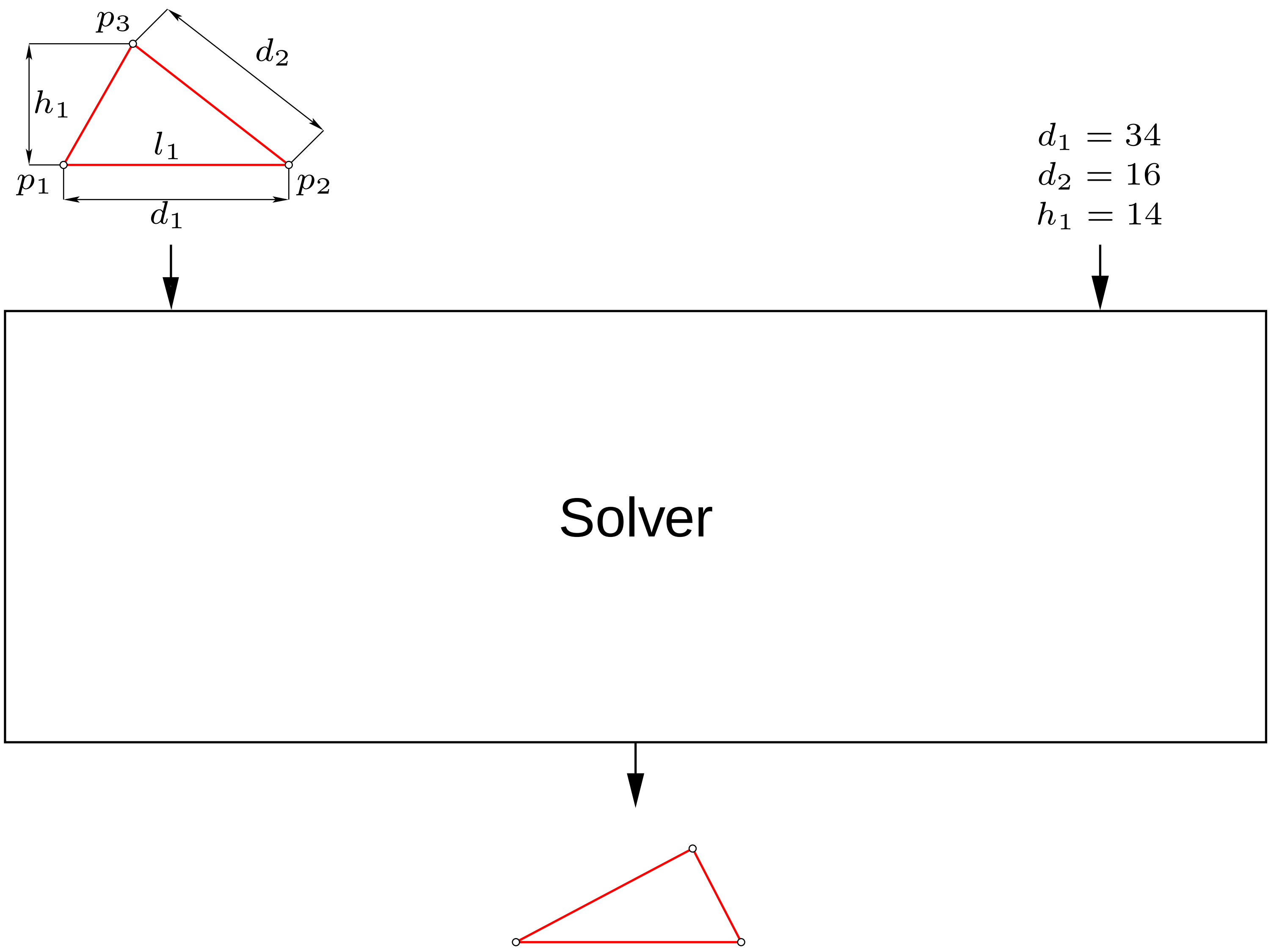
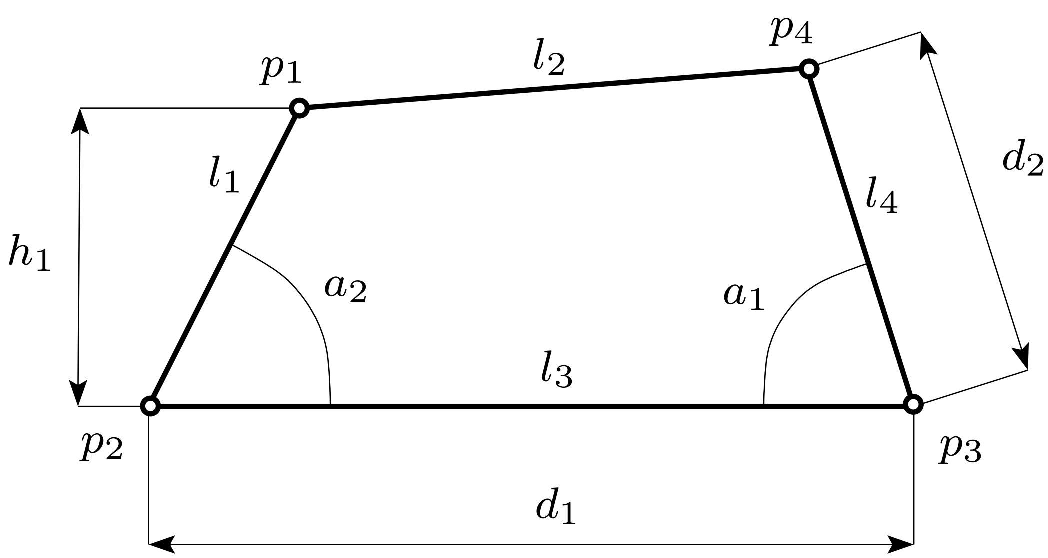


Abstract problems

- An abstract problem $A = \langle G, C, P \rangle$ describes the geometric elements G, the constraints C and the parameters P of the problem.



$$
\begin{aligned}
G &= \{p_1, p_2, p_3, p_4, l_1, l_2, l_3, l_4\} \\
P &= \{d_1, d_2, a_1, a_2, h_1\} \\
C &= \{onPL(p_1, l_1), onPL(p_1, l_2), \\
&\quad onPL(p_2, l_1), onPL(p_2, l_3), \\
&\quad onPL(p_3, l_3), onPL(p_3, l_4), \\
&\quad onPL(p_4, l_2), onPL(p_4, l_4), \\
&\quad distPP(p_2, p_3, d_1), \\
&\quad distPP(p_3, p_4, d_2), \\
&\quad distPL(p_1, l_3, h_1), \\
&\quad angleLL(l_3, l_1, a_2), \\
&\quad angleLL(l_3, l_4, a_1)\}
\end{aligned}
$$

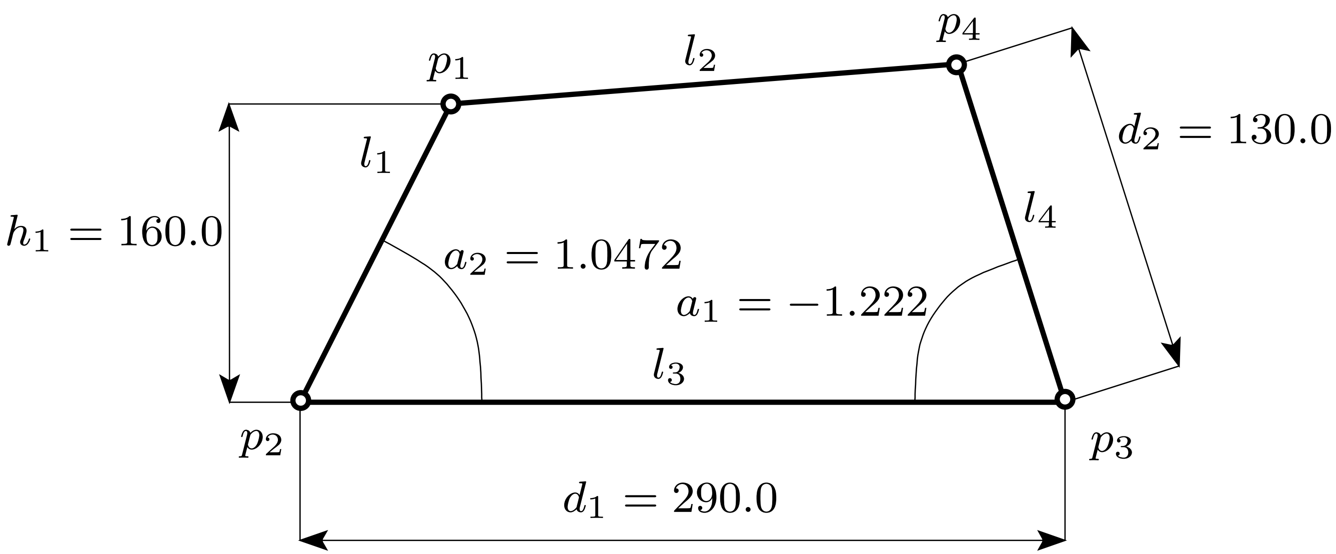
What does an abstract problem mean?

- Characteristic formula Ψ.

$$
\begin{array}{rcl}
\Psi(A) & \equiv & (onPL(p_1, l_1) \\
& \wedge & onPL(p_1, l_2) \\
& \wedge & onPL(p_2, l_1) \\
& \wedge & onPL(p_2, l_3) \\
& \wedge & onPL(p_3, l_3) \\
& \wedge & onPL(p_3, l_4) \\
& \wedge & onPL(p_4, l_2) \\
& \wedge & onPL(p_4, l_4) \\
& \wedge & distPP(p_2, p_3, d_1) \\
& \wedge & distPP(p_3, p_4, d_2) \\
& \wedge & distPL(p_1, l_3, h_1) \\
& \wedge & angleLL(l_3, l_1, a_2) \\
& \wedge & angleLL(l_3, l_4, a_1))
\end{array}
$$

Parameters assignments

- A *parameters assignment* α assigns values to parameters symbols.



$$
\begin{aligned}
\alpha(a_1) &= -1.222 \\
\alpha(a_2) &= 1.0472 \\
\alpha(h_1) &= 160.0 \\
\alpha(d_1) &= 290.0 \\
\alpha(d_2) &= 130.0
\end{aligned}
$$

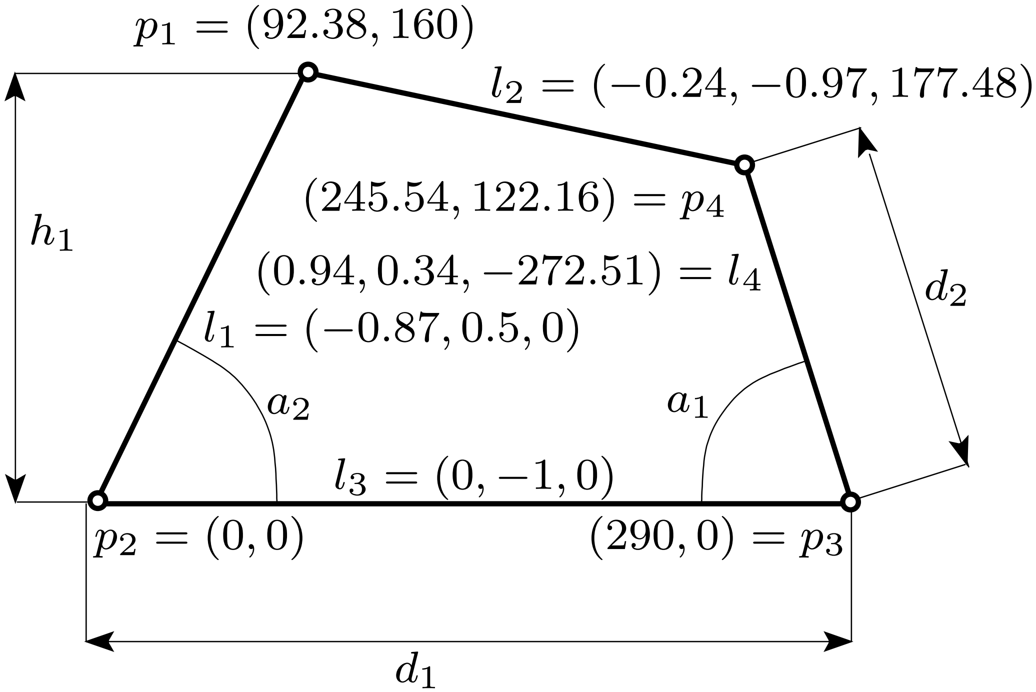
Instance problems

- $\alpha.A = \langle G, \alpha.C, P \rangle$ is an *instance problem*.

$$
\begin{aligned}
\alpha.C \quad = \quad \{ & onPL(p_1, l_1), \\
& onPL(p_1, l_2), \\
& onPL(p_2, l_1), \\
& onPL(p_2, l_3), \\
& onPL(p_3, l_3), \\
& onPL(p_3, l_4), \\
& onPL(p_4, l_2), \\
& onPL(p_4, l_4), \\
& distPP(p_2, p_3, 290.0), \\
& distPP(p_3, p_4, 130.0), \\
& distPL(p_1, l_3, 160.0), \\
& angleLL(l_3, l_1, 1.0472), \\
& angleLL(l_3, l_4, -1.222) \}
\end{aligned}
$$

Geometry assignments

- A *geometry assignment* κ assigns coordinates to geometric elements.



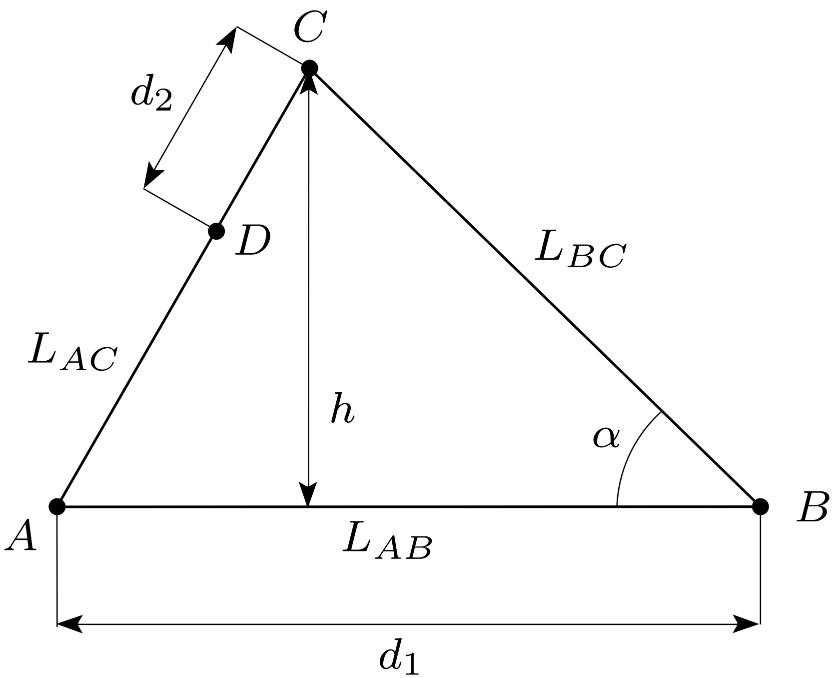
$$
\begin{aligned}
\kappa(p_1) &= (92.38, 160) \\
\kappa(p_2) &= (0, 0) \\
\kappa(p_3) &= (290, 0) \\
\kappa(p_4) &= (245.54, 122.16) \\
\kappa(l_1) &= (-0.87, 0.5, 0) \\
\kappa(l_2) &= (-0.24, -0.97, 177.48) \\
\kappa(l_3) &= (0, -1, 0) \\
\kappa(l_4) &= (0.94, 0.34, -272.51)
\end{aligned}
$$

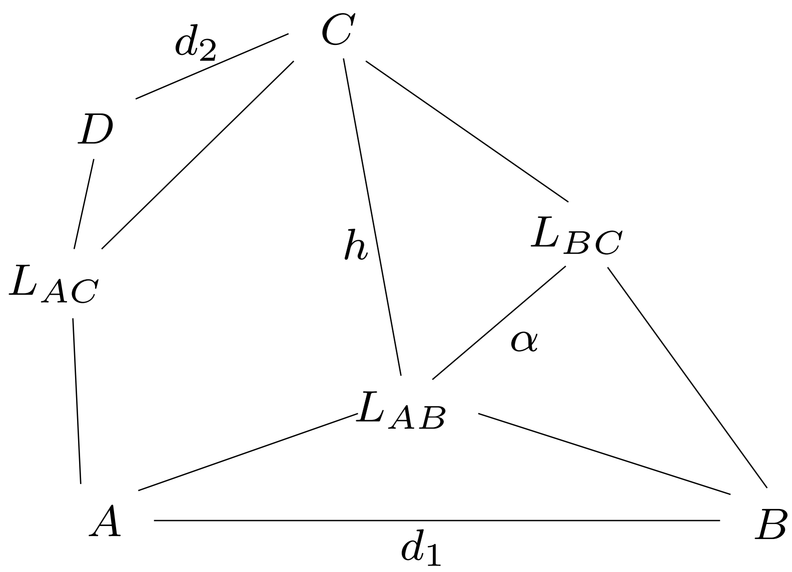
Which are the solutions of an abstract problem?

- A *realization* of an instance problem $\alpha.A$ is a geometry assignment κ for which the formula $\Psi(\kappa.\alpha.A)$ holds.
- $V(\alpha.A)$ is the set of realizations of the instance problem $\alpha.A$.

$$V(\alpha.A) = \{\kappa \quad | \quad \Psi(\kappa.\alpha.A)\}$$

Geometric Constraint Graph



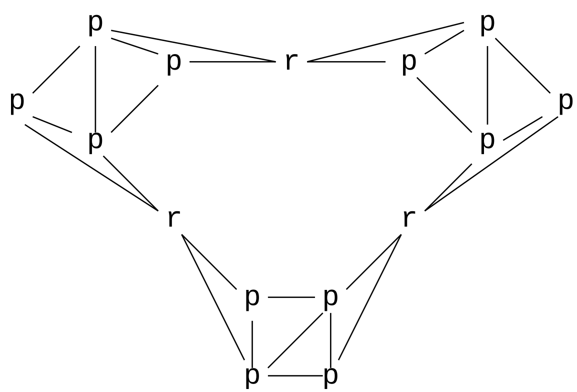


A geometric constraint problem can also be represented by means of a *geometric constraint graph* $G = (V, E)$ where the nodes in V are geometric elements with two degrees of freedom and the edges in $E \subseteq V \times V$ are geometric constraints such that each of them cancels one degree of freedom.

Well-constrained graphs

Theorem 0 (Laman, 1970) *Let $G = (P, D)$ be a geometric constraint graph such that the vertices in P are points in the two-dimensional Euclidean space and the edges in $D \subseteq P \times P$ are distance constraints. G is* generically well-constrained *if and only if for all $G' = (P', D')$, induced subgraph of G by the set of vertices $P' \subseteq P$,*

1. $|D'| \leq 2\,|P'| - 3$, and
2. $|D| = 2\,|P| - 3$.



Structurally well-constrained graphs

A necessary condition for a geometric constraint problem to be solvable is that the associated constraint graph must be structurally well-constrained.

Let $G = (V, E)$ be a geometric constraint graph.

1. G is *structurally over-constrained* if there is an induced subgraph with $m \leq |V|$ nodes and more than $2m - 3$ edges.
2. G is *structurally under-constrained* if it is not structurally over-constrained and $|E| < 2\,|V| - 3$.
3. G is *structurally well-constrained* if it is not structurally over-constrained and $|E| = 2\,|V| - 3$.