Constructive Geometric Constraint Solving

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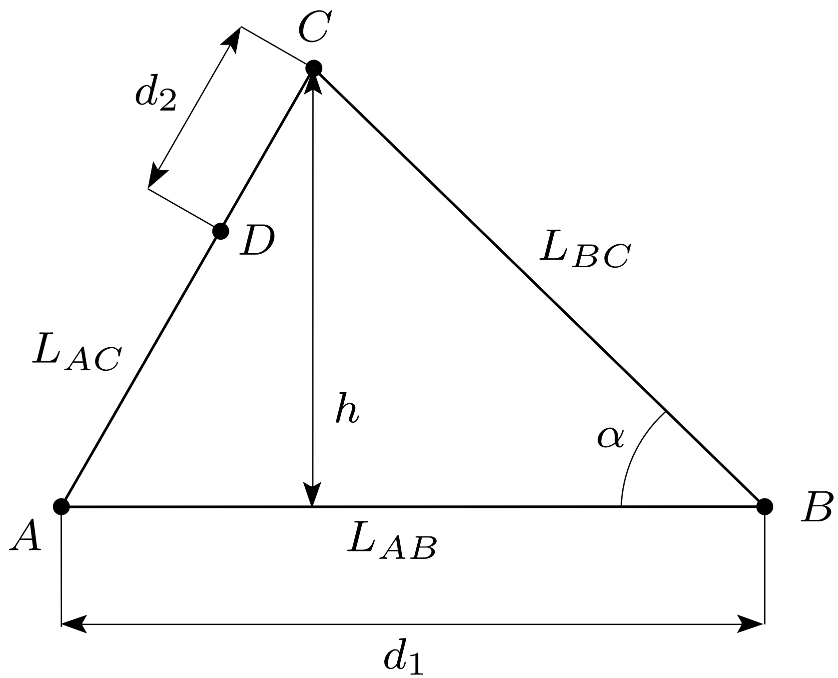
Preliminaries

Geometric constraint problem



A *geometric constraint problem* consists of

- a set of geometric elements, $\{A, B, C, D, L_{AB}, L_{AC}, L_{BC}\}$,
- a set of geometric constraints defined between them, and
- a set of parameters, $\{d_1, d_2, \alpha, h\}$.

Geometric constraint solving

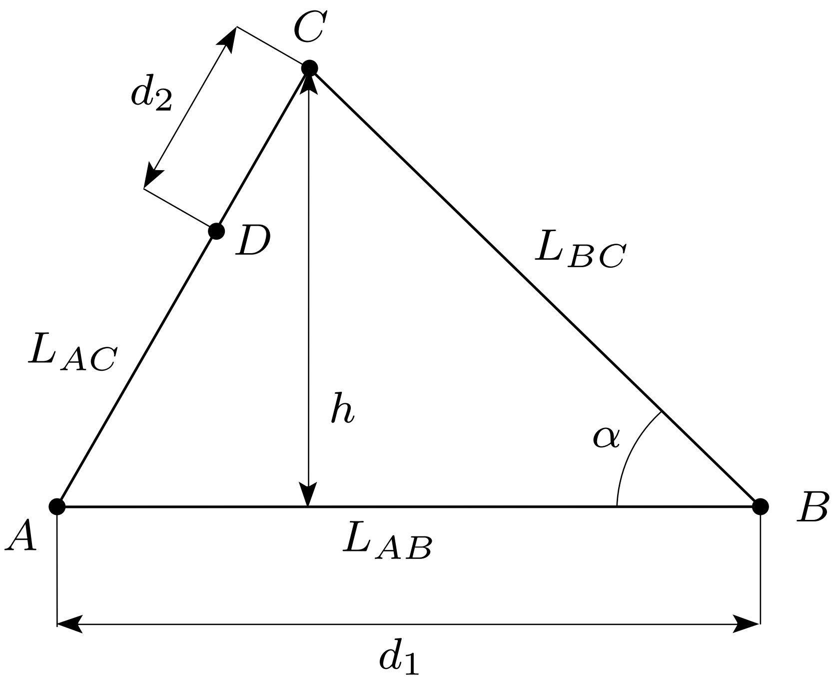
A geometric constraint problem can be represented by a predicate φ in first order logic.

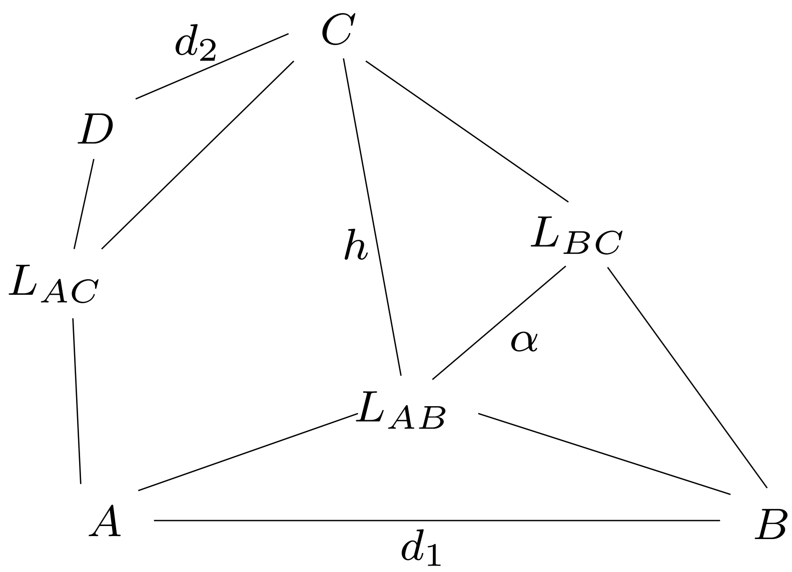
$$\begin{aligned} &\varphi(A, B, C, D, L_{AB}, L_{AC}, L_{BC}) \\ &\quad \equiv \quad \mathrm{d}(A, B) = d_1 \wedge \mathrm{on}(A, L_{AB}) \wedge \mathrm{on}(B, L_{AB}) \wedge \mathrm{on}(A, L_{AC}) \wedge \\ &\qquad\quad \mathrm{on}(C, L_{AC}) \wedge \mathrm{on}(D, L_{AC}) \wedge \mathrm{on}(B, L_{BC}) \wedge \mathrm{on}(C, L_{BC}) \wedge \\ &\qquad\quad \mathrm{h}(C, L_{AB}) = h \wedge \mathrm{a}(L_{AB}, L_{BC}) = \alpha \wedge \mathrm{d}(C, D) = d_2 \end{aligned}$$

Geometric constraint solving consists in proving the truth of the existentially quantified predicate φ that represents the geometric constraint problem.

$$\exists A \; \exists B \; \exists C \; \exists D \; \exists L_{AB} \; \exists L_{AC} \; \exists L_{BC} \quad \varphi(A, B, C, D, L_{AB}, L_{AC}, L_{BC})$$

Geometric Constraint Graph



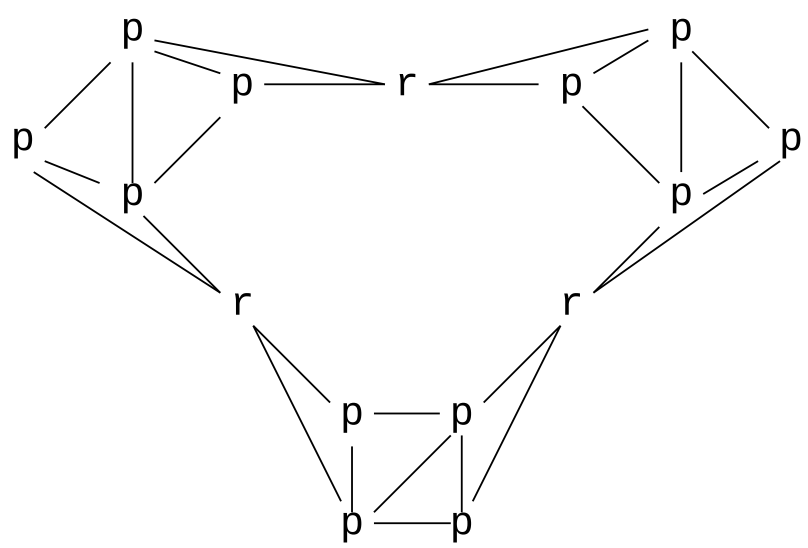


A geometric constraint problem can also be represented by means of a *geometric constraint graph* $G = (V, E)$ where the nodes in V are geometric elements with two degrees of freedom and the edges in $E \subseteq V \times V$ are geometric constraints such that each of them cancels one degree of freedom.

Well-constrained graphs

Theorem 1 (Laman, 1970) *Let $G = (P, D)$ be a geometric constraint graph such that the vertices in P are points in the two-dimensional Euclidean space and the edges in $D \subseteq P \times P$ are distance constraints. G is* generically well-constrained *if and only if for all $G' = (P', D')$, induced subgraph of G by the set of vertices $P' \subseteq P$,*

1. $|D'| \leq 2\,|P'| - 3$, and
2. $|D| = 2\,|P| - 3$.



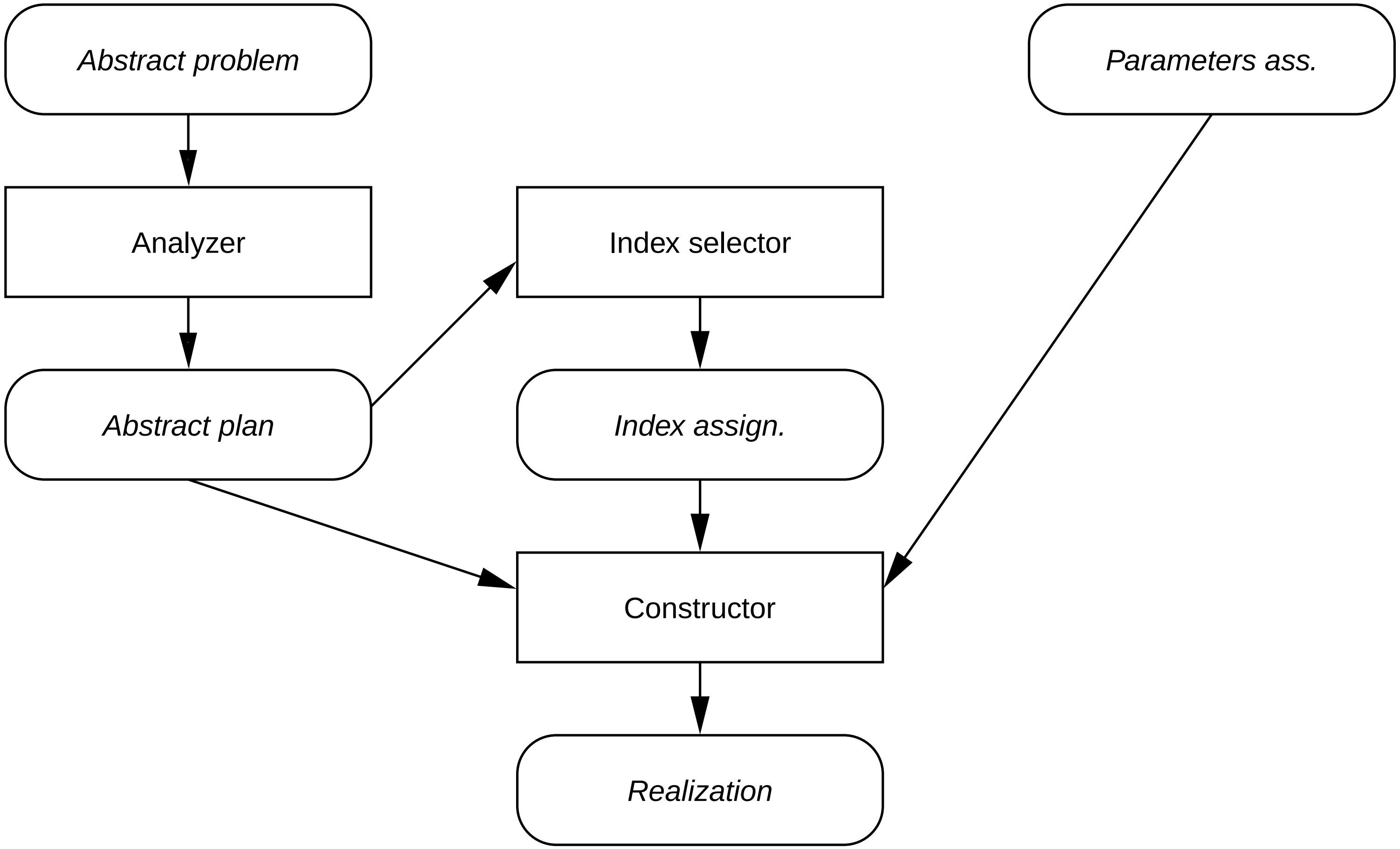
Structurally well-constrained graphs

A necessary condition for a geometric constraint problem to be solvable is that the associated constraint graph must be structurally well-constrained. Let $G = (V, E)$ be a geometric constraint graph.

1. G is *structurally over-constrained* if there is an induced subgraph with $m \leq |V|$ nodes and more than $2m - 3$ edges.
2. G is *structurally under-constrained* if it is not structurally over-constrained and $|E| < 2\,|V| - 3$.
3. G is *structurally well-constrained* if it is not structurally over-constrained and $|E| = 2\,|V| - 3$.

Constructive Geometric Constraint Solvers

Architecture for Constructive Geometric Constraint Solvers



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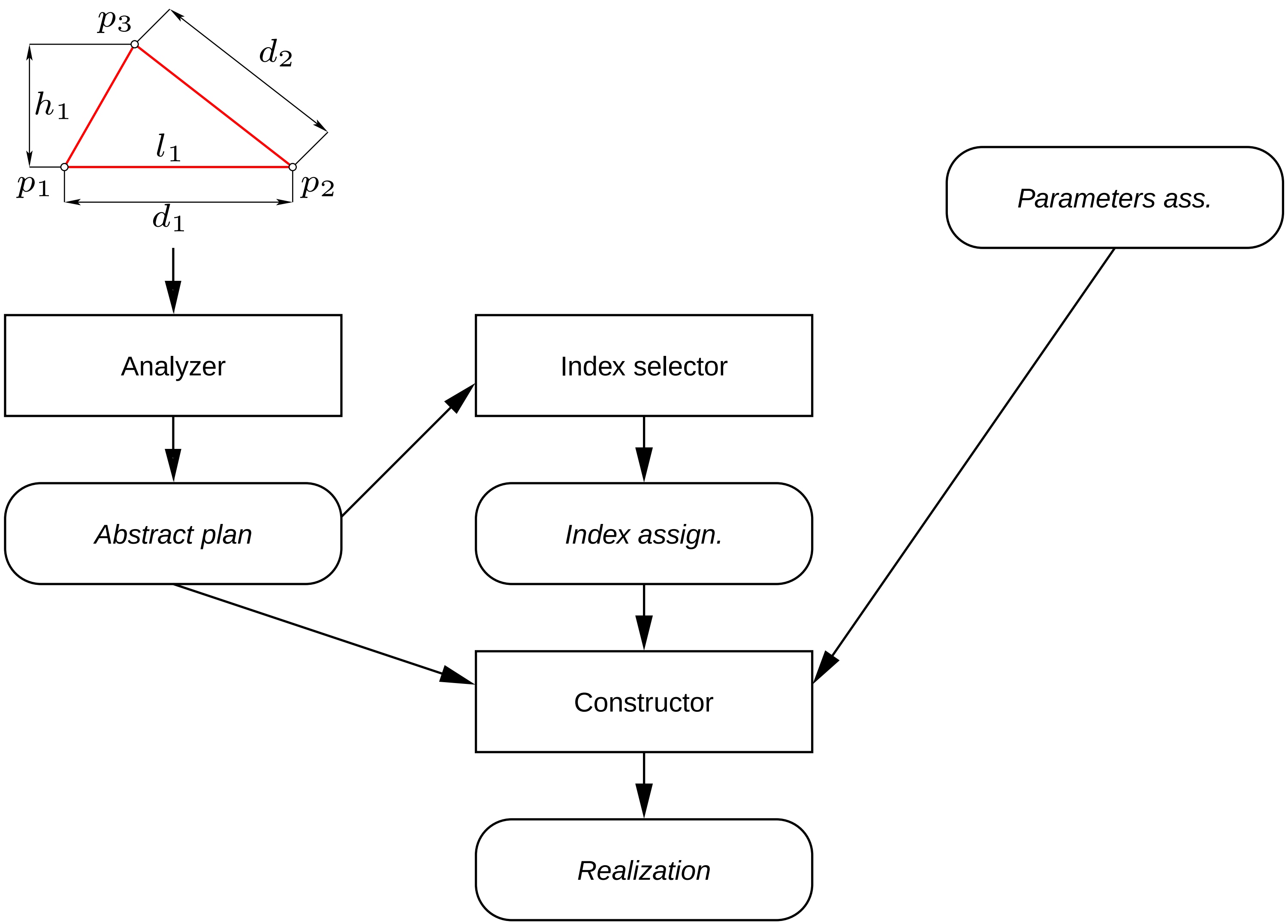

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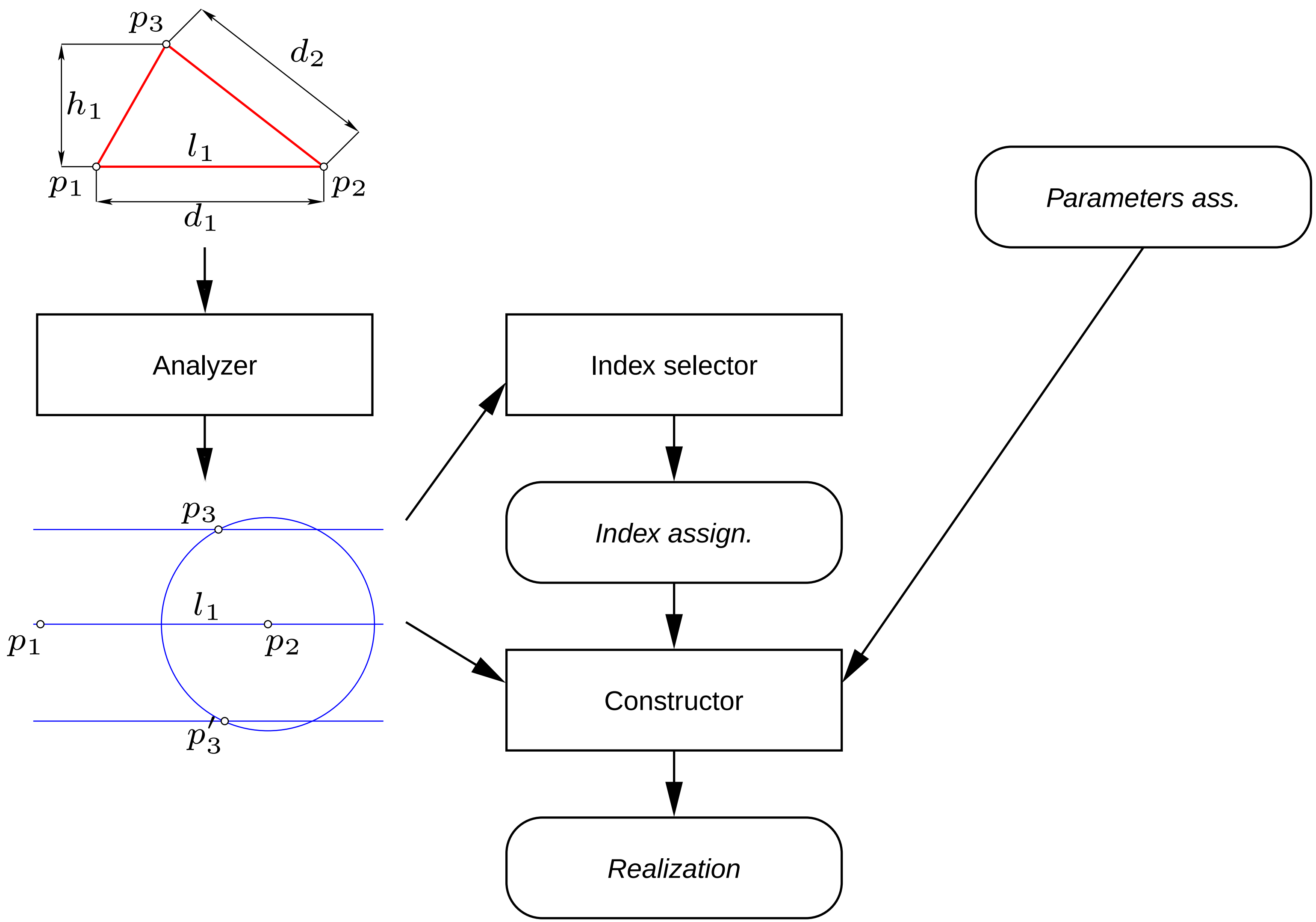

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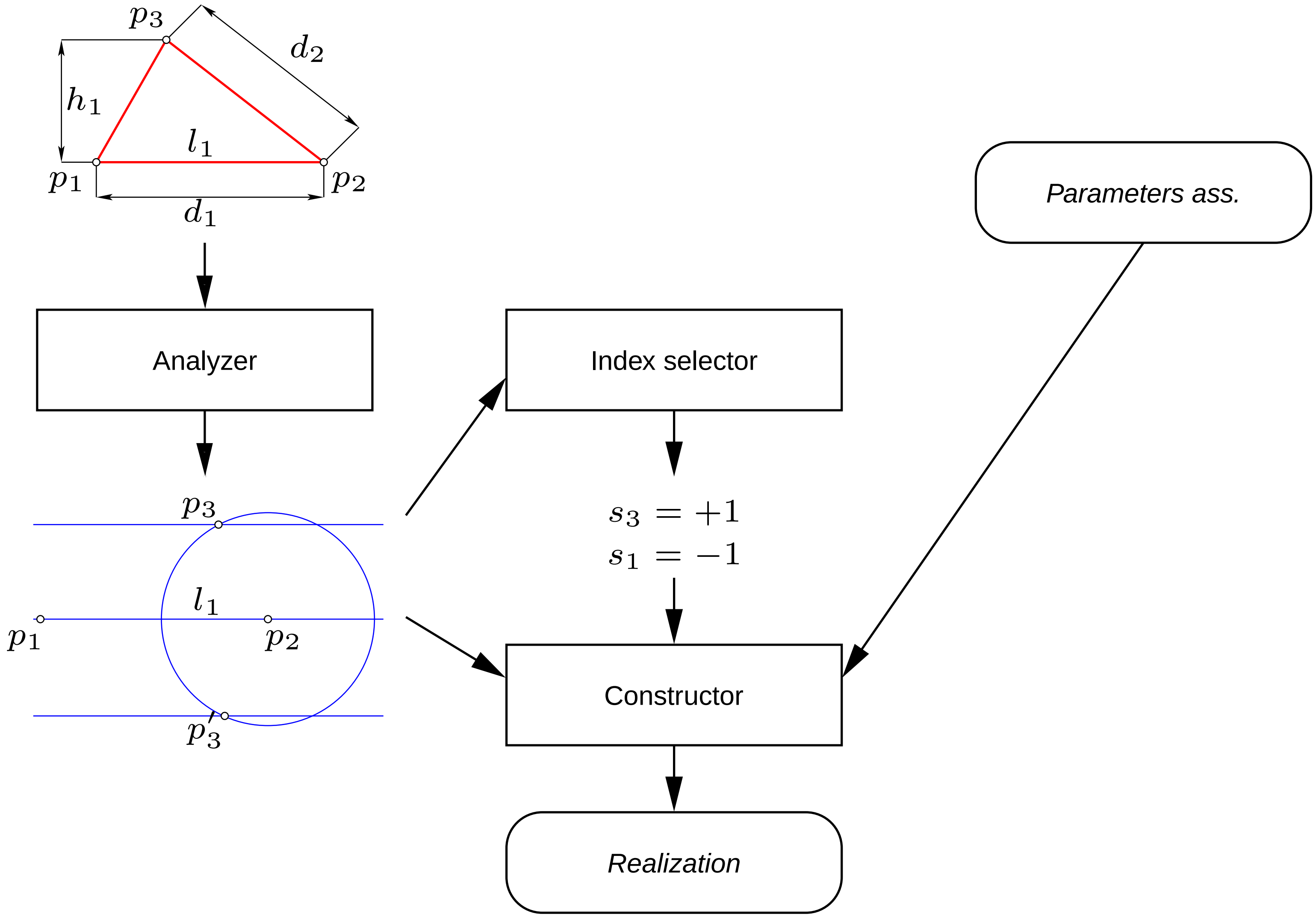

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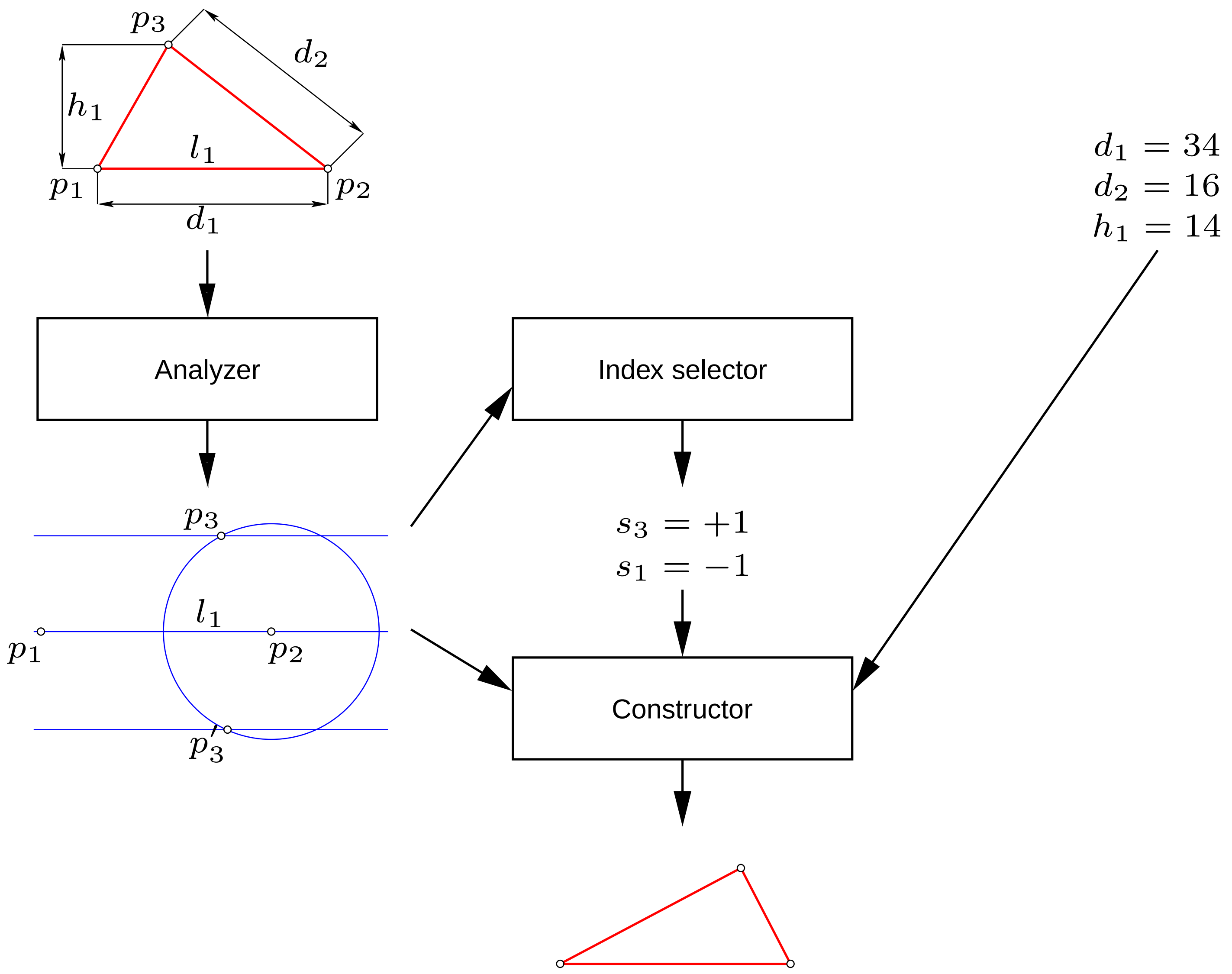
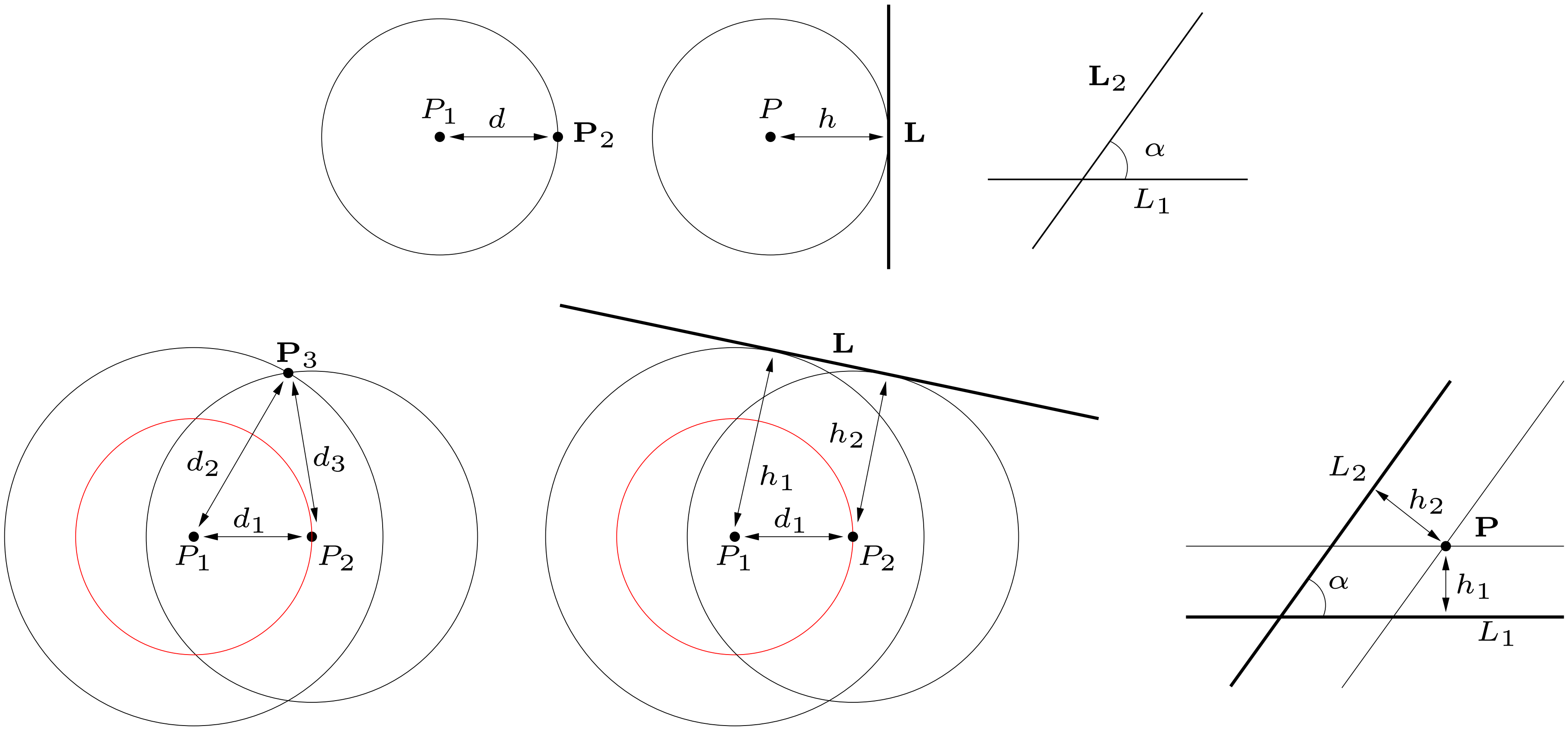


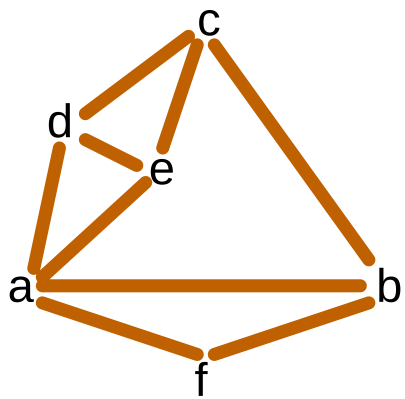
Clusters

A *cluster* is a set of two dimensional geometric elements with known positions with respect to a local coordinate system.



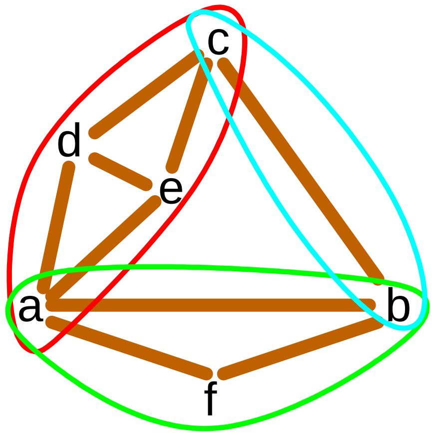
Tree decomposition

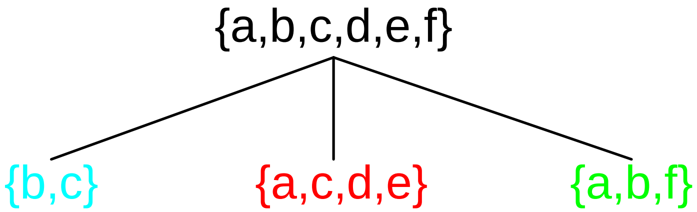
There are graphs that can be tree decomposed



{a,b,c,d,e,f}

There are graphs that can be tree decomposed





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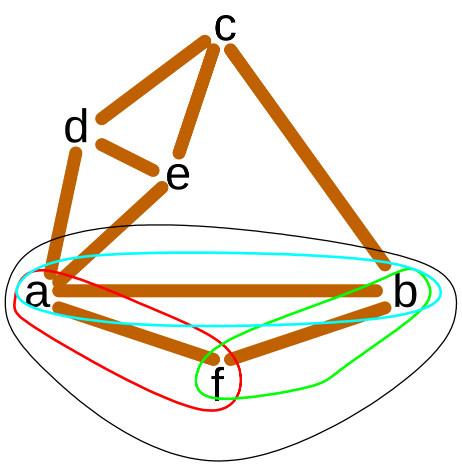

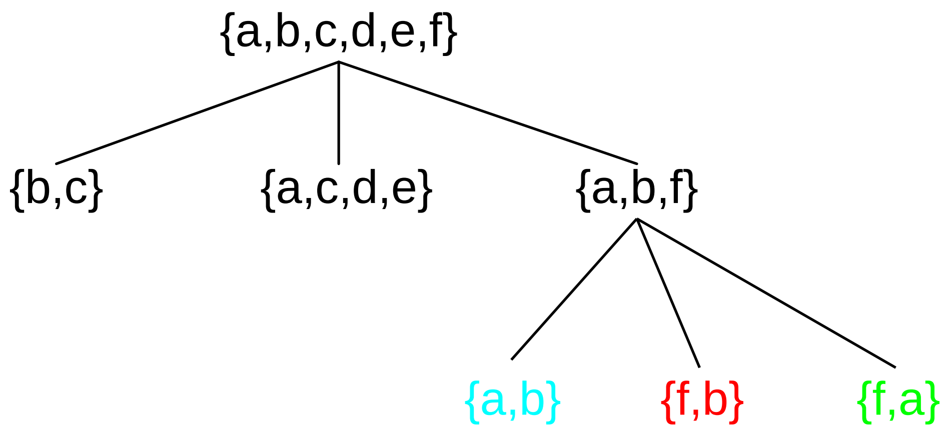

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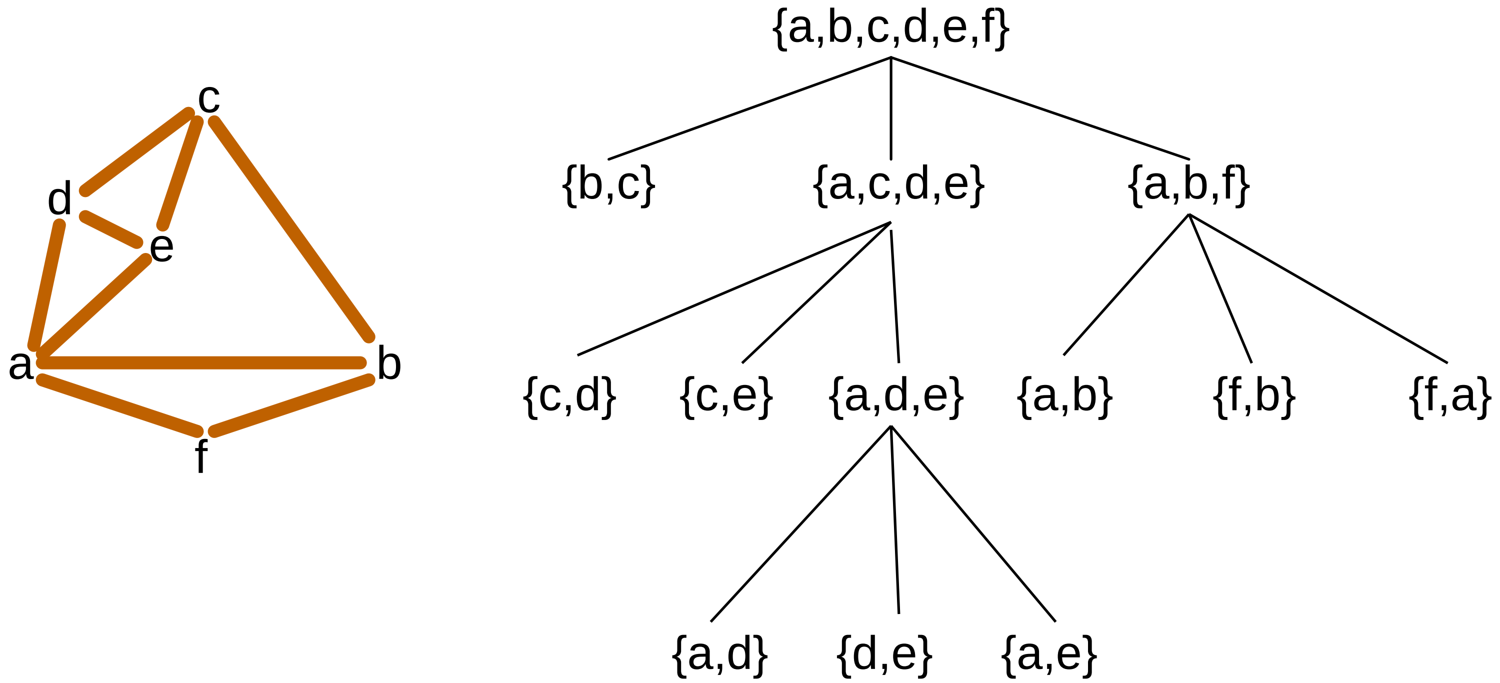
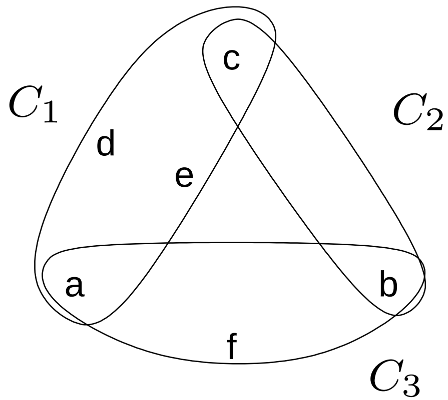
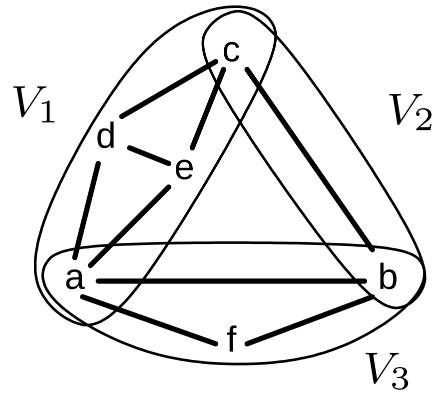


Set decompositions



Let C be a set with, at least, three different members, say a, b, c. Let $\{C_1, C_2, C_3\}$ be three subsets of C. We say that $\{C_1, C_2, C_3\}$ is a *set decomposition* of C if

1. $C_1 \cup C_2 \cup C_3 = C$,
2. $C_1 \cap C_2 = \{a\}$,
3. $C_2 \cap C_3 = \{b\}$ and
4. $C_1 \cap C_3 = \{c\}$



Let $G = (V, E)$ be a graph and let $\{V_1, V_2, V_3\}$ be three subsets of V. $\{V_1, V_2, V_3\}$ is a *set decomposition* of G if it is a set decomposition of V and for every edge e in E, $V(e) \subseteq V_i$ for some i, $1 \leq i \leq 3$.

Tree decomposition

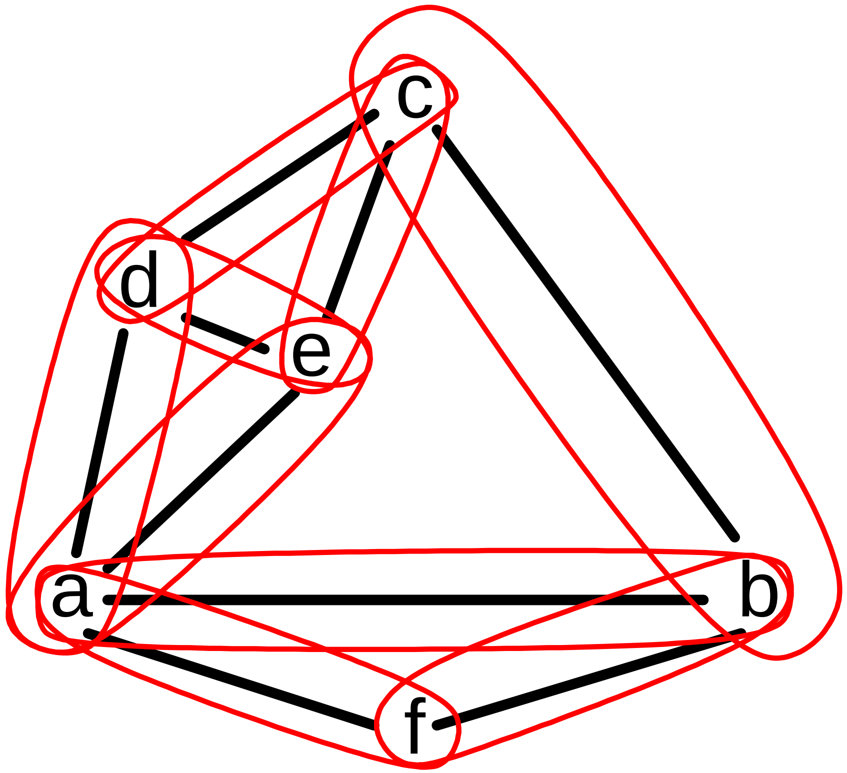
Let $G = (V, E)$ be a graph. A 3-ary tree T is a *tree decomposition* of G if

1. V is the root of T,
2. Each internal node $V' \subset V$ of T is the father of exactly three nodes, say $\{V_1', V_2', V_3'\}$, which are a set decomposition of the subgraph induced by V', and
3. Each leaf node contains exactly two vertices of V.

A graph G is *tree decomposable* if there is a tree decomposition of G.

Reduction analysis

There are graphs that can be reduced



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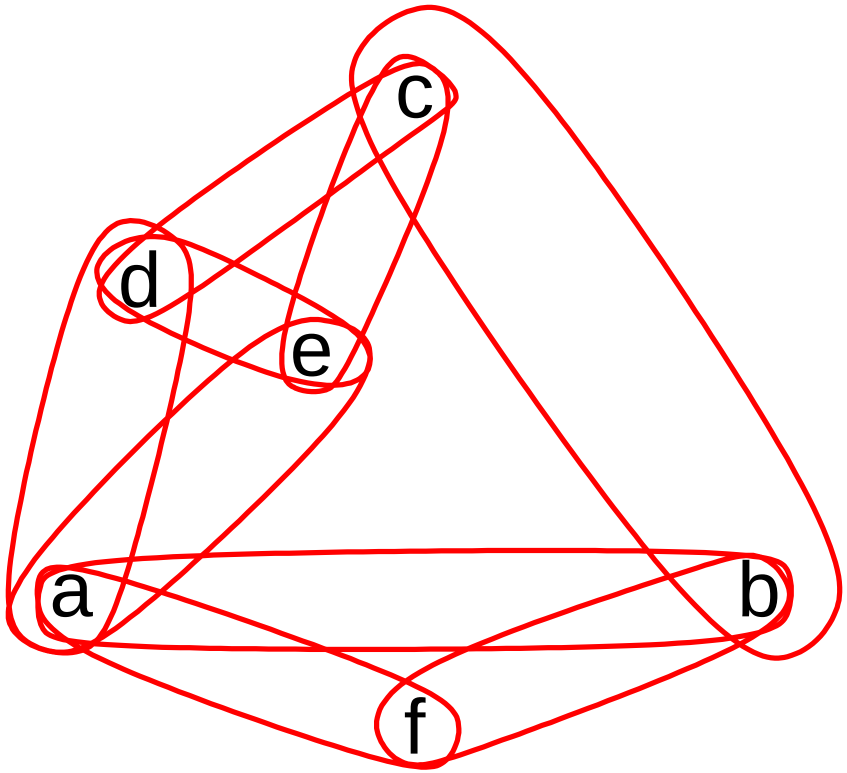

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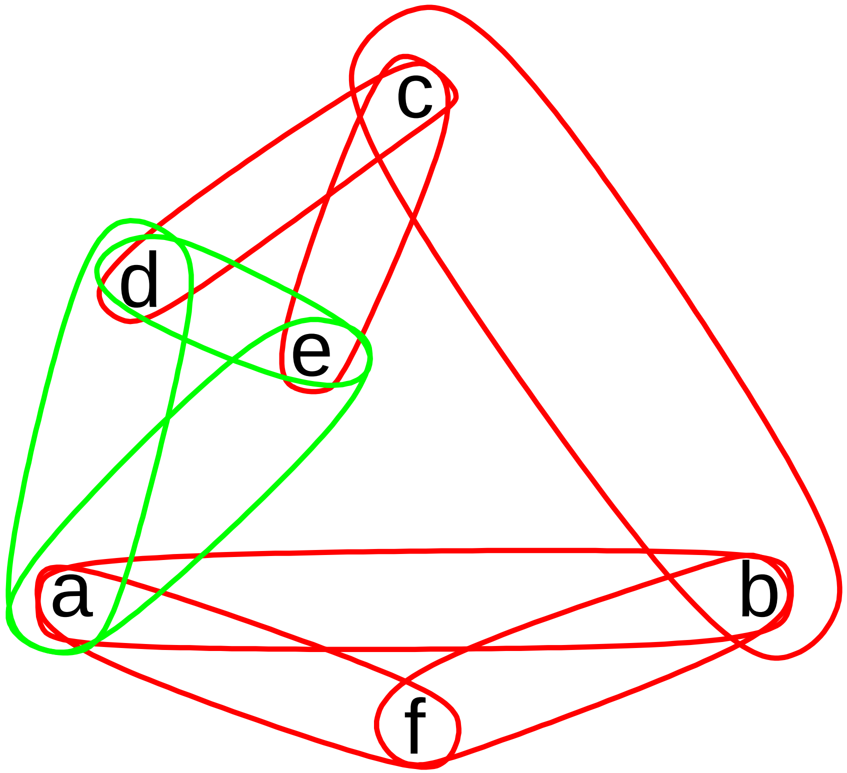

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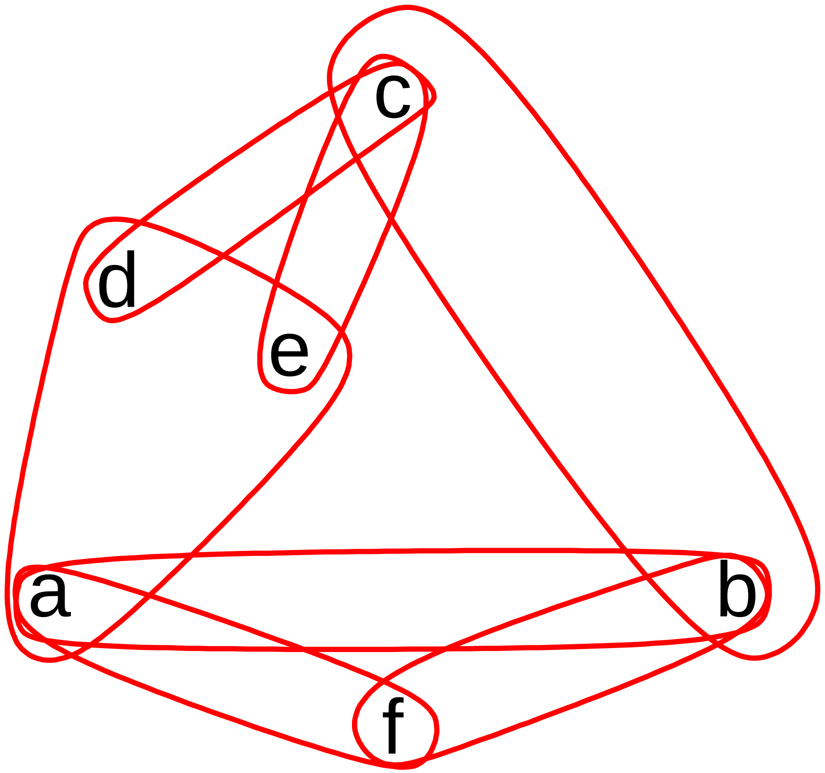

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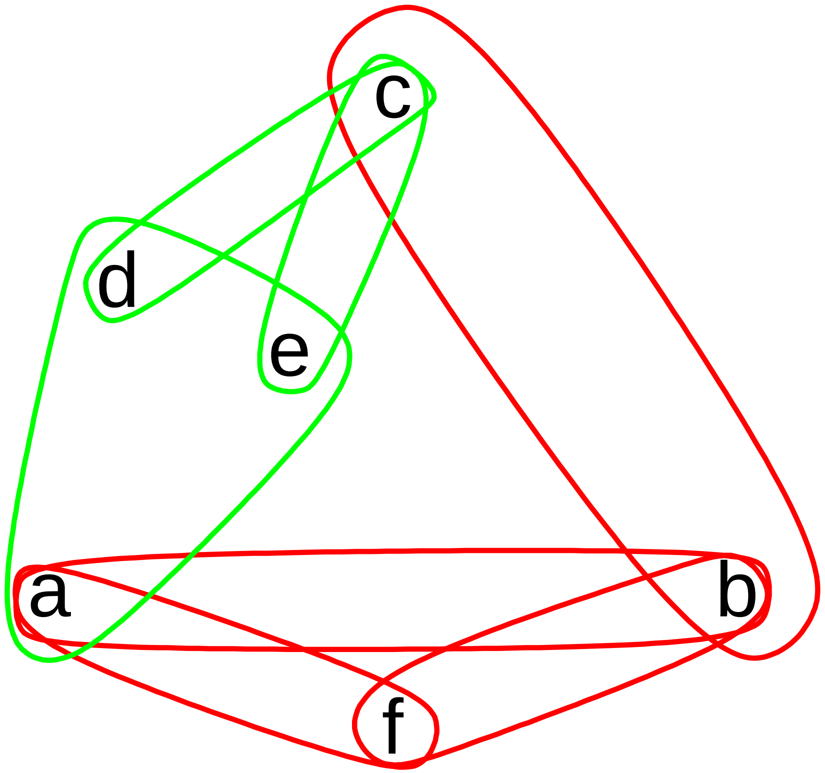

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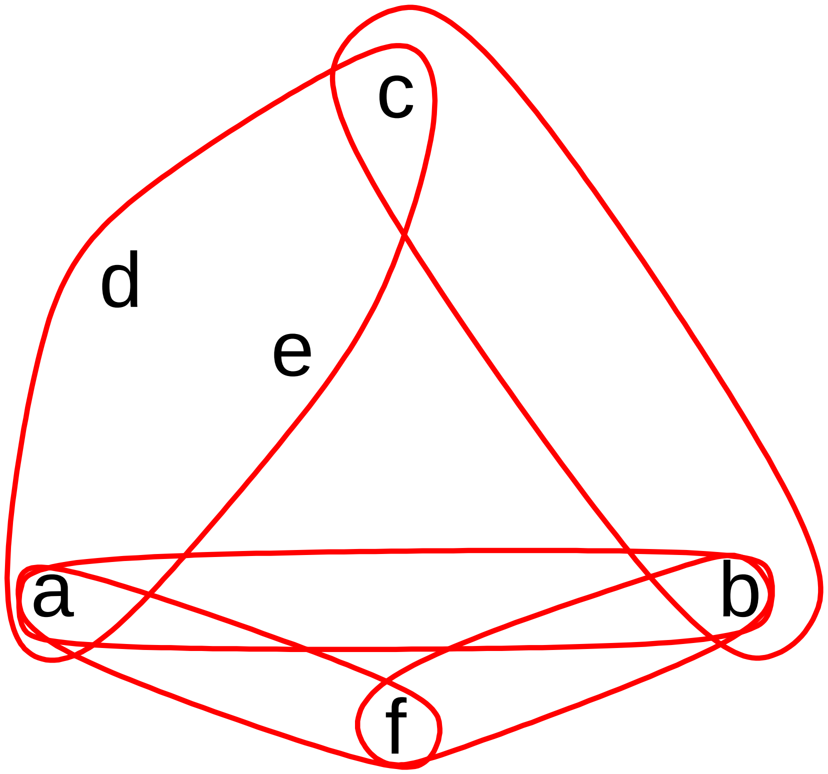

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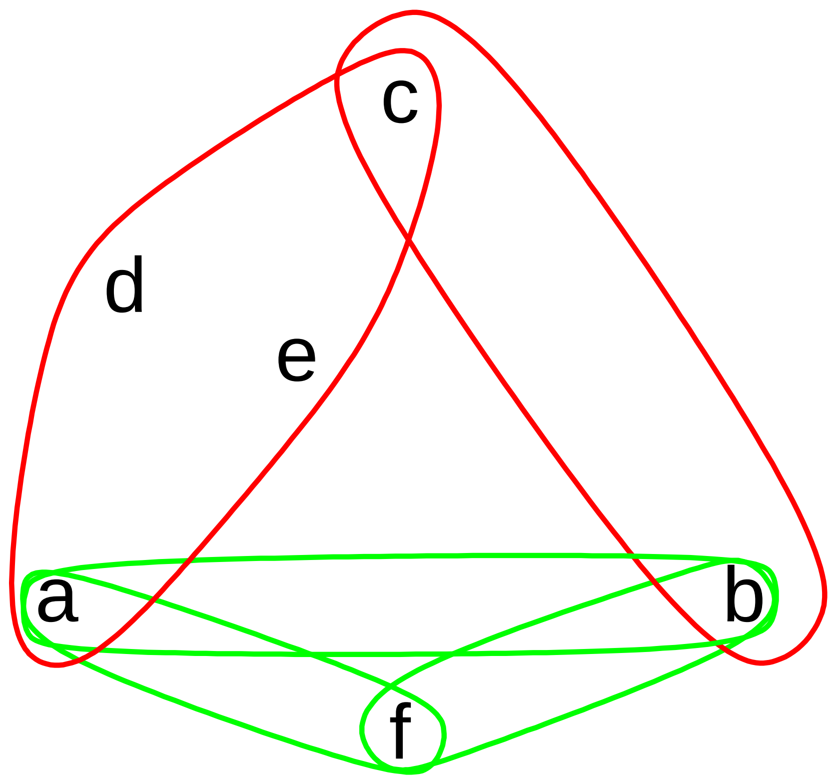

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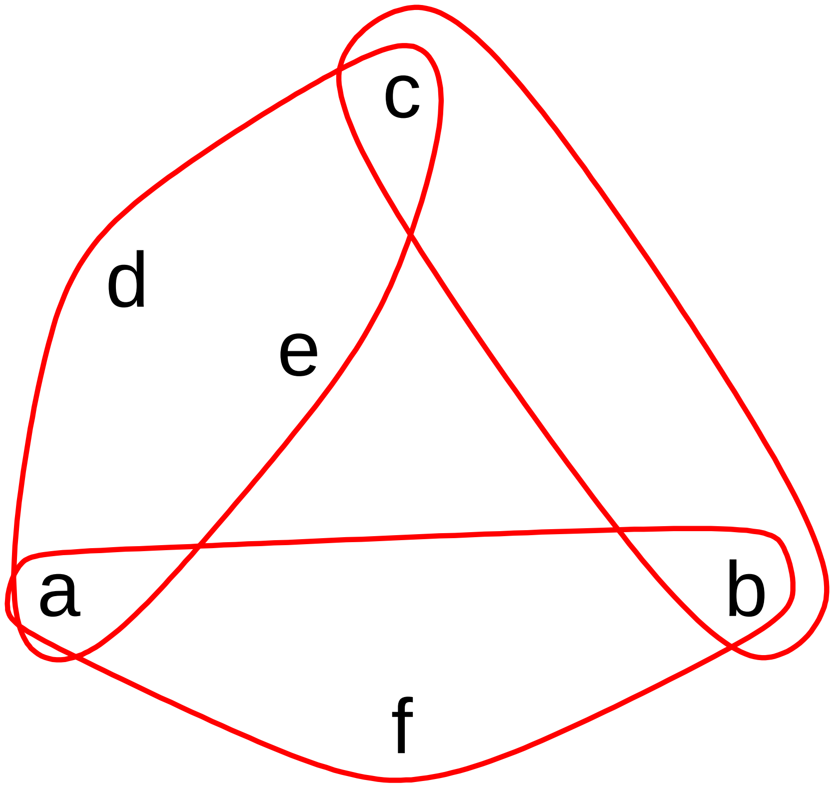

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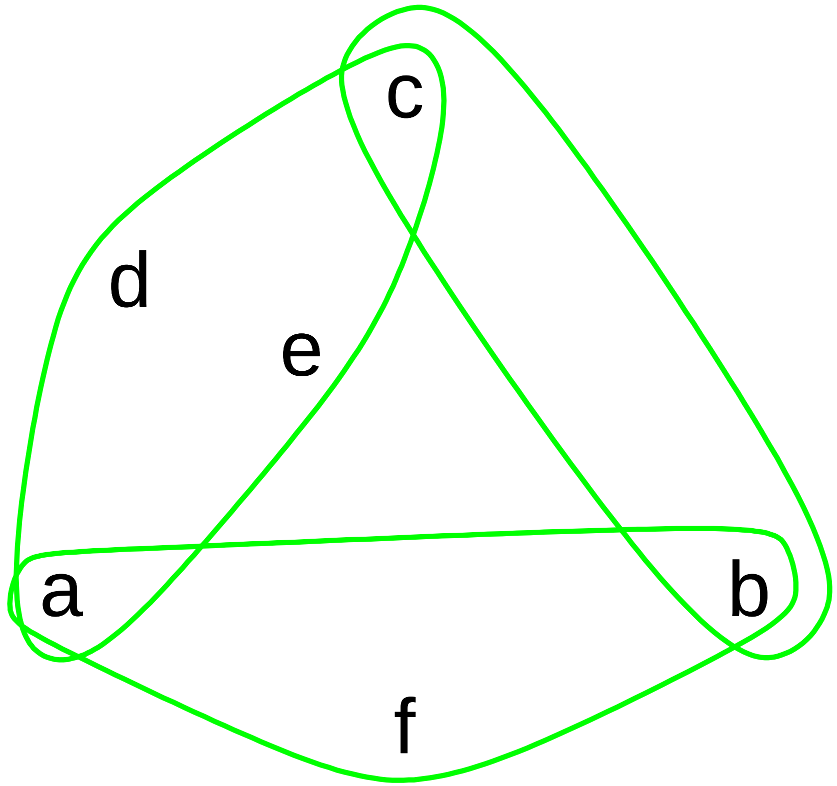

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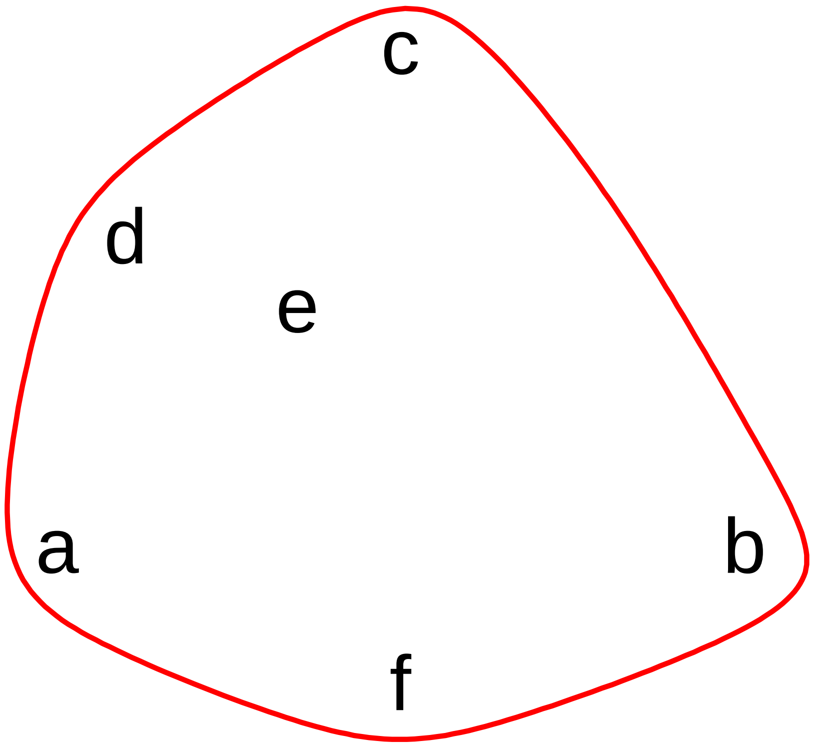


Reduction analysis

Let $G = (V, E)$ be a geometric constraint graph.

We define the *initial set of clusters* $\mathbf{S}_G = \{\{u, v\} \mid (u, v) \in E\}$.

Let $\mathbf{S}$ be a set of clusters in which there are three clusters C_1, C_2, C_3 such that $\{C_1, C_2, C_3\}$ is a set decomposition of C.

$\mathbf{S} \longrightarrow_r \mathbf{S}'$ is a reduction rule where $\mathbf{S}' = (\mathbf{S} - \{C_1, C_2, C_3\}) \cup C$.

The geometric constraint problem represented by the geometric constraint graph G is *solvable by reduction analysis* if $\mathbf{S}_G$ reduces to the singleton $\{V\}$.

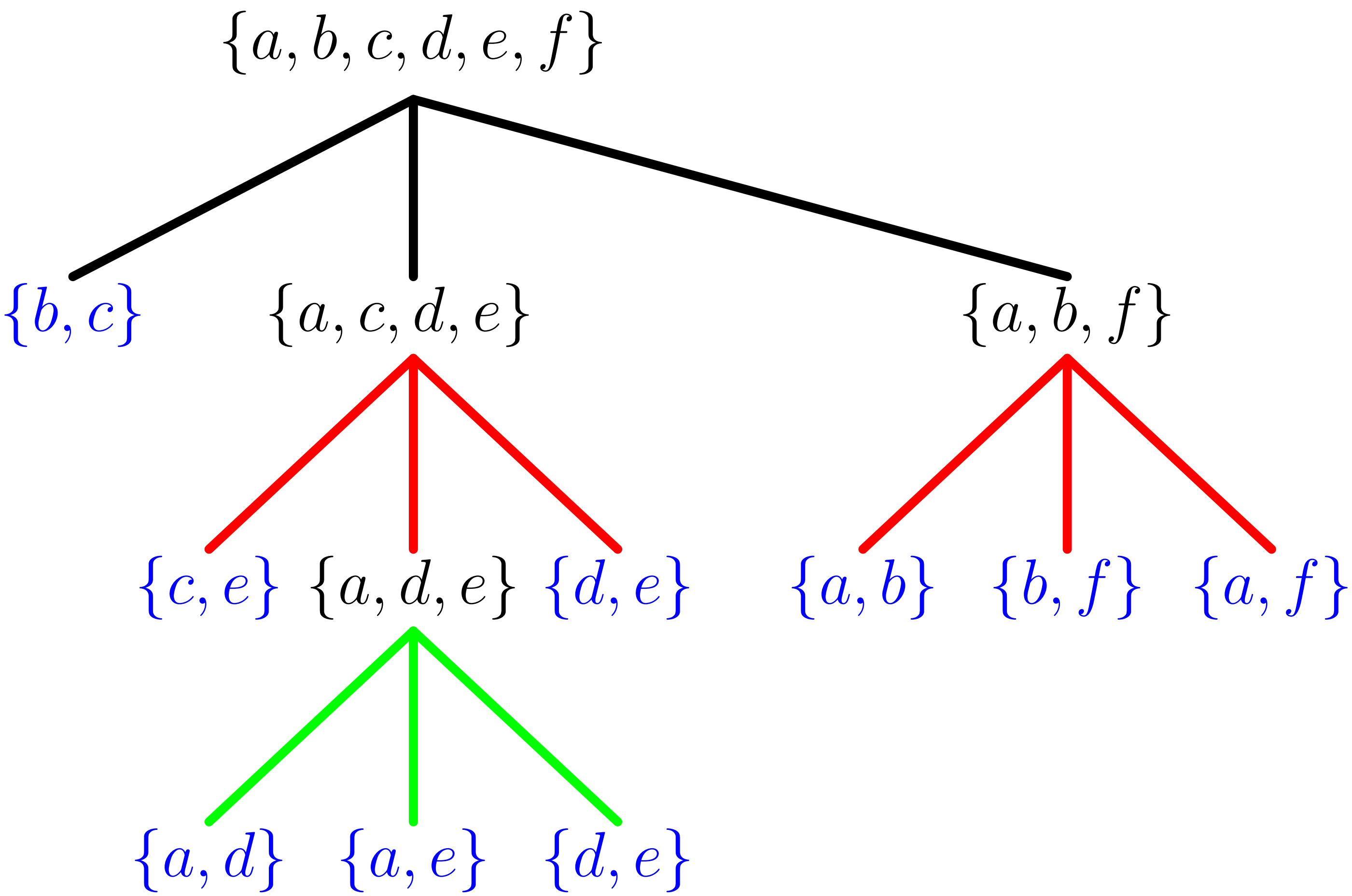
If G is not structurally over-constrained, the abstract reduction system induced by the reduction rule $\longrightarrow_r$ is terminating and confluent which implies the unique normal form property and canonicity.

The domain of solvable graphs by reduction analysis

Let $G = (V, E)$ be a well-constrained geometric constraint graph. The following assertions are equivalent:

1. G is tree decomposable.
2. G is solvable by reduction analysis.

The domain of solvable graphs by reduction analysis



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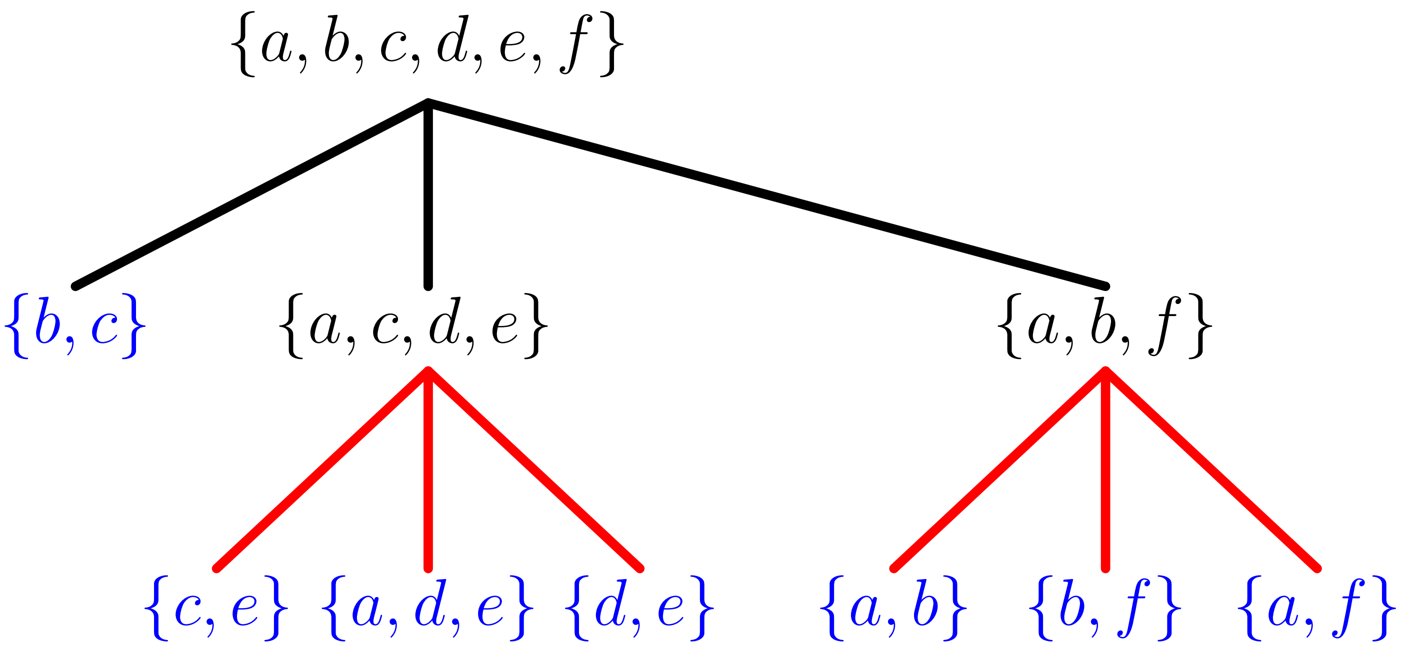

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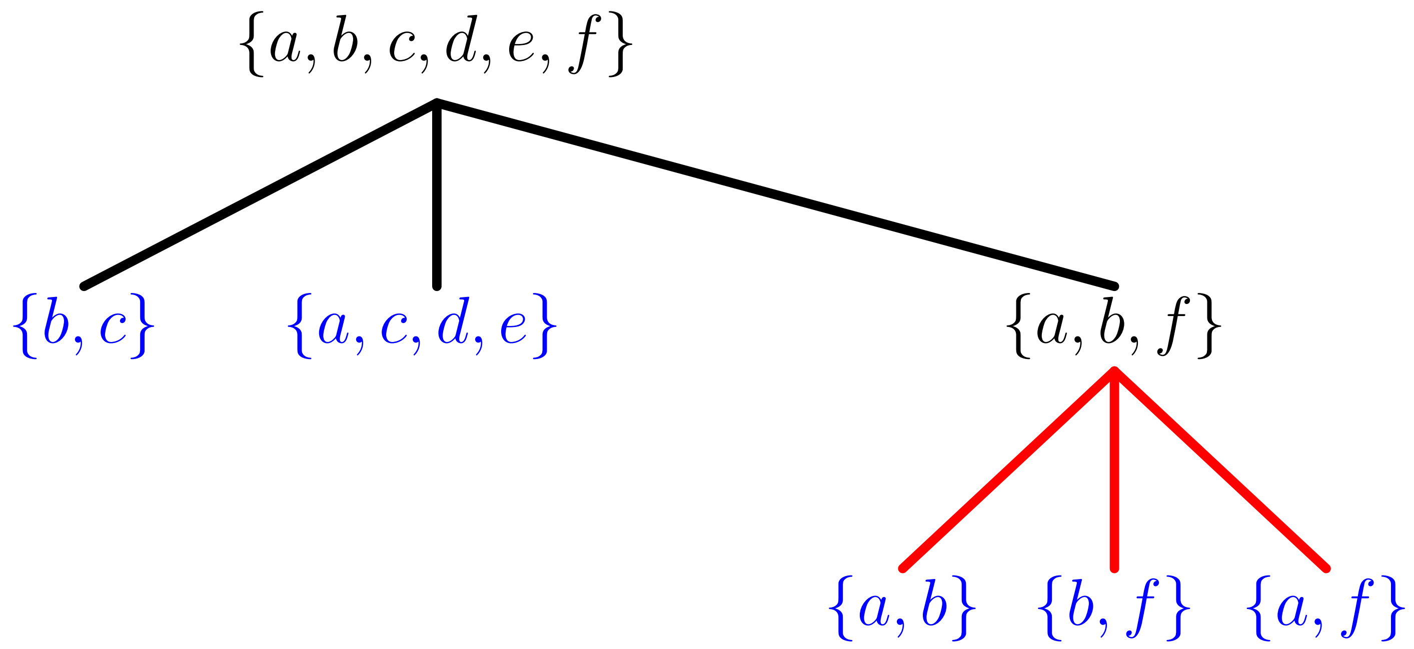

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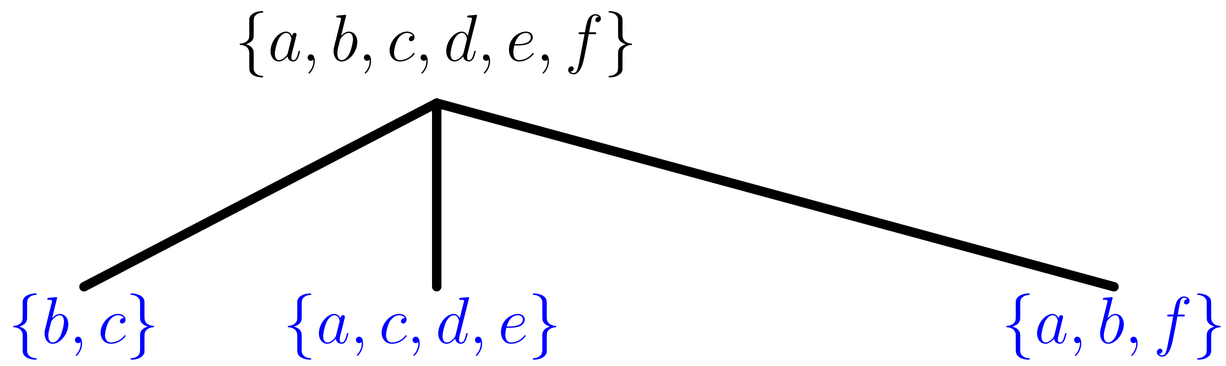
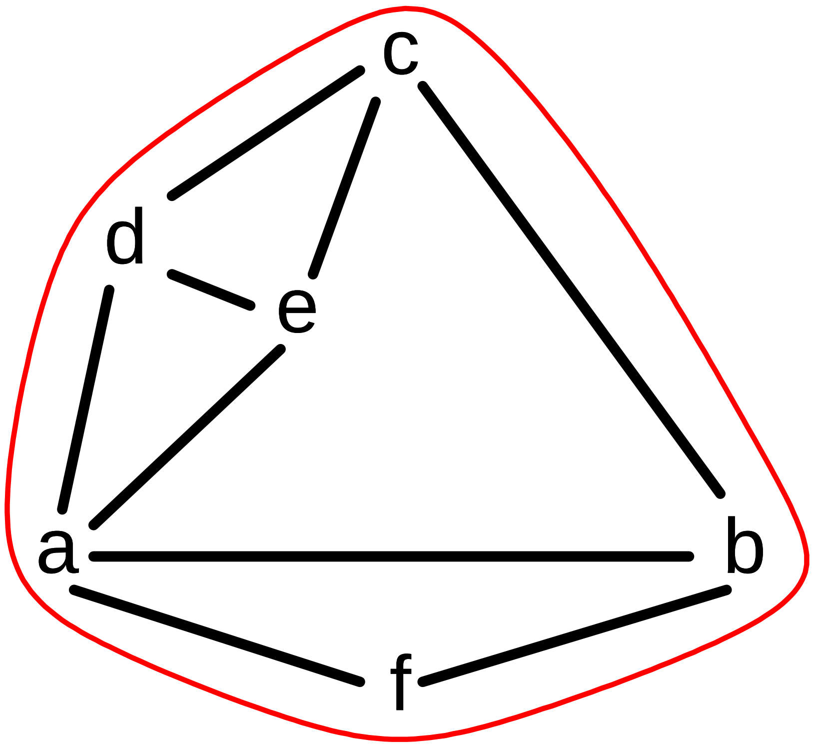


The domain of solvable graphs by reduction analysis

$\{a, b, c, d, e, f\}$

Decomposition analysis

There are graphs that can be decomposed



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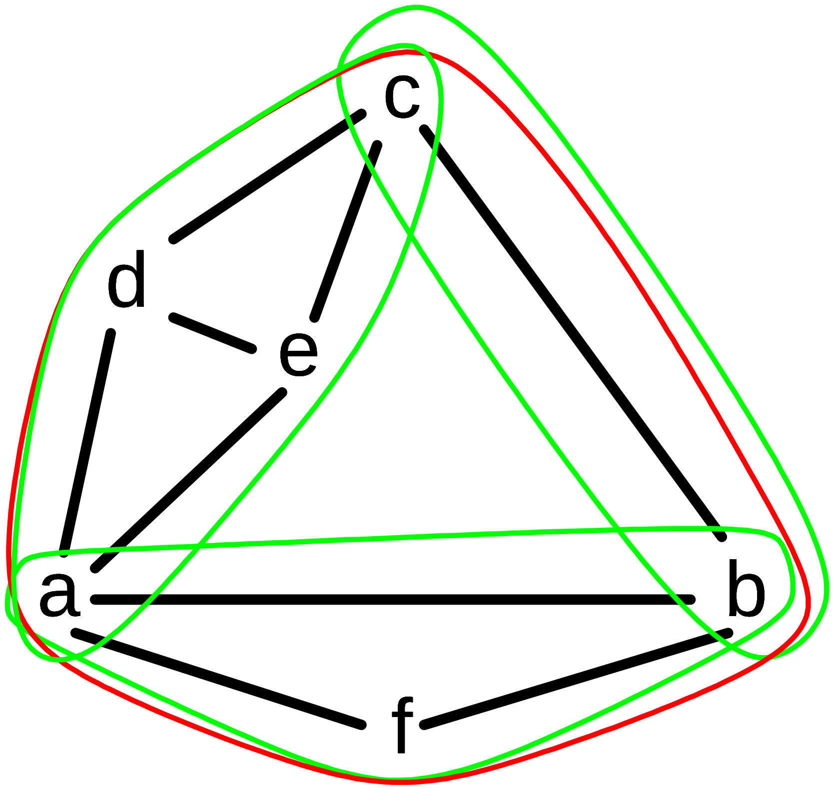

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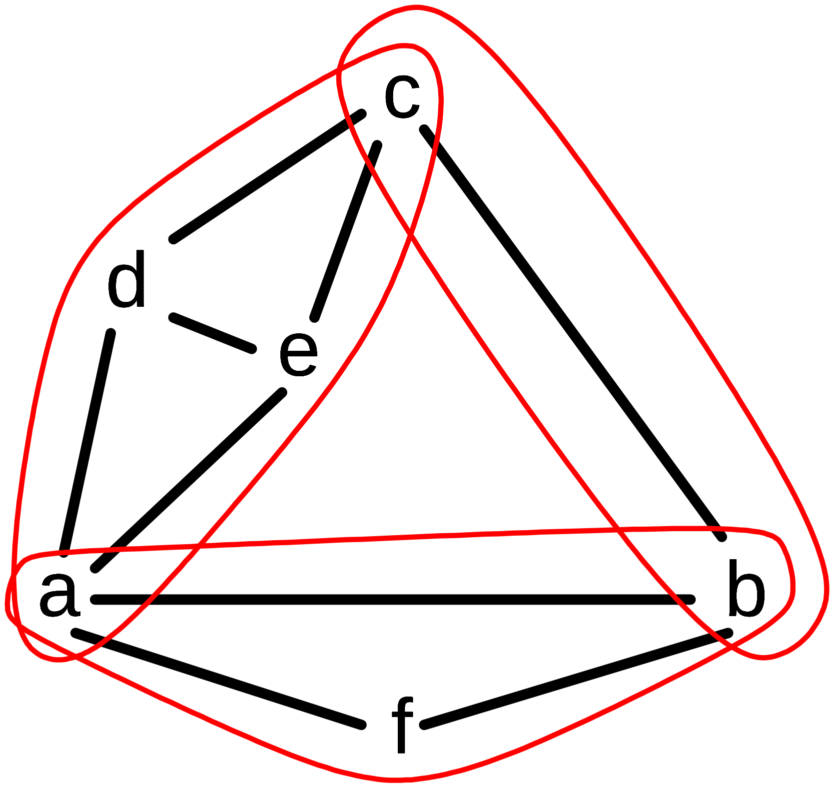

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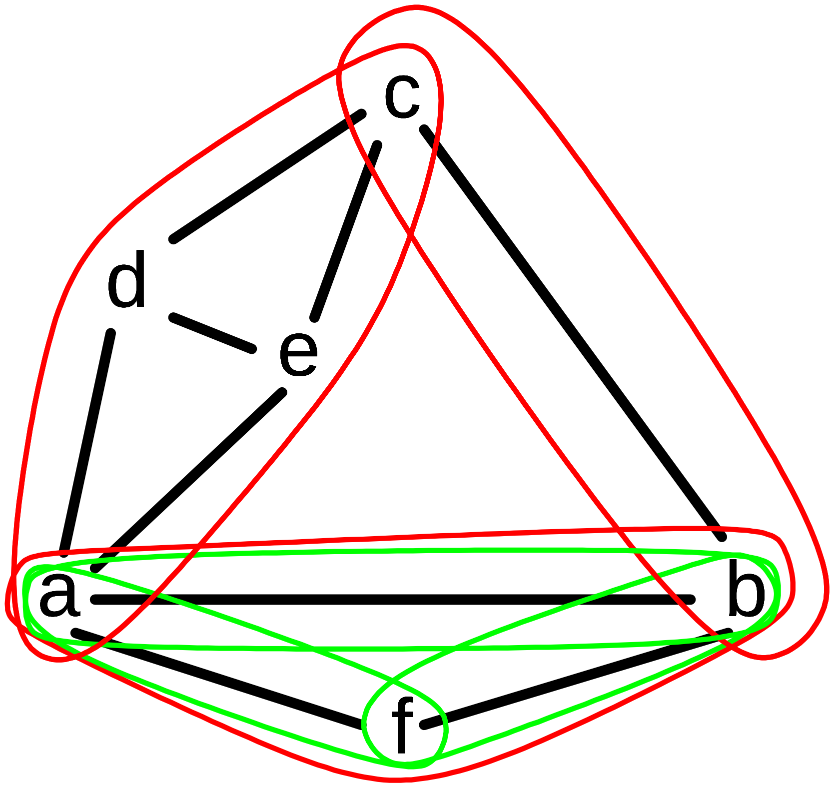

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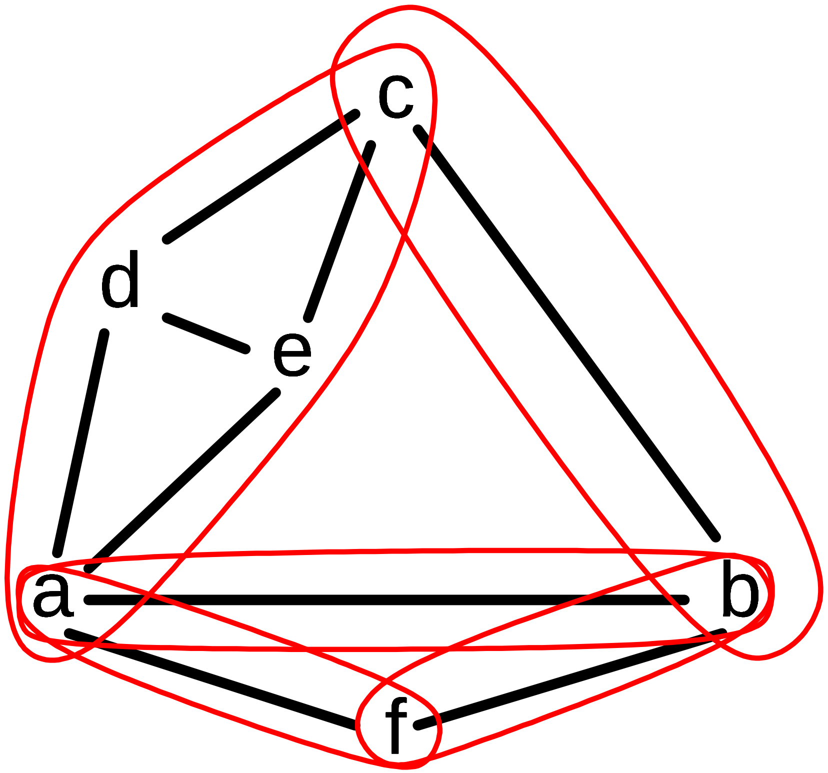

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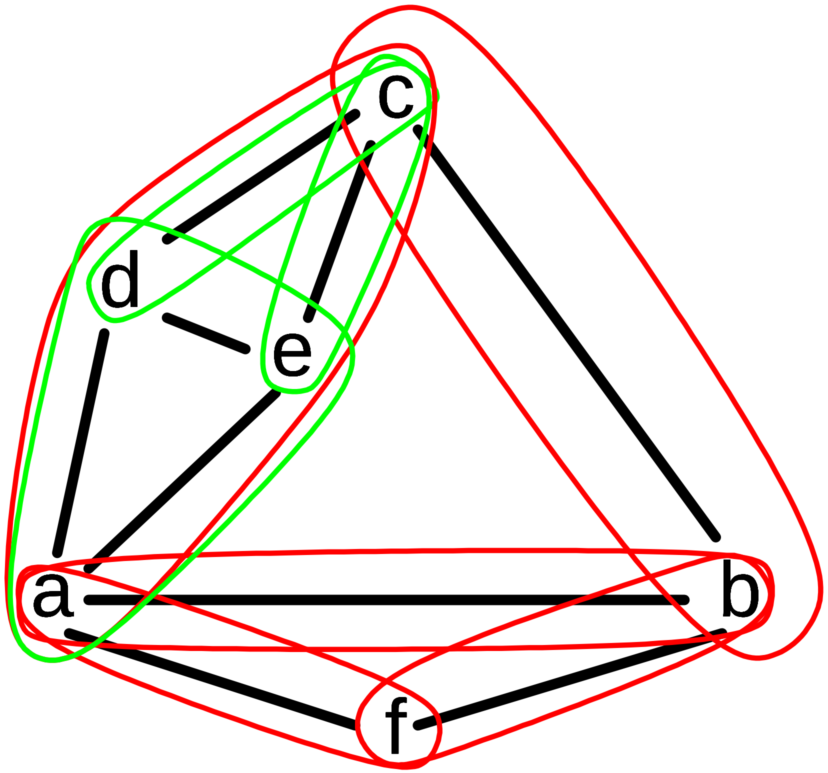

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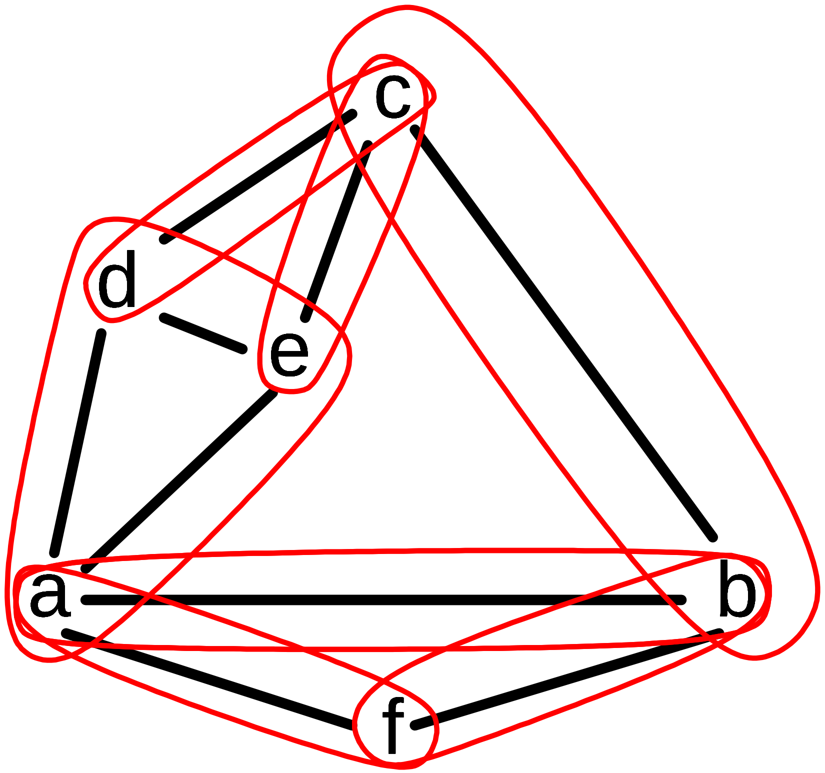

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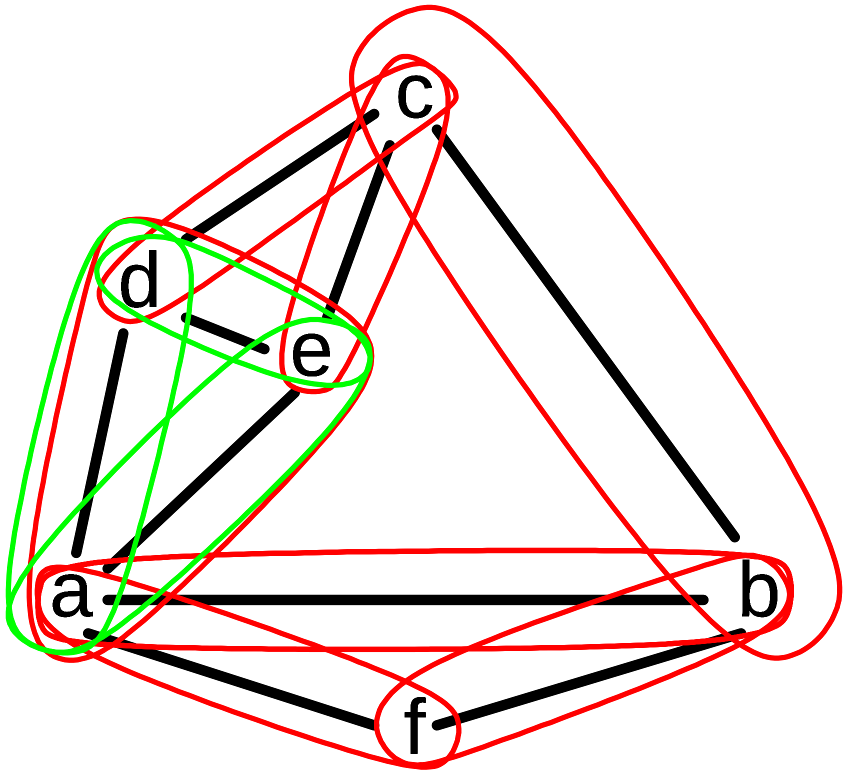

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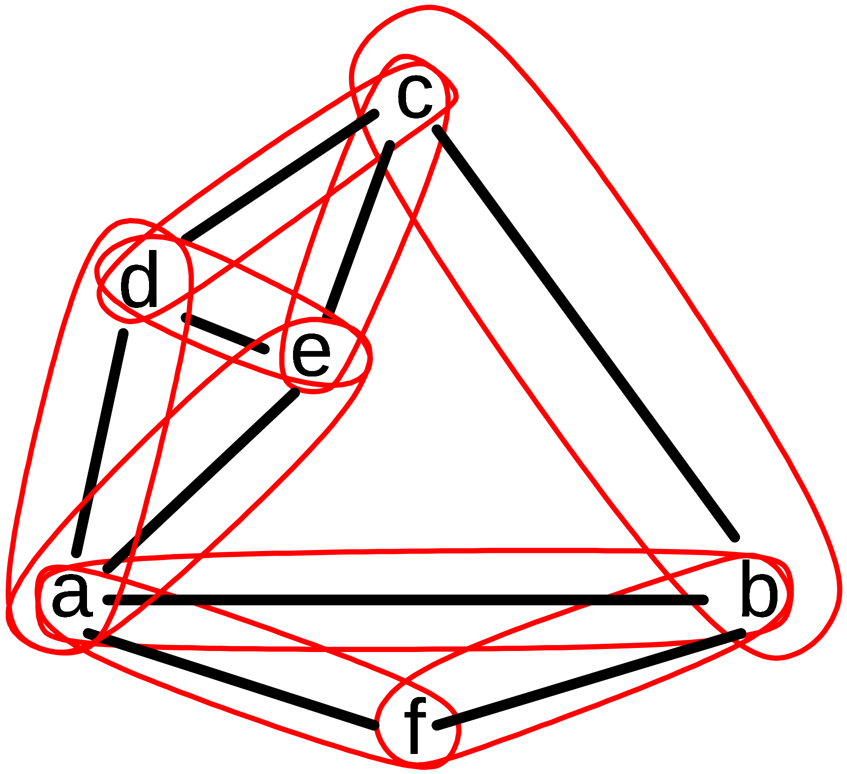


Decomposition analysis

Let $G = (V, E)$ be a geometric constraint graph.

We define the *initial set of clusters* $\mathbf{O}_G = \{V\}$.

Let $\mathbf{O}$ be a set of clusters in which there is a cluster C such that $\{C_1, C_2, C_3\}$ is a set decomposition of the subgraph of G induced by C.

$\mathbf{O} \longrightarrow_o \mathbf{O}'$ is a reduction rule where $\mathbf{O}' = (\mathbf{O} - C) \cup \{C_1, C_2, C_3\}$.

The geometric constraint problem represented by the geometric constraint graph G is *solvable by decomposition analysis* if $\mathbf{O}_G$ reduces to $\mathbf{S}_G$.

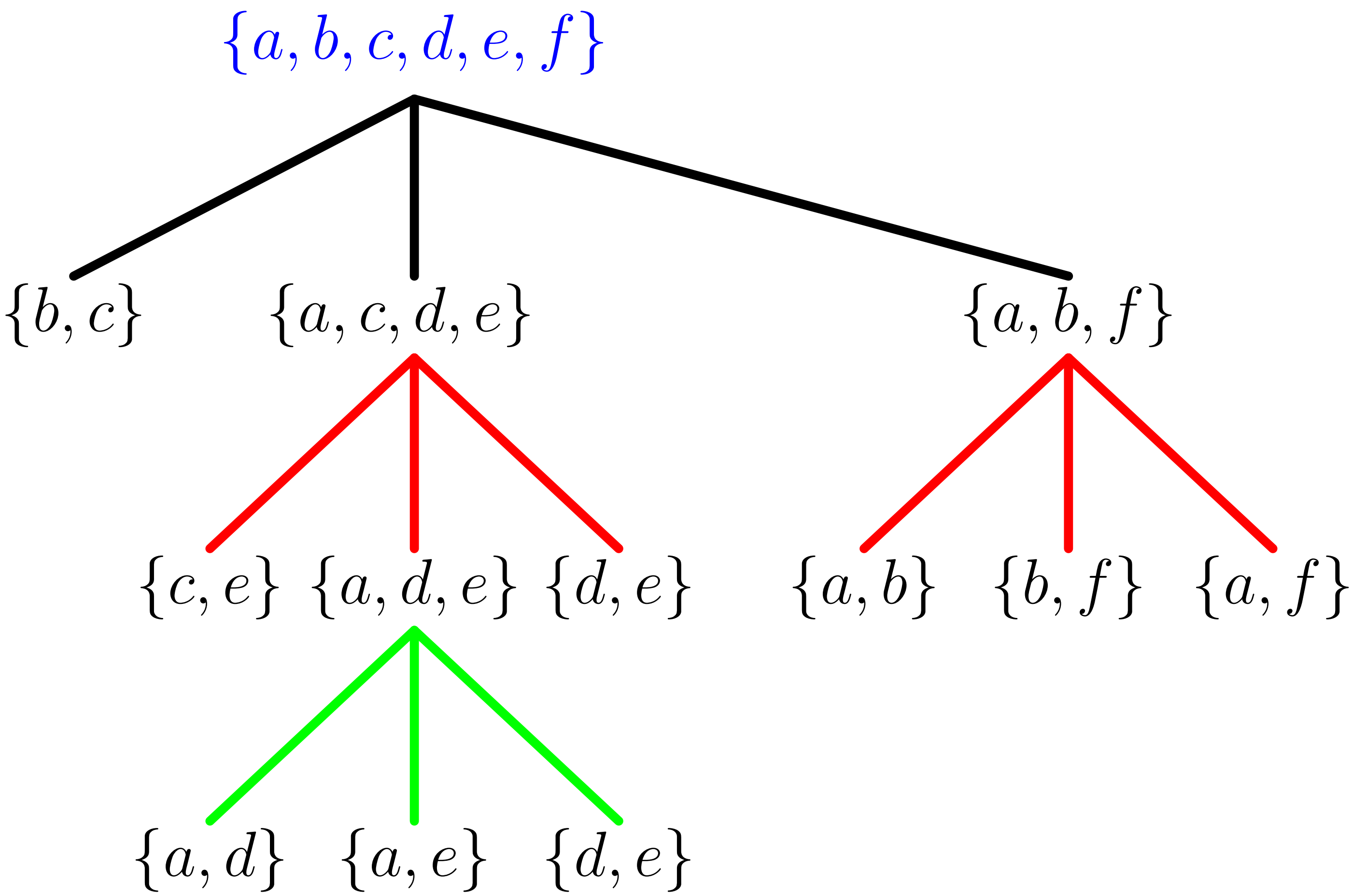
The reduction relation $\longrightarrow_o$ induces an abstract reduction system.

The domain of solvable graphs by decomposition analysis

Let $G = (V, E)$ be a well-constrained geometric constraint graph. The following assertions are equivalent:

1. G is tree decomposable.
2. G is solvable by decomposition analysis.

The domain of solvable graphs by decomposition analysis



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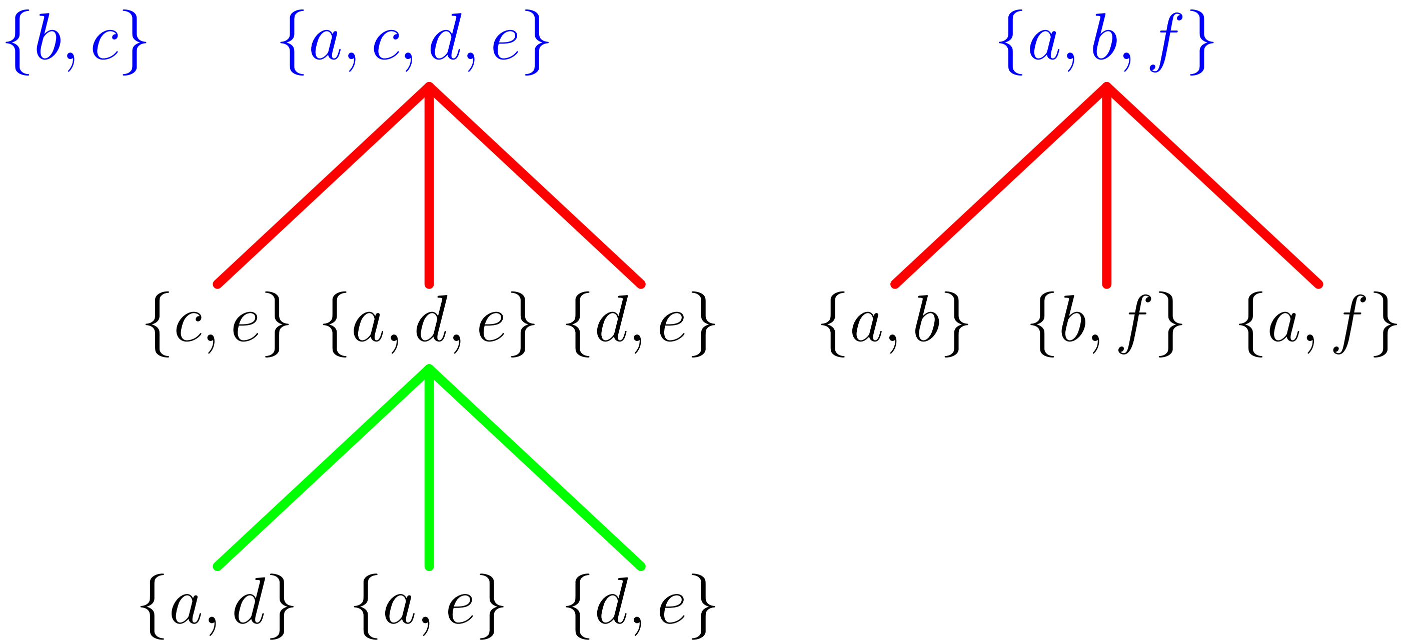

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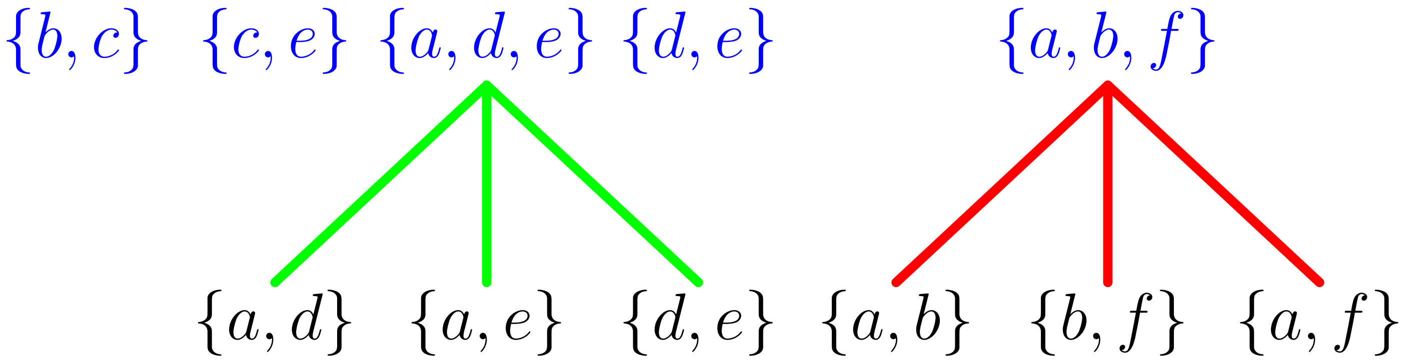

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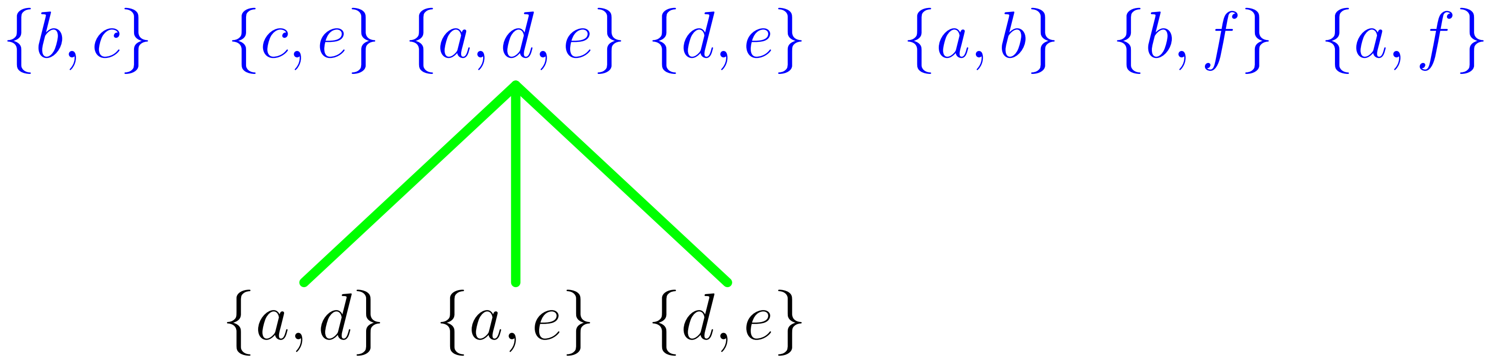
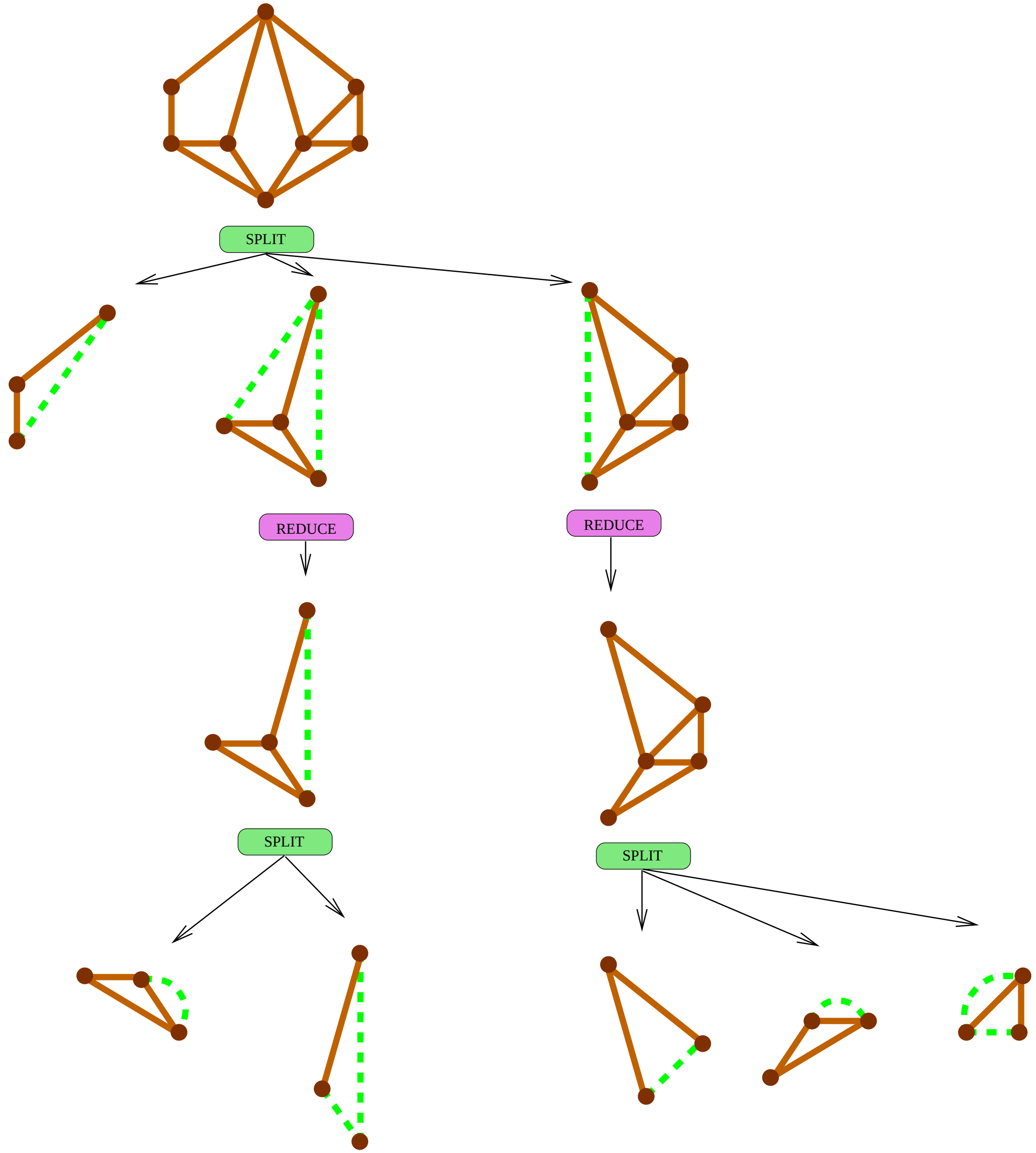


The domain of solvable graphs by decomposition analysis

$$\{b, c\} \quad \{c, e\} \quad \{a, d\} \quad \{a, e\} \quad \{d, e\} \quad \{d, e\} \quad \{a, b\} \quad \{b, f\} \quad \{a, f\}$$

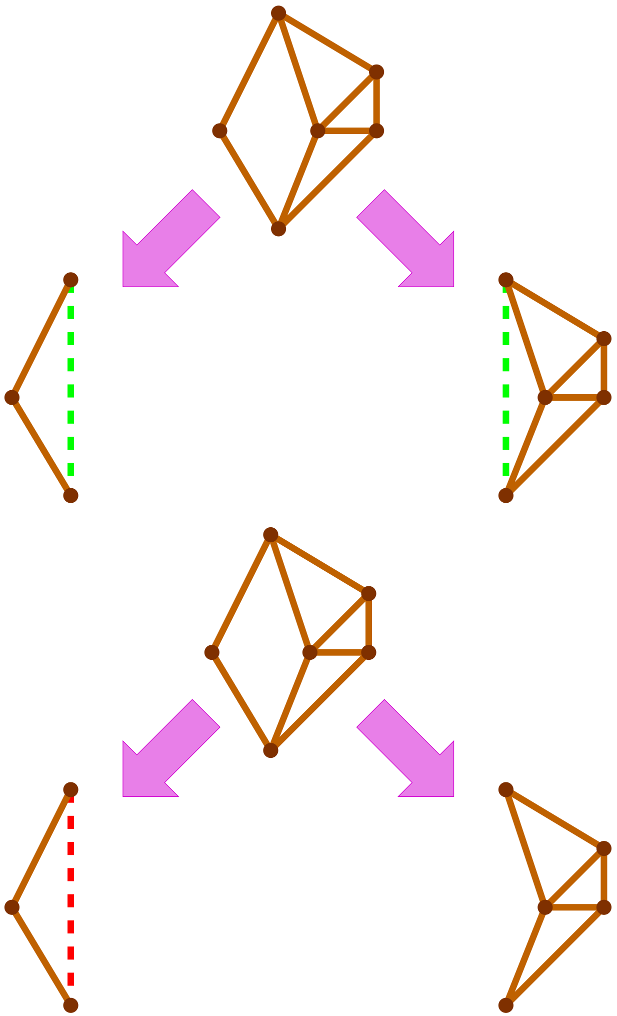
Reformulating Owen's algorithm

Owen's algorithm relies on computing triconnected components . . .



- . . . but after each split some well chosen edges should be removed to continue the process.
- It is difficult to understand which edges should be removed and the reason why they should be removed.

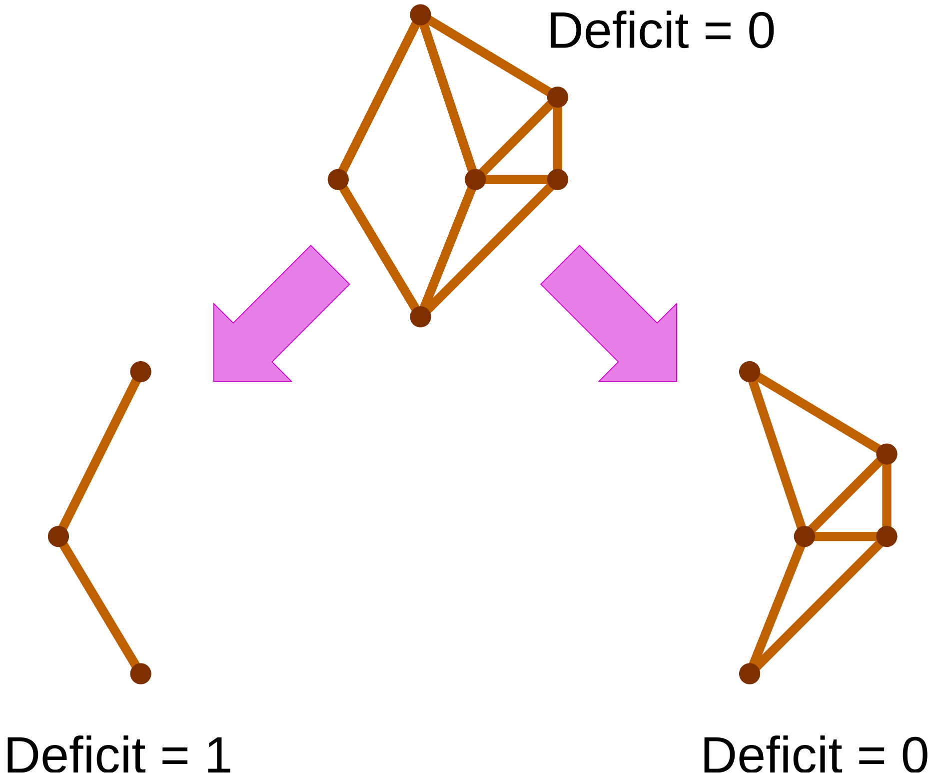
Which edges and why should they be removed?



- The triconnected components algorithm subdivides the graph and adds virtual edges to preserve connectivity properties.
- To further subdivide, Owen's algorithm removes virtual edges *"at any articulation pair with no single edge and exactly one more complex subgraph"*.

The property to be preserved in decomposition algorithms is the deficit

- What is essential to preserve in the graph subdivision process is *rigidity properties*, not connectivity properties.



- Deficit function of a graph $G = (V, E)$ is defined as

$$\text{Deficit}(G) = (2|V| - 3) - |E|$$

- At every graph split, deficit value should be maintained. Thus new edges must be added to fulfill this requirement.

Two results show how deficit can be maintained

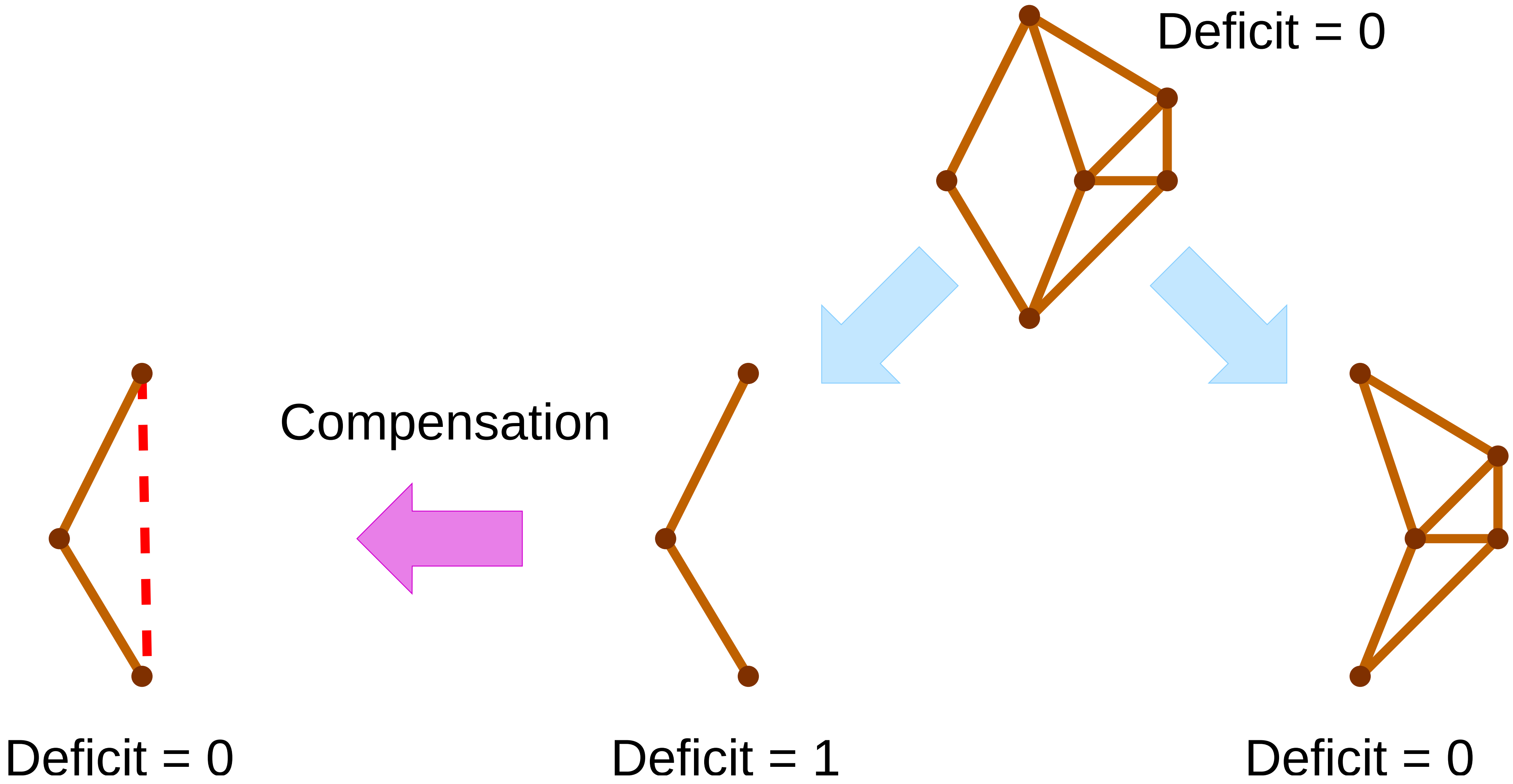
Let G be a well-constrained constraint graph and G' and G'' separating graphs of G. Then

- Deficit$(G) =$ Deficit$(G') +$ Deficit$(G'') - 1$
- If Deficit(G') $>$ Deficit(G''), G' is under-constrained and G'' is well-constrained.

Therefore

> To maintain well-constraintness one virtual edge must be added to the separating graph G'.
>
> The virtual edge subsumes the rigidity properties due to the separating graph G''

Example of deficit compensation



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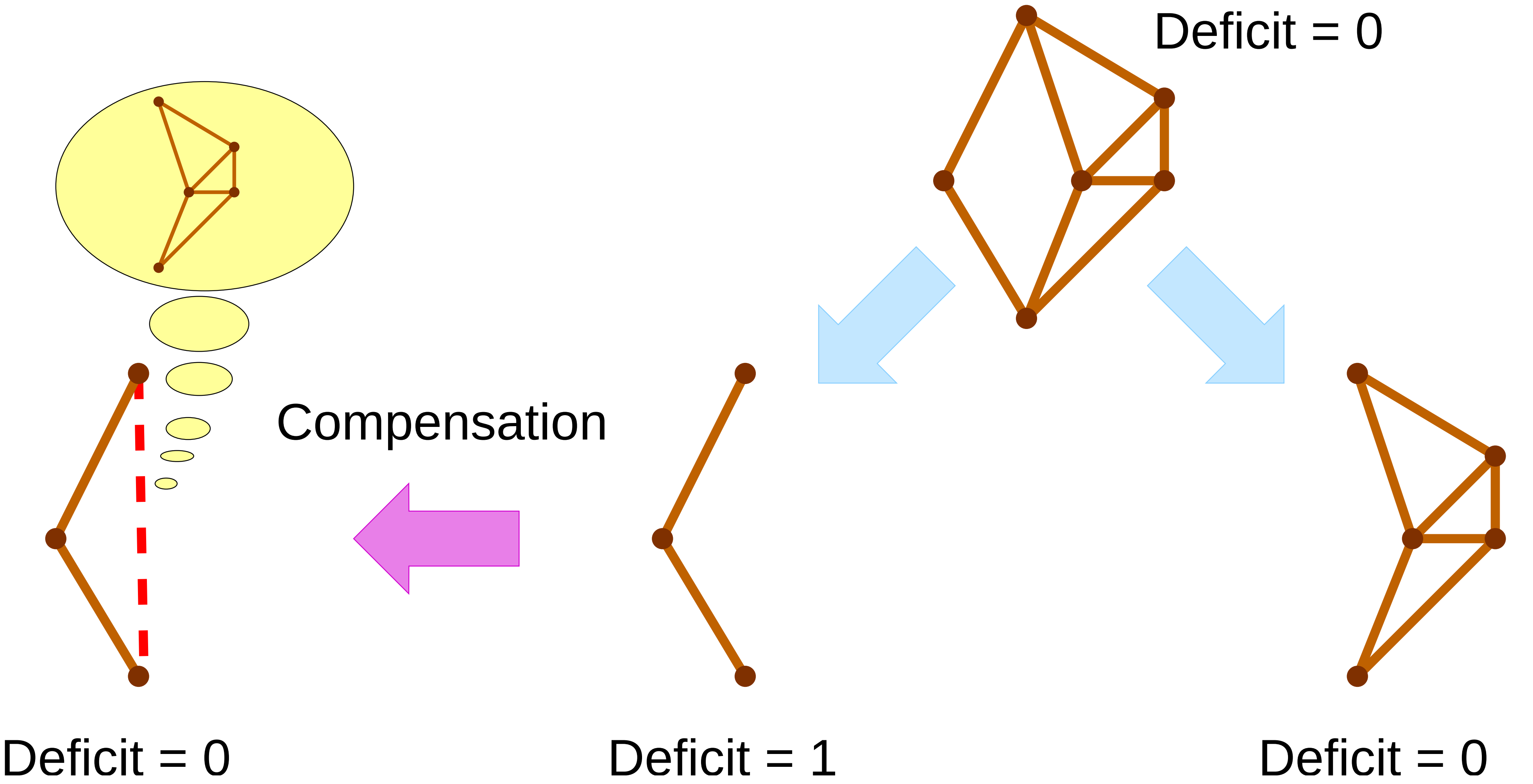


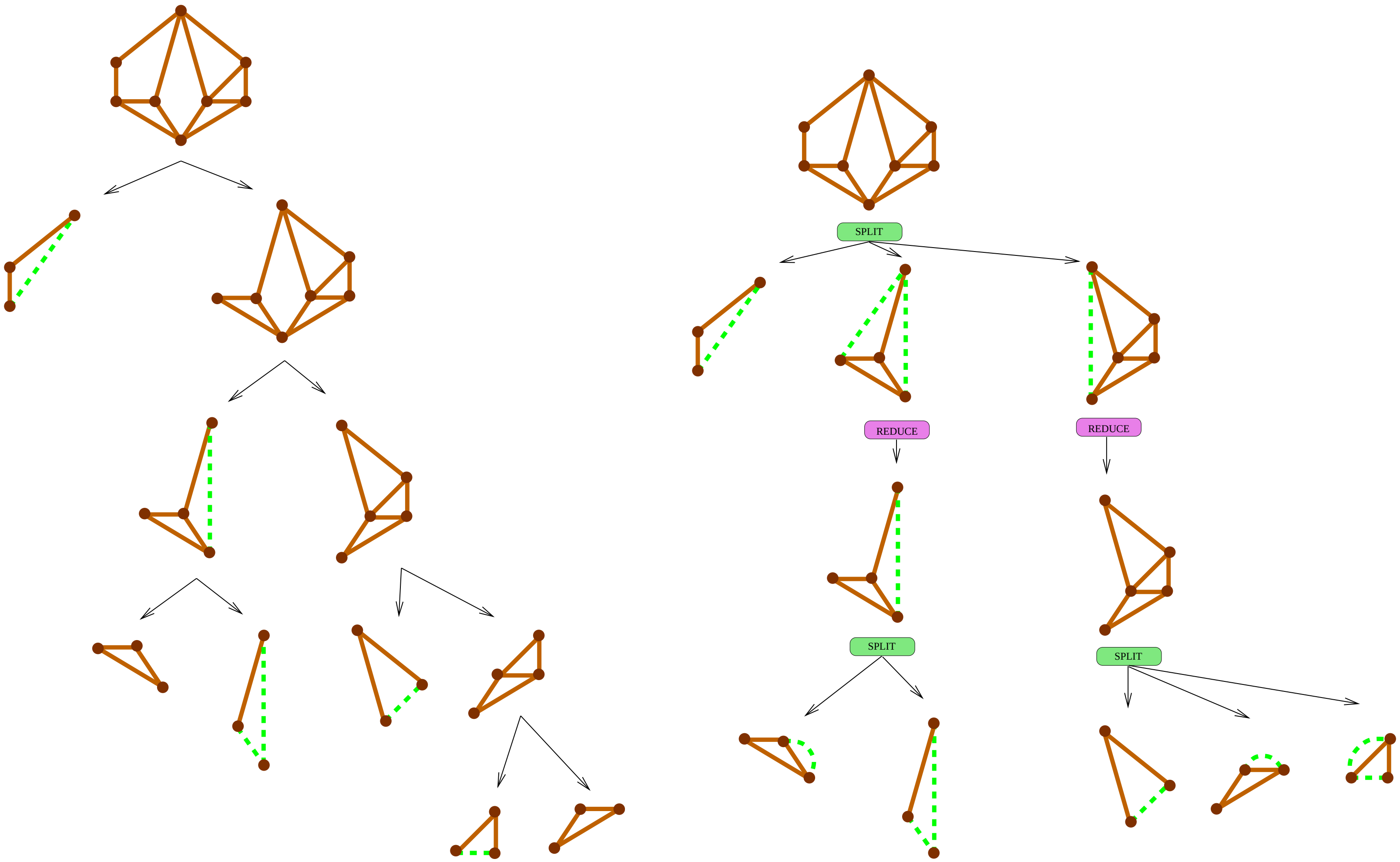
A new formulation of Owen's decomposition algorithm

- A clear and simple application of divide-and-conquer.
- Uses separating pairs to subdivide the graph.
- Applies deficit compensation to maintain rigidity structure.

func Analysis(G)
  if Triconnected(G) **then**
    S := BinaryTree(G, nullTree, nullTree)
  else
    G_1,G_2 := SeparatingGraphs(G)
    if Deficit(G_1) $>$ Deficit(G_2) **then**
      G_1 := AddVirtualEdge(G_1)
    else
      G_2 := AddVirtualEdge(G_2)
    fi
    S := BinaryTree(G, Analysis(G_1),
                Analysis(G_2))
  fi
  return S
end

The result of the new formulation is an s-tree

- The new algorithm yields a binary form of the Owen's tree. We name it a *s-tree*.

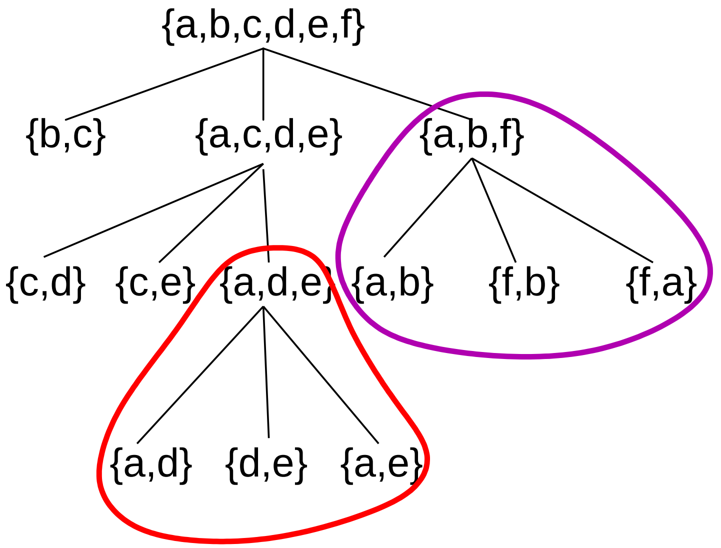


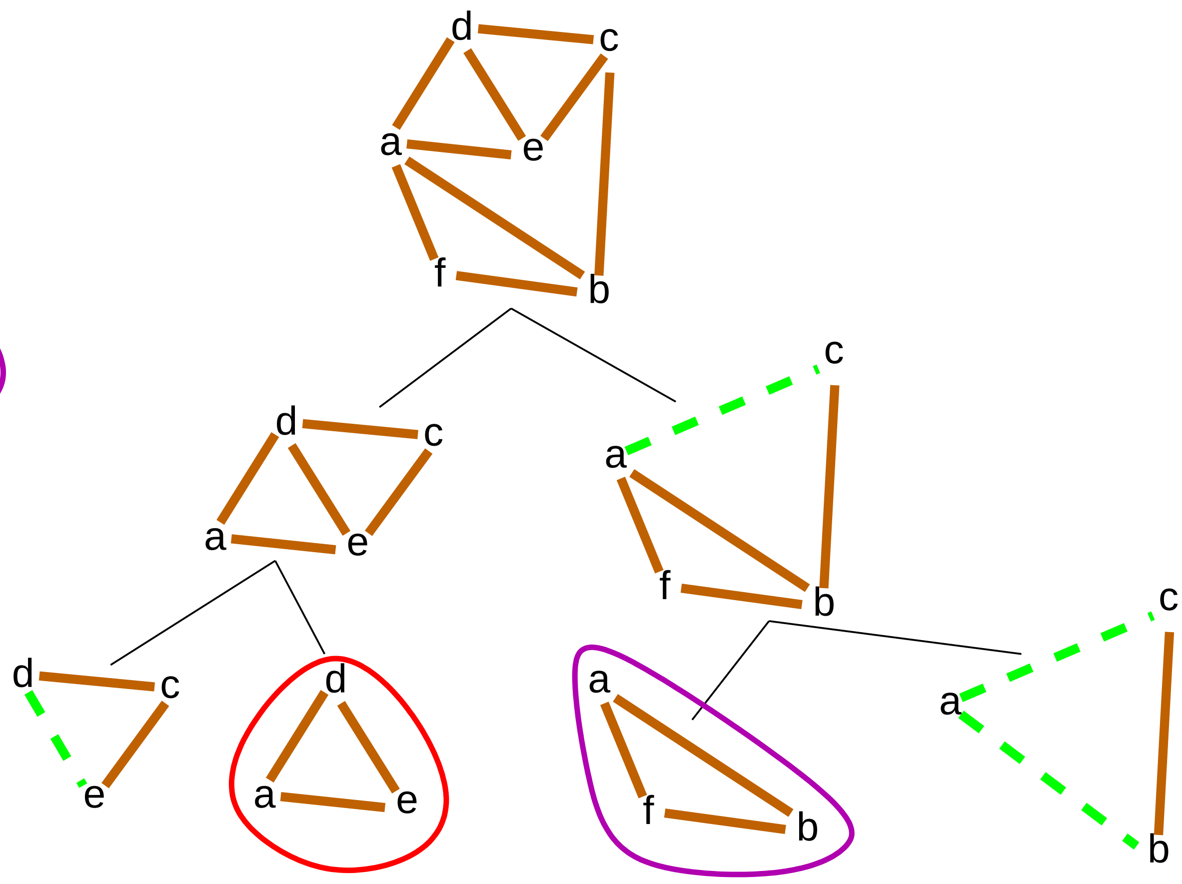
The domain of Owen's method

Let $G = (V, E)$ be a well-constrained geometric constraint graph. The following assertions are equivalent:

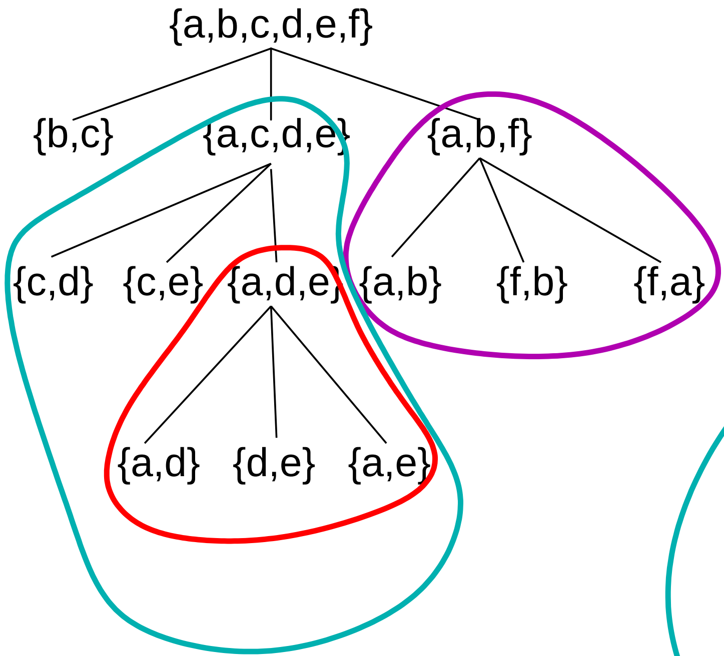
1. G is tree decomposable.
2. G is s-tree decomposable.

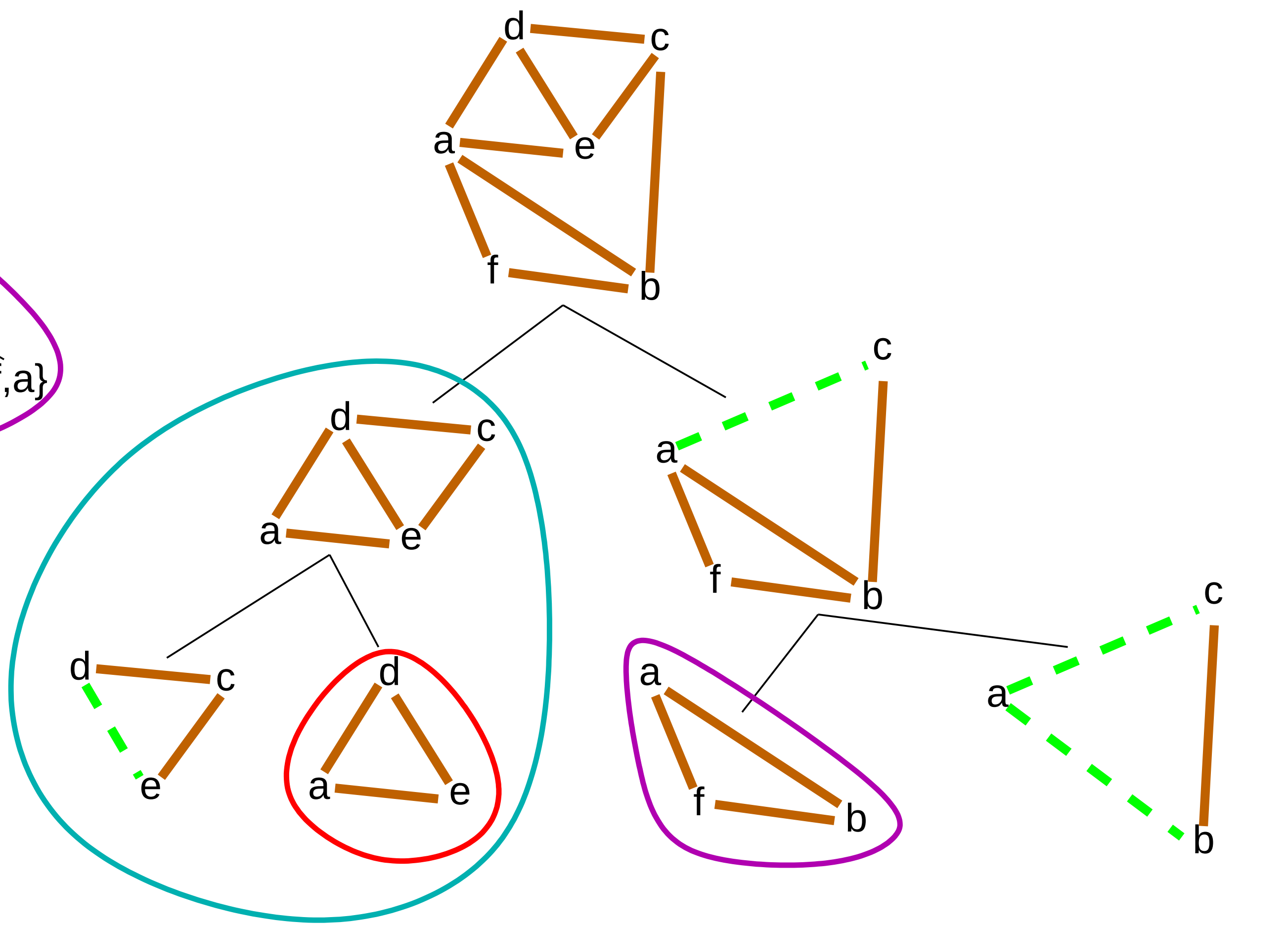
The domain of Owen's method



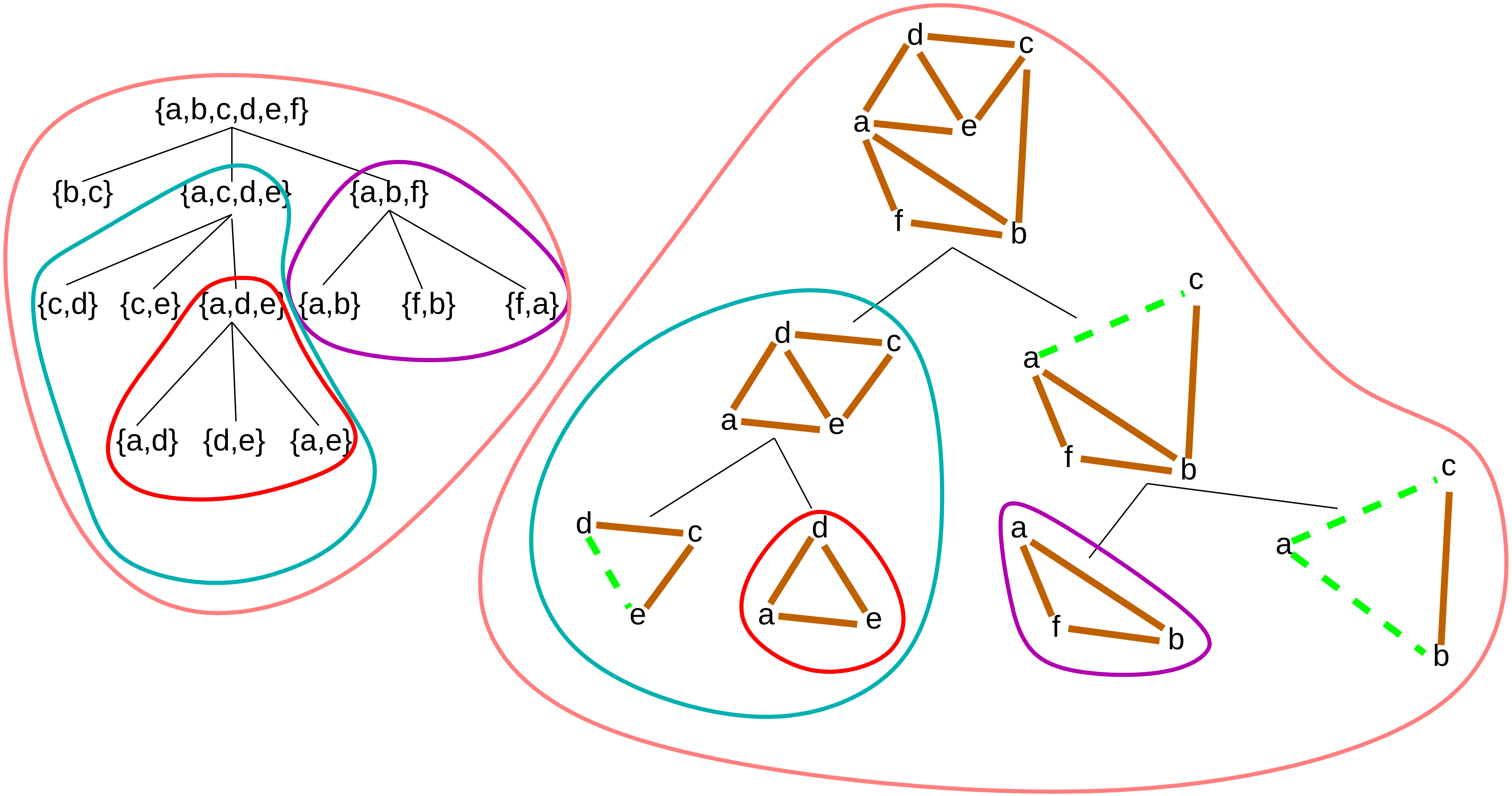


The domain of Owen's method





The domain of Owen's method



Domain equivalence of constructive methods

Constructive methods have the same domain

Let $G = (V, E)$ be a well-constrained geometric constraint graph.
The following assertions are equivalent:

1. G is tree decomposable.
2. G is s-tree decomposable.
3. G is solvable by reduction analysis.
4. G is solvable by decomposition analysis.

The class of graphs fullfiling the above properties is named the *constructively solvable graphs class*.

Summary

- We have introduced the tree decomposition of a graph.
- Tree decomposable graphs characterize the domain of reduction analysis, decomposition analysis and Owen’s method.
- The domains of constructive methods are the same.
- We have clarified and reformulated Owen’s algorithm.
- The reformulated algorithm applies a divide-and-conquer schema and it is conceptually simpler.
- The output of this algorithm is an s-tree.