Multiplicative-Additive Focusing for Parsing as Deduction

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Spurious ambiguity is the phenomenon whereby distinct derivations in grammar may assign the same structural reading, resulting in redundancy in the parse search space and inefficiency in parsing. Understanding the problem depends on identifying the essential mathematical structure of derivations. This is trivial in the case of context free grammar, where the parse structures are ordered trees; in the case of type logical categorial grammar, the parse structures are proof nets. However, with respect to multiplicatives intrinsic proof nets have not yet been given for displacement calculus, and proof nets for additives, which have applications to polymorphism, are not easy to characterise. Here we approach multiplicative-additive spurious ambiguity by means of the proof-theoretic technique of focalisation.

1 Introduction

In context free grammar (CFG) sequential rewriting derivations exhibit spurious ambiguity: distinct rewriting derivations may correspond to the same parse structure (tree) and the same structural reading.[1] In this case it is transparent to develop parsing algorithms avoiding spurious ambiguity by reference to parse trees. In categorial grammar (CG) the problem is more subtle. The Cut-free Lambek sequent proof search space is finite, but involves a combinatorial explosion of spuriously ambiguous sequential proofs. This can be understood, analogously to CFG, as inessential rule reorderings, which we parallelise in underlying geometric parse structures which are (planar) proof nets.

The planarity of Lambek proof nets reflects that the formalism is continuous or concatenative. But the challenge of natural grammar is discontinuity or apparent displacement, whereby there is syntactic/semantic mismatch, or elements appearing out of place. Hence the subsumption of Lambek calculus by displacement calculus **D** including intercalation as well as concatenation [17].

Proof nets for **D** must be partially non-planar; steps towards intrinsic correctness criteria for displacement proof nets are made in [5] and [13]. Additive proof nets are considered in [7] and [1]. However, even in the case of Lambek calculus, parsing by reference to intrinsic criteria [14], [18], appendix B, is not more efficient than parsing by reference to extrinsic criteria of normalised sequent calculus [6]. In its turn, on the other hand, normalisation does not extend to product left rules and product unit left rules nor to additives. The focalisation of [2] is a methodology midway between proof nets and normalisation. Here we apply the focusing discipline to the parsing as deduction of **D** with additives.

In [4] multifocusing is defined for unit-free MALL,[2] providing canonical sequent proofs; an eventual goal would be to formulate multifocusing for multiplicative-additive categorial logic and for categorial

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[2]Here we include units, which are linguistically relevant.

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logic generally. In this respect the present paper represents an intermediate step. Note that [19] develops focusing for Lambek calculus with additives, but not for displacement logic, for which we show completeness of focusing here.

1.1 Spurious ambiguity in CFG

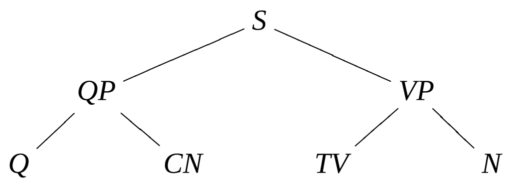
Consider the following production rules:

$$\begin{array}{l} S \to QP\ VP \\ QP \to Q\ CN \\ VP \to TV\ N \end{array}$$

These generate the following sequential rewriting derivations:

$$\begin{array}{l} S \to QP\ VP \to Q\ CN\ VP \to Q\ CN\ TV\ N \\ S \to QP\ VP \to QP\ TV\ N \to Q\ CN\ TV\ N \end{array}$$

These sequential rewriting derivations correspond to the same parellelised parse structure:



And they correspond to the same structural reading; sequential rewriting has *spurious ambiguity*.

1.2 Spurious ambiguity in CG

Lambek calculus is a logic of strings with the operation $+$ of concatenation. Recall the definitions of types, configurations and sequents in the Lambek calculus **L** [11], in terms of a set $\mathscr{P}$ of primitive types (the original Lambek calculus did not include the product unit):

(1)			
Types	$\mathscr{F}$	$::=$	$\mathscr{P} \mid \mathscr{F}/\mathscr{F} \mid \mathscr{F}\backslash\mathscr{F} \mid \mathscr{F}\bullet\mathscr{F}$
Configurations	$\mathscr{O}$	$::=$	$\Lambda \mid \mathscr{F},\mathscr{O}$
Sequents	Σ	$::=$	$\mathscr{O} \Rightarrow \mathscr{F}$

Lambek calculus types have the following interpretation:

$$\begin{array}{lcl} [[C/B]] & = & \{s_1 \mid \forall s_2 \in [[B]], s_1{+}s_2 \in [[C]]\} \\ [[A\backslash C]] & = & \{s_2 \mid \forall s_1 \in [[A]], s_1{+}s_2 \in [[C]]\} \\ [[A\bullet B]] & = & \{s_1{+}s_2 \mid s_1 \in [[A]]\ \&\ s_2 \in [[B]]\} \end{array}$$

The logical rules of **L** are as follows:

$$\frac{\Gamma \Rightarrow A \qquad \Delta(C) \Rightarrow D}{\Delta(\Gamma, A\backslash C) \Rightarrow D}\backslash L \qquad \frac{A,\Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\backslash R$$

$$\frac{\Gamma \Rightarrow B \qquad \Delta(C) \Rightarrow D}{\Delta(C/B, \Gamma) \Rightarrow D}/L \qquad \frac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow C/B}/R$$

$$\frac{\Delta(A,B) \Rightarrow D}{\Delta(A\bullet B) \Rightarrow D}\bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1,\Gamma_2 \Rightarrow A\bullet B}\bullet R$$

$$\dfrac{CN \Rightarrow CN \qquad \dfrac{\dfrac{\dfrac{N \Rightarrow N \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\backslash L}{N, (N\backslash S)/N, N \Rightarrow S}/L}{(N\backslash S)/N, N \Rightarrow N\backslash S}\backslash R \qquad S \Rightarrow S}{S/(N\backslash S), (N\backslash S)/N, N \Rightarrow S}/L}{(S/(N\backslash S))/CN, CN, (N\backslash S)/N, N \Rightarrow S}/L$$

$$\dfrac{N \Rightarrow N \qquad \dfrac{CN \Rightarrow CN \qquad \dfrac{\dfrac{\dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\backslash L}{N\backslash S \Rightarrow N\backslash S}\backslash R \qquad S \Rightarrow S}{S/(N\backslash S), N\backslash S \Rightarrow S}/L}{(S/(N\backslash S))/CN, CN, N\backslash S \Rightarrow S}/L}{(S/(N\backslash S))/CN, CN, (N\backslash S)/N, N \Rightarrow S}/L$$

Figure 1: Spurious ambiguity

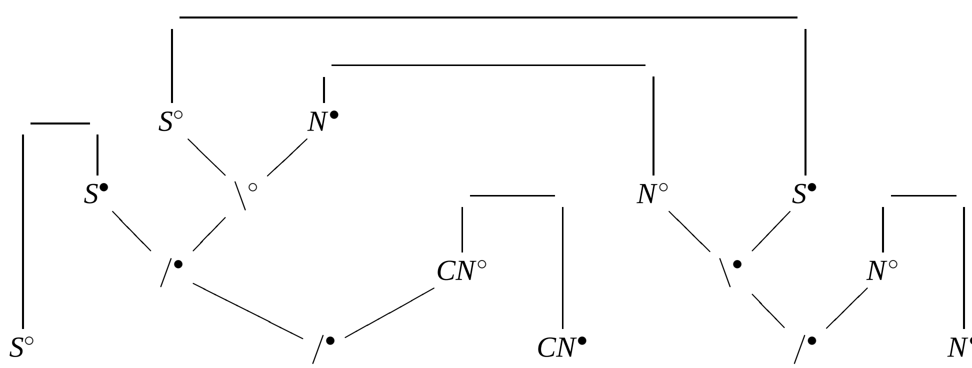


Figure 2: Proof net

Even amongst Cut-free proofs there is spurious ambiguity; consider for example the sequential derivations of Figure 1. These have the same parallelised parse structure (proof net) of Figure 2.

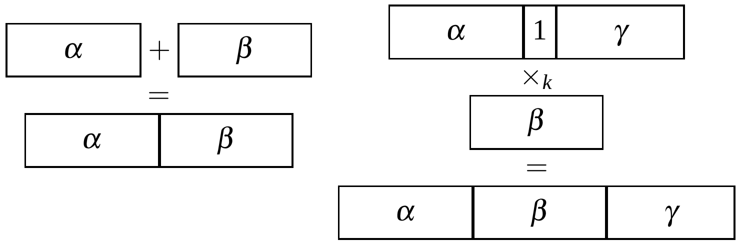
Lambek proof structures are planar graphs which must satisfy certain global and local properties to be correct as proofs (proof nets). Proof nets provide a geometric perspective on derivational equivalence. Alternatively we may identify the same algebraic parse structure (Curry-Howard term):

$$((x_Q\, x_{CN})\, \lambda x((x_{TV}\, x_N)\, x))$$

But Lambek calculus is continuous (planarity). A major issue issue in grammar is discontinuity, hence the displacement calculus.

2 D with additives, DA

In this section we present displacement calculus **D**, and a displacement logic **DA** comprising **D** with additives. Although **D** is indeed a conservative extension of **L**, we think of it not just as an *extension* of Lambek calculus but as a *generalisation*, because it involves a whole new machinery of sequent calculus to deal with discontinuity. Displacement calculus is a logic of discontinuous strings — strings punctuated by a *separator* 1 and subject to operations of append and plug; see Figure 3. Recall the definition of types and their sorts, configurations and their sorts, and sequents, for the displacement calculus with additives:



append $+ : L_i, L_j \to L_{i+j}$ plug $\times_k : L_{i+1}, L_j \to L_{i+j}$

Figure 3: Append and plug

(2) Types
$$
\begin{array}{rcll}
\mathscr{F}_i & ::= & \mathscr{F}_{i+j}/\mathscr{F}_j & \\
\mathscr{F}_j & ::= & \mathscr{F}_i\backslash\mathscr{F}_{i+j} & \\
\mathscr{F}_{i+j} & ::= & \mathscr{F}_i \bullet \mathscr{F}_j & \\
\mathscr{F}_0 & ::= & I & \\
\mathscr{F}_{i+1} & ::= & \mathscr{F}_{i+j}{\uparrow_k}\mathscr{F}_j & 1 \le k \le i+1 \\
\mathscr{F}_j & ::= & \mathscr{F}_{i+1}{\downarrow_k}\mathscr{F}_{i+j} & 1 \le k \le i+1 \\
\mathscr{F}_{i+j} & ::= & \mathscr{F}_{i+1}\odot_k\mathscr{F}_j & 1 \le k \le i+1 \\
\mathscr{F}_1 & ::= & J & \\
\mathscr{F}_i & ::= & \mathscr{F}_i \& \mathscr{F}_i & \\
\mathscr{F}_i & ::= & \mathscr{F}_i \oplus \mathscr{F}_i &
\end{array}
$$

Sort $sA =$ the i s.t. $A \in \mathscr{F}_i$
For example, $s(S{\uparrow_1}N){\uparrow_2}N = s(S{\uparrow_1}N){\uparrow_1}N = 2$ where $sN = sS = 0$

Configurations
$$
\begin{array}{rcl}
\mathscr{O} & ::= & \Lambda \mid \mathscr{T}, \mathscr{O} \\
\mathscr{T} & ::= & 1 \mid \mathscr{F}_0 \mid \mathscr{F}_{i>0}\{\underbrace{\mathscr{O} : \ldots : \mathscr{O}}_{i\ \mathscr{O}'\mathrm{s}}\}
\end{array}
$$
For example, there is the configuration $(S{\uparrow_1}N){\uparrow_2}N\{N, 1 : S{\uparrow_1}N, S\}, 1, N, 1$

Sort $s\mathscr{O} = |\mathscr{O}|_1$
For example $s(S{\uparrow_1}N){\uparrow_2}N\{N, 1 : S{\uparrow_1}N, S\}, 1, N, 1 = 3$

Sequents $\Sigma ::= \mathscr{O} \Rightarrow A$ s.t. $s\mathscr{O} = sA$

The figure $\overrightarrow{A}$ of a type A is defined by:

$$
\overrightarrow{A} = \begin{cases} A & \text{if } sA = 0 \\ A\{\underbrace{1 : \ldots : 1}_{sA\ 1's}\} & \text{if } sA > 0 \end{cases}
$$

Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle$ is the result of replacing the successive 1's in Γ by $\Delta_1, \ldots, \Delta_i$ respectively.

Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth metalinguistic wrap $\Delta|_k \Gamma$, $1 \le k \le i$, is given by

(3) $\Delta|_k\Gamma =_{df} \Delta\otimes\langle\underbrace{1:\ldots:1}_{k-1\ 1\text{'s}}:\Gamma:\underbrace{1:\ldots:1}_{i-k\ 1\text{'s}}\rangle$

i.e. $\Delta|_k\Gamma$ is the configuration resulting from replacing by Γ the kth separator in Δ.

In broad terms, syntactical interpretation of displacement calculus is as follows:

$$\begin{array}{rcl}
[[C/B]] & = & \{s_1|\ \forall s_2\in[[B]], s_1{+}s_2\in[[C]]\}\\
[[A\backslash C]] & = & \{s_2|\ \forall s_1\in[[A]], s_1{+}s_2\in[[C]]\}\\
[[A{\bullet}B]] & = & \{s_1{+}s_2|\ s_1\in[[A]]\ \&\ s_2\in[[B]]\}\\
[[I]] & = & \{0\}\\
\\
[[C{\uparrow_k}B]] & = & \{s_1|\ \forall s_2\in[[B]], s_1\times_k s_2\in[[C]]\}\\
[[A{\downarrow_k}C]] & = & \{s_2|\ \forall s_1\in[[A]], s_1\times_k s_2\in[[C]]\}\\
[[A{\odot_k}B]] & = & \{s_1\times_k s_2|\ s_1\in[[A]]\ \&\ s_2\in[[B]]\}\\
[[J]] & = & \{1\}
\end{array}$$

The logical rules of the displacement calculus with additives are as follows, where $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0(\Gamma\otimes\langle\Delta_1:\ldots:\Delta_i\rangle)$:

$$\frac{\Gamma\Rightarrow B \qquad \Delta\langle\overrightarrow{C}\rangle\Rightarrow D}{\Delta\langle\overrightarrow{C/B},\Gamma\rangle\Rightarrow D}\,/L \qquad \frac{\Gamma,\overrightarrow{B}\Rightarrow C}{\Gamma\Rightarrow C/B}\,/R$$

$$\frac{\Gamma\Rightarrow A \qquad \Delta\langle\overrightarrow{C}\rangle\Rightarrow D}{\Delta\langle\Gamma,\overrightarrow{A\backslash C}\rangle\Rightarrow D}\,\backslash L \qquad \frac{\overrightarrow{A},\Gamma\Rightarrow C}{\Gamma\Rightarrow A\backslash C}\,\backslash R$$

$$\frac{\Delta\langle\overrightarrow{A},\overrightarrow{B}\rangle\Rightarrow D}{\Delta\langle\overrightarrow{A{\bullet}B}\rangle\Rightarrow D}\,{\bullet}L \qquad \frac{\Gamma_1\Rightarrow A \qquad \Gamma_2\Rightarrow B}{\Gamma_1,\Gamma_2\Rightarrow A{\bullet}B}\,{\bullet}R$$

$$\frac{\Delta\langle\Lambda\rangle\Rightarrow A}{\Delta\langle\overrightarrow{I}\rangle\Rightarrow A}\,IL \qquad \frac{}{\Lambda\Rightarrow I}\,IR$$

$$\frac{\Gamma\Rightarrow B \qquad \Delta\langle\overrightarrow{C}\rangle\Rightarrow D}{\Delta\langle\overrightarrow{C{\uparrow_k}B}|_k\Gamma\rangle\Rightarrow D}\,{\uparrow_k}L \qquad \frac{\Gamma|_k\overrightarrow{B}\Rightarrow C}{\Gamma\Rightarrow C{\uparrow_k}B}\,{\uparrow_k}R$$

$$\frac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Rightarrow D}{\Delta\langle\overrightarrow{A{\odot_k}B}\rangle\Rightarrow D}\,{\odot_k}L \qquad \frac{\Gamma_1\Rightarrow A \qquad \Gamma_2\Rightarrow B}{\Gamma_1|_k\Gamma_2\Rightarrow A{\odot_k}B}\,{\odot_k}R$$

$$\frac{\Gamma\Rightarrow A \qquad \Delta\langle\overrightarrow{C}\rangle\Rightarrow D}{\Delta\langle\Gamma|_k\overrightarrow{A{\downarrow_k}C}\rangle\Rightarrow D}\,{\downarrow_k}L \qquad \frac{\overrightarrow{A}|_k\Gamma\Rightarrow C}{\Gamma\Rightarrow A{\downarrow_k}C}\,{\downarrow_k}R$$

$$\frac{\Delta\langle 1\rangle\Rightarrow A}{\Delta\langle\overrightarrow{J}\rangle\Rightarrow A}\,JL \qquad \frac{}{1\Rightarrow J}\,JR$$

$$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\&B}\rangle \Rightarrow C}\,\&L_1 \qquad \frac{\Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\&B}\rangle \Rightarrow C}\,\&L_2$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\&B}\,\&R$$

$$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow C \qquad \Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\oplus B}\rangle \Rightarrow C}\,\oplus L$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A\oplus B}\,\oplus R_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A\oplus B}\,\oplus R_2$$

The continuous multiplicatives $\{/, \backslash, \bullet, I\}$ of Lambek (1958[11]; 1988[10]), are the basic means of categorial (sub)categorization. The directional divisions over, $/$, and under, $\backslash$, are exemplified by assignments such as *the*: N/CN for *the man*: N and *sings*: $N\backslash S$ for *John sings*: S, and *loves*: $(N\backslash S)/N$ for *John loves Mary*: S. Hence, for *the man*:

$$\frac{CN \Rightarrow CN \qquad N \Rightarrow N}{N/CN, N \Rightarrow N}\,/L$$

And for *John sings* and *John loves Mary*:

$$\frac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\,\backslash L \qquad \frac{N \Rightarrow N \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\,\backslash L}{N, (N\backslash S)/N, N \Rightarrow S}\,/L$$

The continuous product $\bullet$ is exemplified by a 'small clause' assignment such as *considers*: $(N\backslash S)/(N\bullet(CN/CN))$ for *John considers Mary socialist*: S.

$$\frac{\dfrac{N \Rightarrow N \qquad \dfrac{\dfrac{CN \Rightarrow CN \qquad CN \Rightarrow CN}{CN/CN, CN \Rightarrow CN}\,/L}{CN/CN \Rightarrow CN/CN}\,/R}{N, CN/CN \Rightarrow N\bullet(CN/CN)}\,\bullet R \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\,\backslash L}{N, (N\backslash S)/(N\bullet(CN/CN)), N, CN/CN \Rightarrow S}\,/L$$

Of course this use of product is not essential: we could just as well have used $((N\backslash S)/(CN/CN))/N$ since in general we have both $A/(C\bullet B) \Rightarrow (A/B)/C$ (currying) and $(A/B)/C \Rightarrow A/(C\bullet B)$ (uncurrying).

The discontinuous multiplicatives $\{\uparrow, \downarrow, \odot, J\}$, the displacement connectives, of Morrill and Valentín (2010[16]), Morrill et al. (2011[17]), are defined in relation to intercalation. When the value of the k subscript is one it may be omitted, i.e. it defaults to one. Circumfixation, or extraction, $\uparrow$, is exemplified by a discontinuous idiom assignment *gives*+1+*the*+*cold*+*shoulder*: $(N\backslash S)\uparrow N$ for *Mary gives the man the cold shoulder*: S:

$$\frac{\dfrac{CN \Rightarrow CN \qquad N \Rightarrow N}{N/CN, CN \Rightarrow N}\,/L \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\,\backslash L}{N, (N\backslash S)\uparrow N\{N/CN, CN\} \Rightarrow S}\,\uparrow L$$

Infixation, $\downarrow$, and extraction together are exemplified by a quantifier assignment *everyone*: $(S{\uparrow}N){\downarrow}S$ simulating Montague's S14 quantifying in:

$$\dfrac{\dfrac{\dots,N,\dots \Rightarrow S}{\dots,1,\dots \Rightarrow S{\uparrow}N}{\uparrow}R \qquad \dfrac{}{S \Rightarrow S}\,id}{\dots,(S{\uparrow}N){\downarrow}S,\dots \Rightarrow S}{\downarrow}L$$

Circumfixation and discontinuous product, $\odot$, are illustrated in an assignment to a relative pronoun *that*: $(CN\backslash CN)/((S{\uparrow}N)\odot I)$ allowing both peripheral and medial extraction, *that John likes*: $CN\backslash CN$ and *that John saw today*: $CN\backslash CN$:

$$\dfrac{\dfrac{\dfrac{N,(N\backslash S)/N,N \Rightarrow S}{N,(N\backslash S)/N,1 \Rightarrow S{\uparrow}N}{\uparrow}R \qquad \dfrac{}{\Rightarrow I}\,IL}{N,(N\backslash S)/N \Rightarrow (S{\uparrow}N)\odot I}\odot R \qquad CN\backslash CN \Rightarrow CN\backslash CN}{(CN\backslash CN)/((S{\uparrow}N)\odot I),N,(N\backslash S)/N \Rightarrow CN\backslash CN}\,/L$$

$$\dfrac{\dfrac{\dfrac{N,(N\backslash S)/N,N,S\backslash S \Rightarrow S}{N,(N\backslash S)/N,1,S\backslash S \Rightarrow S{\uparrow}N}{\uparrow}R \qquad \dfrac{}{\Rightarrow I}\,IL}{N,(N\backslash S)/N,S\backslash S \Rightarrow (S{\uparrow}N)\odot I}\odot R \qquad CN\backslash CN \Rightarrow CN\backslash CN}{(CN\backslash CN)/((S{\uparrow}N)\odot I),N,(N\backslash S)/N,S\backslash S \Rightarrow CN\backslash CN}\,/L$$

The additive conjunction and disjunction $\{\&, \oplus\}$ of Lambek (1961[9]), Morrill (1990[15]), and Kanazawa (1992[8]), capture polymorphism. For example the additive conjunction & can be used for *rice*: $N\&CN$ as in *rice grows*: S and *the rice grows*: S:

$$\dfrac{\dfrac{N \Rightarrow N}{N\&CN \Rightarrow N}\&L_1 \qquad S \Rightarrow S}{N\&CN,N\backslash S \Rightarrow S}\backslash L \qquad\qquad \dfrac{N/CN,CN,N\backslash S \Rightarrow S}{N/CN,N\&CN,N\backslash S \Rightarrow S}\&L_2$$

The additive disjunction $\oplus$ can be used for *is*: $(N\backslash S)/(N\oplus(CN/CN))$ as in *Tully is Cicero*: S and *Tully is humanist*: S:

$$\dfrac{\dfrac{N \Rightarrow N}{N \Rightarrow N\oplus(CN/CN)}\oplus R_1 \qquad N\backslash S \Rightarrow N\backslash S}{(N\backslash S)/(N\oplus(CN/CN)),N \Rightarrow N\backslash S}\,/L \qquad \dfrac{\dfrac{CN/CN \Rightarrow CN/CN}{CN/CN \Rightarrow N\oplus(CN/CN)}\oplus R_2 \qquad N\backslash S \Rightarrow N\backslash S}{(N\backslash S)/(N\oplus(CN/CN)),CN/CN \Rightarrow N\backslash S}\,/L$$

3 Focalisation for DA

In focalisation situated (antecedent, input, $^\bullet$/ succedent, output, $^\circ$) non-atomic types are classified as of negative (asynchronous) or positive (synchronous) *polarity* according as their rule is reversible or not; situated atoms are positive or negative according to their bias. The table below summarizes the notational convention on formulas P, Q, M and N:

	input	output
sync.	**Q**	**P**
async.	**M**	**N**

The grammar of these types polarised with respect to input and output occurrences is as follows; Q and P denote synchronous formulas in input and output position respectively, whereas M and N denote asynchronous formulas in input and output position respectively (in the nonatomic case we will abbreviate thus: *left sync., right synch., left async., and right async.*):

(4) Positive output $P ::= \mathrm{At}^+ \mid A{\bullet}B^\circ \mid I^\circ \mid A{\odot_k}B^\circ \mid J^\circ \mid A{\oplus}B^\circ$
Positive input $Q ::= \mathrm{At}^- \mid C/B^\bullet \mid A\backslash C^\bullet \mid C{\uparrow_k}B^\bullet \mid A{\downarrow_k}C^\bullet \mid A\&B^\bullet$
Negative output $N ::= \mathrm{At}^- \mid C/B^\circ \mid A\backslash C^\circ \mid C{\uparrow_k}B^\circ \mid A{\downarrow_k}C^\circ \mid A\&B^\circ$
Negative input $M ::= \mathrm{At}^+ \mid A{\bullet}B^\bullet \mid I^\bullet \mid A{\odot_k}B^\bullet \mid J^\bullet \mid A{\oplus}B^\bullet$

Notice that if P occurs in the antecedent then this occurrence of P is negative, and so forth.

There are alternating phases of don't-care nondeterministic negative rule application, and positive rule application locking on to *focalised* formulas.

Given a sequent with no occurrences of negative formulas, one chooses a positive formula as principal formula (which is boxed; we say it is focalised) and applies proof search to its subformulas while these remain positive. When one finds a negative formula or a literal, invertible rules are applied in a don't care nondeterminitic fashion until no longer possible, when another positive formula is chosen, and so on.

A sequent is either unfocused and as before, or else focused and has exactly one type boxed. The focalised logical rules are given in Figures 4-11 including Curry-Howard categorial semantic labelling. Occurrences of P, Q, M and N are supposed not to be focalised, which means that their focalised occurrence *must* be signalled with a box. By contrast, occurrences of A, B, C may be focalised or not.

4 Completeness of focalisation for DA

We shall be dealing with three systems: the displacement calculus **DA** with sequents notated $\Delta \Rightarrow A$, the *weakly focalised* displacement calculus with additives $\mathbf{DA_{foc}}$ with sequents notated $\Delta \Longrightarrow_w A$, and the *strongly focalised* displacement calculus with additives $\mathbf{DA_{Foc}}$ with sequents notated $\Delta \Longrightarrow A$. Sequents of both $\mathbf{DA_{foc}}$ and $\mathbf{DA_{Foc}}$ may contain at most one focalised formula, possibly A. When a $\mathbf{DA_{foc}}$ sequent is notated $\Delta \Longrightarrow_w A \diamondsuit \text{foc}$, it means that the sequent possibly contains a (unique) focalised formula. Otherwise, $\Delta \Longrightarrow_w A$ means that the sequent does not contain a focus.

In this section we prove the strong focalisation property for the displacement calculus with additives **DA**.

The focalisation property for Linear Logic was discovered by [2]. In this paper we follow the proof idea from [12], which we adapt to the intuitionistic non-commutative case **DA** with twin multiplicative modes of combination, the continuous (concatenation) and the discontinuous (intercalation) products. The proof relies heavily on the Cut-elimination property for weakly focalised **DA** which is proved in

$$\frac{\overrightarrow{A}: x, \Gamma \Rightarrow C: \chi}{\Gamma \Rightarrow A\backslash C: \lambda x \chi}\ \backslash R \qquad \frac{\Gamma, \overrightarrow{B}: y \Rightarrow C: \chi}{\Gamma \Rightarrow C/B: \lambda y \chi}\ /R$$

$$\frac{\Delta\langle \overrightarrow{A}: x, \overrightarrow{B}: y\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{A \bullet B}: z\rangle \Rightarrow D: \omega\{\pi_1 z/x, \pi_2 z/y\}}\ \bullet L$$

$$\frac{\Delta\langle \Lambda\rangle \Rightarrow A: \phi}{\Delta\langle \overrightarrow{I}: x\rangle \Rightarrow A: \phi}\ IL$$

$$\frac{\overrightarrow{A}: x|_k \Gamma \Rightarrow C: \chi}{\Gamma \Rightarrow A|_k C: \lambda x \chi}\ \downarrow_k R \qquad \frac{\Gamma|_k \overrightarrow{B}: y \Rightarrow C: \chi}{\Gamma \Rightarrow C\uparrow_k B: \lambda y \chi}\ \uparrow_k R$$

$$\frac{\Delta\langle \overrightarrow{A}: x|_k \overrightarrow{B}: y\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{A \odot_k B}: z\rangle \Rightarrow D: \omega\{\pi_1 z/x, \pi_2 z/y\}}\ \odot_k L$$

$$\frac{\Delta\langle 1\rangle \Rightarrow A: \phi}{\Delta\langle \overrightarrow{J}: x\rangle \Rightarrow A: \phi}\ JL$$

Figure 4: Asynchronous multiplicative rules

$$\frac{\Gamma \Rightarrow A: \phi \qquad \Gamma \Rightarrow B: \psi}{\Gamma \Rightarrow A\&B: (\phi, \psi)}\ \&R$$

$$\frac{\Gamma\langle \overrightarrow{A}: x\rangle \Rightarrow C: \chi_1 \qquad \Gamma\langle \overrightarrow{B}: y\rangle \Rightarrow C: \chi_2}{\Gamma\langle \overrightarrow{A \oplus B}: z\rangle \Rightarrow C: z \to x.\chi_1; y.\chi_2}\ \oplus L$$

Figure 5: Asynchronous additive rules

$$\frac{\Gamma \Rightarrow \boxed{P}: \phi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma, \overrightarrow{\boxed{P\backslash Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ \backslash L \qquad \frac{\Gamma \Rightarrow \boxed{P}: \phi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma, \overrightarrow{\boxed{P\backslash M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ \backslash L$$

$$\frac{\Gamma \Rightarrow N: \phi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma, \overrightarrow{\boxed{N\backslash Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ \backslash L \qquad \frac{\Gamma \Rightarrow N: \phi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma, \overrightarrow{\boxed{N\backslash M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ \backslash L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{Q/P}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ /L \qquad \frac{\Gamma \Rightarrow N: \psi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{Q/N}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ /L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{M/P}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ /L \qquad \frac{\Gamma \Rightarrow M: \psi \qquad \Delta\langle \overrightarrow{N}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{N/M}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ /L$$

Figure 6: Left synchronous continuous multiplicative rules

$$\frac{\Gamma \Rightarrow \boxed{P}: \phi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma|_k \overrightarrow{\boxed{P{\downarrow_k}Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ {\downarrow_k}L \qquad \frac{\Gamma \Rightarrow \boxed{P}: \phi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma|_k \overrightarrow{\boxed{P{\downarrow_k}M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ {\downarrow_k}L$$

$$\frac{\Gamma \Rightarrow N: \phi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma|_k \overrightarrow{\boxed{N{\downarrow_k}Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ {\downarrow_k}L \qquad \frac{\Gamma \Rightarrow N: \phi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma|_k \overrightarrow{\boxed{N{\downarrow_k}M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ {\downarrow_k}L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{Q{\uparrow_k}P}}: x|_k \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ {\uparrow_k}L \qquad \frac{\Gamma \Rightarrow N: \psi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{Q{\uparrow_k}N}}: x|_k \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ {\uparrow_k}L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{M{\uparrow_k}P}}: x|_k \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ {\uparrow_k}L \qquad \frac{\Gamma \Rightarrow N: \psi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{M{\uparrow_k}N}}: x|_k \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ {\uparrow_k}L$$

Figure 7: Left synchronous discontinuous multiplicative rules

$$\frac{\Gamma\langle\overrightarrow{\boxed{Q}}:x\rangle \Rightarrow C:\chi}{\Gamma\langle\overrightarrow{\boxed{Q\&B}}:z\rangle \Rightarrow C:\chi\{\pi_1 z/x\}}\&L_1 \qquad \frac{\Gamma\langle\overrightarrow{M}:x\rangle \Rightarrow C:\chi}{\Gamma\langle\overrightarrow{\boxed{M\&B}}:z\rangle \Rightarrow C:\chi\{\pi_1 z/x\}}\&L_1$$

$$\frac{\Gamma\langle\overrightarrow{\boxed{Q}}:y\rangle \Rightarrow C:\chi}{\Gamma\langle\overrightarrow{\boxed{A\&Q}}:z\rangle \Rightarrow C:\chi\{\pi_2 z/y\}}\&L_2 \qquad \frac{\Gamma\langle\overrightarrow{M}:y\rangle \Rightarrow C:\chi}{\Gamma\langle\overrightarrow{\boxed{A\&M}}:z\rangle \Rightarrow C:\chi\{\pi_2 z/y\}}\&L_2$$

Figure 8: Left synchronous additive rules

$$\frac{\Gamma_1 \Rightarrow \boxed{P_1}:\phi \qquad \Gamma_2 \Rightarrow \boxed{P_2}:\psi}{\Gamma_1,\Gamma_2 \Rightarrow \boxed{P_1{\bullet}P_2}:(\phi,\psi)}\bullet R \qquad \frac{\Gamma_1 \Rightarrow \boxed{P}:\phi \qquad \Gamma_2 \Rightarrow N:\psi}{\Gamma_1,\Gamma_2 \Rightarrow \boxed{P{\bullet}N}:(\phi,\psi)}\bullet R$$

$$\frac{\Gamma_1 \Rightarrow N:\phi \qquad \Gamma_2 \Rightarrow \boxed{P}:\psi}{\Gamma_1,\Gamma_2 \Rightarrow \boxed{N{\bullet}P}:(\phi,\psi)}\bullet R \qquad \frac{\Gamma_1 \Rightarrow N_1:\phi \qquad \Gamma_2 \Rightarrow N_2:\psi}{\Gamma_1,\Gamma_2 \Rightarrow \boxed{N_1{\bullet}N_2}:(\phi,\psi)}\bullet R$$

$$\frac{}{\Lambda \Rightarrow \boxed{I}:0}IR$$

Figure 9: Right synchronous continuous multiplicative rules

$$\frac{\Gamma_1 \Rightarrow \boxed{P_1}:\phi \qquad \Gamma_2 \Rightarrow \boxed{P_2}:\psi}{\Gamma_1\,|_k\,\Gamma_2 \Rightarrow \boxed{P_1{\odot_k}P_2}:(\phi,\psi)}\odot_k R \qquad \frac{\Gamma_1 \Rightarrow \boxed{P}:\phi \qquad \Gamma_2 \Rightarrow N:\psi}{\Gamma_1\,|_k\,\Gamma_2 \Rightarrow \boxed{P{\odot_k}N}:(\phi,\psi)}\odot_k R$$

$$\frac{\Gamma_1 \Rightarrow N:\phi \qquad \Gamma_2 \Rightarrow \boxed{P}:\psi}{\Gamma_1\,|_k\,\Gamma_2 \Rightarrow \boxed{N{\odot_k}P}:(\phi,\psi)}\odot_k R \qquad \frac{\Gamma_1 \Rightarrow N_1:\phi \qquad \Gamma_2 \Rightarrow N_2:\psi}{\Gamma_1\,|_k\,\Gamma_2 \Rightarrow \boxed{N_1{\odot_k}N_2}:(\phi,\psi)}\odot_k R$$

$$\frac{}{1 \Rightarrow \boxed{J}:0}JR$$

Figure 10: Right synchronous discontinuous multiplicative rules

$$\frac{\Gamma \Rightarrow \boxed{P}:\phi}{\Gamma \Rightarrow \boxed{P{\oplus}B}:\iota_1\phi}\oplus R_1 \qquad \frac{\Gamma \Rightarrow N:\phi}{\Gamma \Rightarrow \boxed{N{\oplus}B}:\iota_1\phi}\oplus R_1$$

$$\frac{\Gamma \Rightarrow \boxed{P}:\psi}{\Gamma \Rightarrow \boxed{A{\oplus}P}:\iota_2\psi}\oplus R_2 \qquad \frac{\Gamma \Rightarrow N:\psi}{\Gamma \Rightarrow \boxed{A{\oplus}N}:\iota_2\psi}\oplus R_2$$

Figure 11: Right synchronous additive rules

the appendix. In our presentation of focalisation we have avoided the *react* rules of [2] and [3], and use instead a simpler, box, notation suitable for non-commutativity.

$\mathbf{DA_{Foc}}$ is a subsystem of $\mathbf{DA_{foc}}$. $\mathbf{DA_{foc}}$ has the focusing rules *foc* and Cut rules $p\text{-}Cut_1$, $p\text{-}Cut_2$, $n\text{-}Cut_1$ and $n\text{-}Cut_2$[3] shown in (5), and the synchronous and asynchronous rules displayed before, which are read as allowing in synchronous rules the occurrence of asynchronous formulas, and in asynchronous rules as allowing arbitrary sequents with possibly one focalised formula. $\mathbf{DA_{Foc}}$ has the focusing rules but not the Cut rules, and the synchronous and asynchronous rules displayed before, which are such that focalised sequents cannot contain any complex asynchronous formulas, whereas sequents with at least one complex asynchronous formula cannot contain a focalised formula. Hence, strongly focalised proof search operates in alternating asynchronous and synchronous phases. The weakly focalised calculus $\mathbf{DA_{foc}}$ is an intermediate logic which we use to prove the completeness of $\mathbf{DA_{Foc}}$ for $\mathbf{DA}$.

(5)
$$\frac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w A}{\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w A}\,foc \qquad \frac{\Delta\Longrightarrow_w \boxed{P}}{\Delta\Longrightarrow_w P}\,foc$$

$$\frac{\Gamma\Longrightarrow_w \boxed{P} \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C\diamond \text{foc}}{\Delta\langle\Gamma\rangle\Longrightarrow_w C\diamond \text{foc}}\,p\text{-}Cut_1 \qquad \frac{\Gamma\Longrightarrow_w N\diamond \text{foc} \qquad \Delta\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\rangle\Longrightarrow_w C\diamond \text{foc}}\,p\text{-}Cut_2$$

$$\frac{\Gamma\Longrightarrow_w P\diamond \text{foc} \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\rangle\Longrightarrow_w C\diamond \text{foc}}\,n\text{-}Cut_1 \qquad \frac{\Gamma\Longrightarrow_w N \qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond \text{foc}}{\Delta\langle\Gamma\rangle\Longrightarrow_w C\diamond \text{foc}}\,n\text{-}Cut_2$$

4.1 Embedding of DA into $\mathbf{DA_{foc}}$

The identity axiom we consider for **DA** and for both $\mathbf{DA_{foc}}$ and $\mathbf{DA_{Foc}}$ is restricted to atomic types; recalling that atomic types are classified into positive bias At^+ and negative bias At^-:

(6) If $P\in At^+$, $P\Longrightarrow_w \boxed{P}$ and $P\Longrightarrow \boxed{P}$
If $Q\in At^-$, $\boxed{Q}\Longrightarrow_w Q$ and $\boxed{Q}\Longrightarrow Q$

In fact, the Identity rule holds of any type A. It has the following formulation in the sequent calculi considered here:

(7)
$$\begin{cases} \overrightarrow{A}\Rightarrow A & & \text{in } \mathbf{DA}\\ \overrightarrow{P}\Longrightarrow_w \boxed{P} & \overrightarrow{\boxed{N}}\Longrightarrow_w N & \text{in } \mathbf{DA_{foc}}\\ \overrightarrow{P}\Longrightarrow P & \overrightarrow{N}\Longrightarrow N & \text{in } \mathbf{DA_{Foc}}\end{cases}$$

The Identity axiom for arbitrary types is also known as *Eta-expansion*. Eta-expansion is easy to prove in both **DA** and $\mathbf{DA_{foc}}$, but the same is not the case for $\mathbf{DA_{Foc}}$. This is the reason to consider what we have called weak focalisation, which helps us to prove smoothly this crucial property for the proof of strong focalisation.

Theorem 4.1 (Embedding of **DA** into $\mathbf{DA_{foc}}$) *For any configuration* Δ *and type A, we have that if* $\Delta\Rightarrow A$ *then* $\Delta\Longrightarrow_w A$.

Proof. We proceed by induction on the length of the derivation of **DA** proofs. In the following lines, we apply the induction hypothesis (i.h.) for each premise of **DA** rules (with the exception of the Identity rule and the right rules of units):

- Identity axiom:

[3] If it is convenient, we may drop the subscripts.

(8) $$\dfrac{\overrightarrow{P}\Longrightarrow_w \boxed{\mathrm{P}}}{\overrightarrow{P}\Longrightarrow_w P}\,foc \qquad \dfrac{\boxed{\overrightarrow{\mathrm{N}}}\Longrightarrow_w N}{\overrightarrow{N}\Longrightarrow_w N}\,foc$$

- Cut rule: just apply *n-Cut*.
- Units

(9) $$\dfrac{}{\Lambda\Rightarrow I}\,IR \quad\rightsquigarrow\quad \dfrac{\dfrac{}{\Lambda\Longrightarrow_w\boxed{I}}\,IR}{\Lambda\Longrightarrow_w I}\,foc$$

(10) $$\dfrac{}{1\Rightarrow J}\,JR \quad\rightsquigarrow\quad \dfrac{\dfrac{}{1\Longrightarrow_w\boxed{J}}\,JR}{1\Longrightarrow_w J}\,foc$$

Left unit rules apply as in the case of **DA**.

- Left discontinuous product: directly translates.
- Right discontinuous product. There are cases $P_1\odot_k P_2$, $N_1\odot_k N_2$, $N\odot_k P$ and $P\odot_k N$. We show one representative example:

$$\dfrac{\Delta\Rightarrow P \qquad \Gamma\Rightarrow N}{\Delta|_k\Gamma\Rightarrow P\odot_k N}\,\odot_k R \quad\rightsquigarrow\quad \dfrac{\dfrac{\Gamma\Longrightarrow_w N \qquad \dfrac{\Delta\Longrightarrow_w P \qquad \dfrac{\overrightarrow{P}\Longrightarrow_w\boxed{P} \qquad \dfrac{\boxed{\overrightarrow{N}}\Longrightarrow_w N}{\overrightarrow{N}\Longrightarrow_w N}\,foc}{\overrightarrow{P}\,|_k\,\overrightarrow{N}\Rightarrow\boxed{P\odot_k N}}\,\odot_k R}{\Delta|_k\,\overrightarrow{N}\Longrightarrow_w\boxed{P\odot_k N}}\,n\text{-}Cut_2}{\Delta|_k\Gamma\Longrightarrow_w\boxed{P\odot_k N}}\,n\text{-}Cut_2}{\Delta|_k\Gamma\Longrightarrow_w P\odot_k N}\,foc$$

- Left discontinuous $\uparrow_k$ rule (the left rule for $\downarrow_k$ is entirely similar). Like in the case for the right discontinuous product $\odot_k$ rule, we only show one representative example:

$$\dfrac{\Gamma\Rightarrow P \qquad \Delta\langle\overrightarrow{N}\rangle\Rightarrow A}{\Delta\langle\overrightarrow{N\uparrow_k P}|_k\Gamma\rangle\Rightarrow A}\,\uparrow_k L \quad\rightsquigarrow\quad \dfrac{\dfrac{\Gamma\Longrightarrow_w P \qquad \dfrac{\dfrac{\overrightarrow{P}\Longrightarrow_w\boxed{\mathrm{P}} \qquad \boxed{\overrightarrow{N}}\Longrightarrow_w N}{\overrightarrow{\boxed{N\uparrow_k P}}\,|_k\,\overrightarrow{P}\Longrightarrow_w N}\,\uparrow_k L \qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w A}{\Delta\langle\overrightarrow{\boxed{N\uparrow_k P}}\,|_k\,\overrightarrow{P}\rangle\Longrightarrow_w A}\,n\text{-}Cut_1}{\Delta\langle\overrightarrow{\boxed{N\uparrow_k P}}\,|_k\,\Gamma\rangle\Longrightarrow_w A}\,n\text{-}Cut_2}{\Delta\langle\overrightarrow{N\uparrow_k P}\,|_k\,\Gamma\rangle\Longrightarrow_w A}\,foc$$

- Right discontinuous $\uparrow_k$ rule (the right discontinuous rule for $\downarrow_k$ is entirely similar):

(11) $$\dfrac{\Delta|_k\,\overrightarrow{A}\Rightarrow B}{\Delta\Rightarrow B\uparrow_k A}\,\uparrow_k R \quad\rightsquigarrow\quad \dfrac{\Delta|_k\,\overrightarrow{A}\Longrightarrow_w B}{\Delta\Longrightarrow_w B\uparrow_k A}\,\uparrow_k R$$

- Product and implicative continuous rules. These follow the same pattern as the discontinuous case. We interchange the metalinguistic k-th intercalation $|_k$ with the metalinguistic concatenation ',', and we interchange $\odot_k$, $\uparrow_k$ and $\downarrow_k$ with $\bullet$, $/$, and $\backslash$ respectively.

Concerning additives, conjunction Right translates directly and we consider then conjunction Left (disjunction is symmetric):

$$(12)\quad \frac{\Delta\langle\overrightarrow{P}\rangle \Rightarrow C}{\Delta\langle\overrightarrow{P\&M}\rangle \Rightarrow C}\&L \quad\rightsquigarrow\quad \frac{\overrightarrow{P\&M}\Longrightarrow_w P \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Delta\langle\overrightarrow{P\&M}\rangle\Longrightarrow_w C}\,n\text{-}Cut_1$$

where by Eta expansion and application of the *foc* rule, we have $\overrightarrow{P\&M}\Longrightarrow_w P$. $\square$

4.2 Embedding of $\mathbf{DA_{foc}}$ into $\mathbf{DA_{Foc}}$

Theorem 4.2 (Embedding of $\mathbf{DA_{foc}}$ into $\mathbf{DA_{Foc}}$) *For any configuration Δ and type A, we have that if $\Delta\Longrightarrow_w A$ with one focalised formula and no asynchronous formula occurrence, then $\Delta\Longrightarrow A$ with the same formula focalised. If $\Delta\Longrightarrow_w A$ with no focalised formula and with at least one asynchronous formula, then $\Delta\Longrightarrow A$.*

Proof. We proceed by induction on the size of $\mathbf{DA_{foc}}$ sequents.[4] We consider Cut-free $\mathbf{DA_{foc}}$ proofs which match the sequents of this theorem. If the last rule is logical (i.e., it is not an instance of the *foc* rule) the i.h. applies directly and we get $\mathbf{DA_{Foc}}$ proofs of the same end-sequent. Now, let us suppose that the last rule is not logical, i.e. it is an instance of the *foc* rule. Let us suppose that the end sequent $\Delta\Longrightarrow_w A$ is a synchronous sequent. Suppose for example that the focalised formula is in the succedent:

$$(13)\quad \frac{\Delta\Longrightarrow_w \boxed{P}}{\Delta\Longrightarrow_w P}\,foc$$

The sequent $\Delta\Longrightarrow_w \boxed{P}$ arises from a synchronous rule to which we can apply i.h.. Let us suppose now that the end-sequent contains at least one asynchronous formula. We see three cases which are illustrative:

(14) a. $\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w \boxed{P}$

b. $\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w B\uparrow_k A$

c. $\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w A\&B$

We have by Eta expansion that $\overrightarrow{A\odot_k B}\Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}}$. We apply to this sequent the invertible $\odot_k$ left rule, whence $\overrightarrow{A}|_k\overrightarrow{B}\Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}}$. In case (14a), we have the following proof in $\mathbf{DA_{foc}}$:

$$(15)\quad \cfrac{\cfrac{\overrightarrow{A}\,|_k\,\overrightarrow{B}\Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}} \qquad \Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w \boxed{P}}{\Delta\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle\Longrightarrow_w \boxed{P}}\,p\text{-}Cut_1}{\Delta\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle\Longrightarrow_w P}\,foc$$

[4]For a given type A, the *size* of A, $|A|$, is the number of connectives in A. By recursion on configurations we have:

$$\begin{array}{rcl} |\Lambda| & ::= & 0 \\ |\overrightarrow{A},\Delta| & ::= & |A|+|\Delta|, \text{for } sA=0 \\ |1,\Delta| & ::= & |\Delta| \\ |A\{\Delta_1:\cdots:\Delta_{sA}\}| & ::= & |A|+\sum_{i=1}^{sA}|\Delta_i| \end{array}$$

Moreover, we have:

$|\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w A| = |\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w A|$

$|\Delta\Longrightarrow_w \boxed{P}| = |\Delta\Longrightarrow_w P|$

To the above $\mathbf{DA_{foc}}$ proof we apply Cut-elimination and we get the Cut-free $\mathbf{DA_{foc}}$ end-sequent $\Delta\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle \Longrightarrow_w P$. We have $|\Delta\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle\Longrightarrow_w P| < |\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w P|$. We can apply then i.h. and we derive the provable $\mathbf{DA_{Foc}}$ sequent $\Delta\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle\Longrightarrow P$ to which we can apply the left $\odot_k$ rule. We have obtained $\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow P$. In the same way, we have that $\overrightarrow{\boxed{B\uparrow_k A}}|_k\overrightarrow{A}\Longrightarrow_w B$. Thus, in case (14b), we have the following proof in $\mathbf{DA_{foc}}$:

(16)
$$\cfrac{\cfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w B\uparrow_k A \qquad \overrightarrow{\boxed{B\uparrow_k A}}\,|_k\,\overrightarrow{A}\Longrightarrow_w B}{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\,|_k\,\overrightarrow{A}\Longrightarrow_w B}\,p\text{-}Cut_2}{\Delta\langle\overrightarrow{Q}\rangle\,|_k\,\overrightarrow{A}\Longrightarrow_w B}\,foc$$

As before, we apply Cut-elimination to the above proof. We get the Cut-free $\mathbf{DA_{foc}}$ end-sequent $\Delta\langle\overrightarrow{Q}\rangle|_k\overrightarrow{A} \Longrightarrow_w B$. It has size less than $|\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w B\uparrow_k A|$. We can apply i.h. and we get the $\mathbf{DA_{Foc}}$ provable sequent $\Delta\langle\overrightarrow{Q}\rangle|_k\overrightarrow{A}\Longrightarrow B$ to which we apply the $\uparrow_k$ right rule.

In case (14c):

(17)
$$\cfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w A\&B}{\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w A\&B}\,foc$$

by applying the *foc* rule and the invertibility of $\&R$ we get the provable $\mathbf{DA_{foc}}$ sequents $\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w A$ and $\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w B$. These sequents have smaller size than $\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w A\&B$. The aforementioned sequents have a Cut-free proof in $\mathbf{DA_{foc}}$. We apply i.h. and we get $\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow A$ and $\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow B$. We apply the & right rule in $\mathbf{DA_{Foc}}$, and we get $\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow A\&B$. $\square$

By this theorem we obtain the completeness of strong focalisation.

5 Example

We can have coordinate unlike types with nominal and adjectival complementation of *is*:

(18) $[\mathbf{Tully}]{+}\mathbf{is}{+}[[\mathbf{Cicero}{+}\mathbf{and}{+}\mathbf{humanist}]] : Sf$

Lexical lookup of types yields:

(19) $[\blacksquare Nt(s(m)) : b], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g(CNg/CN)) :$
$\lambda A\lambda B(Pres\ (A\rightarrow C.[B=C];D.((D\ \lambda E[E=B])\ B))), [[\blacksquare\forall gNt(s(g)) : 007,$
$\blacksquare\forall f\forall a((\blacksquare((((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}((((\langle\rangle Na\backslash Sf)/$
$(\exists bNb\oplus\exists g(CNg/CN))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare((((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf))) :$
$\lambda F\lambda G\lambda H\lambda I[((G\ H)\ I)\wedge((F\ H)\ I)], \square\forall n(CNn/CNn) : \hat{}\lambda J\lambda K[(J\ K)\wedge(\check{}teetotal\ K)]]] \ \Rightarrow\ Sf$

The bracket modalities $\langle\rangle$ and $[]^{-1}$mark as syntactic domains subjects and coordinate structures which are weak and strong islands respectively. The quantifiers and first-order structure mark agreement features such as third person singular for any gender for *is*. The normal modality $\square$ marks semantic intensionality and $\blacksquare$ marks rigid designator semantic intensionality. The example has positive and negative additive disjunction so that the derivation in Figures 12–16 illustrates both synchronous and asynchronous focusing additives. This delivers the correct semantics: $[(Pres\ [t=c])\wedge(Pres\ (\check{}humanist\ t))]$.

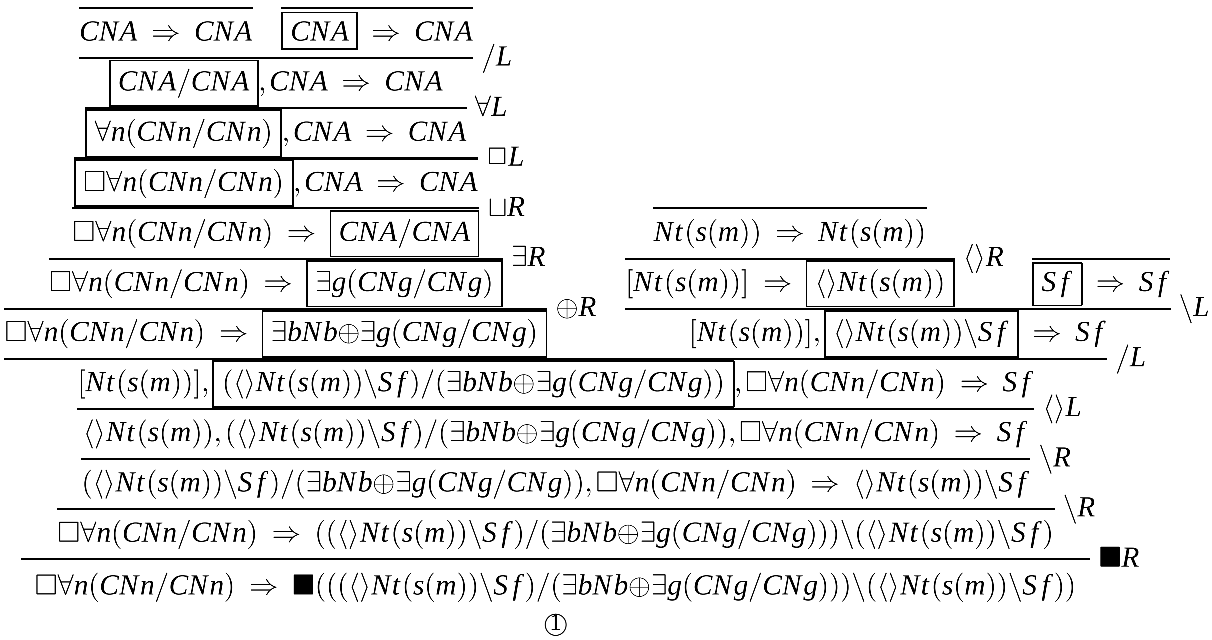


Figure 12: Coordination of unlike types, Part I

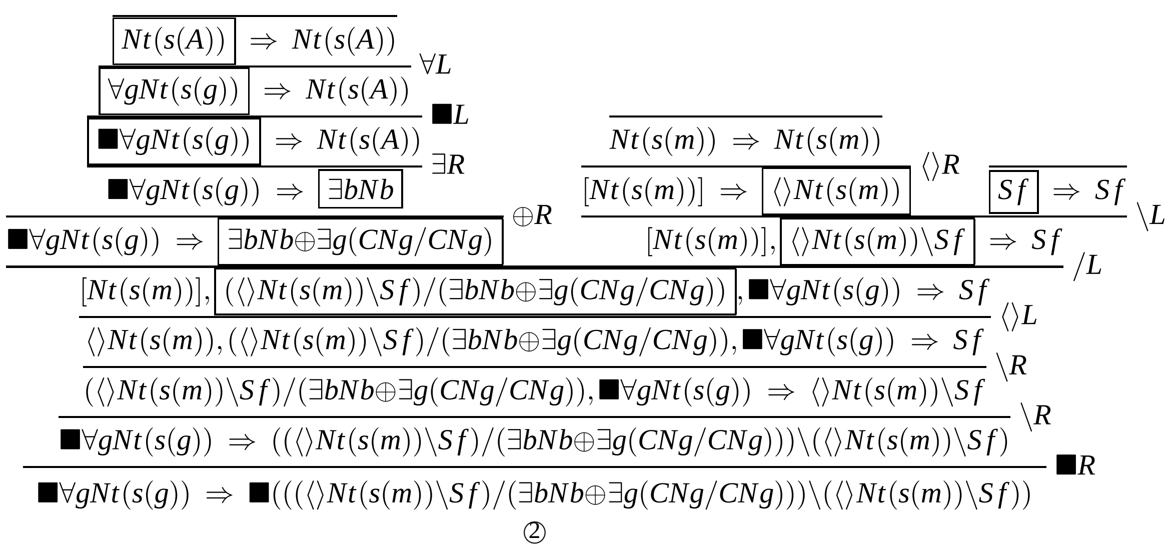


Figure 13: Coordination of unlike types, Part II

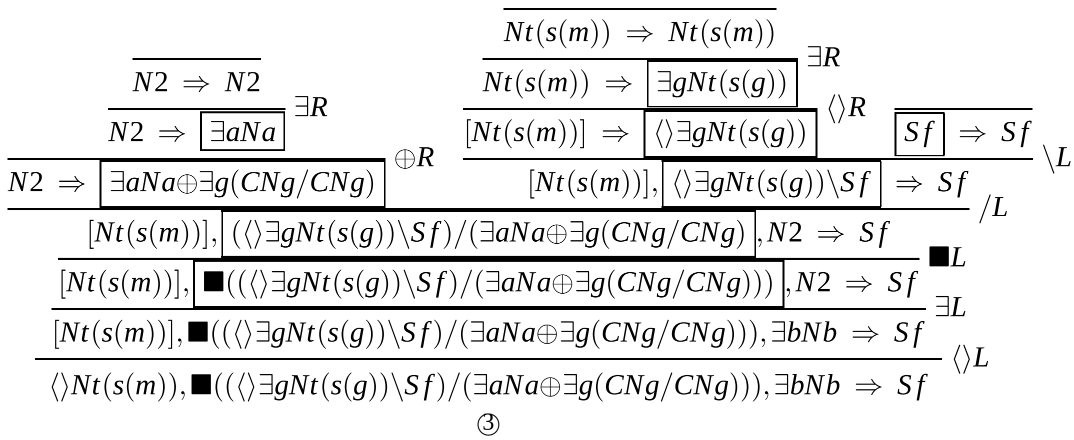


Figure 14: Coordination of unlike types, Part III

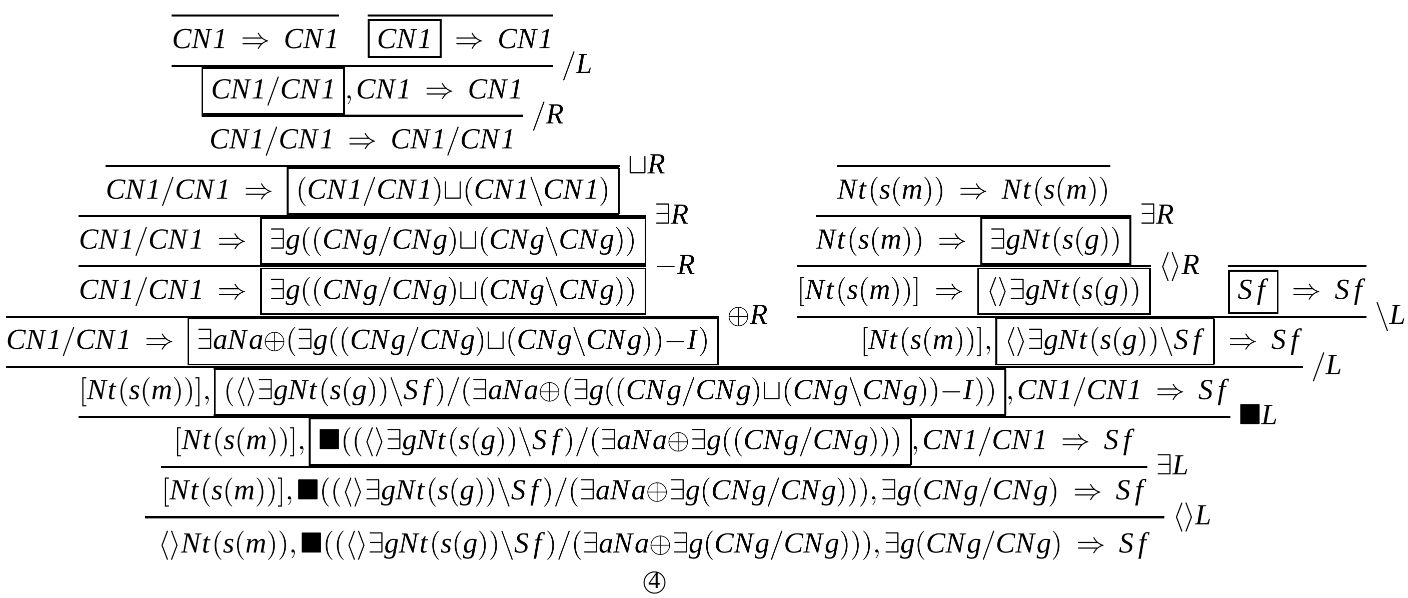


Figure 15: Coordination of unlike types, Part IV

$$\dfrac{}{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}$$

$$\dfrac{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}\ \blacksquare L$$

$$\dfrac{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\ \langle\rangle R \qquad \dfrac{}{\boxed{Sf} \Rightarrow Sf}$$

$$\dfrac{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))} \qquad \boxed{Sf} \Rightarrow Sf}{[\blacksquare Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}\ \backslash L$$

③ ④

$$\dfrac{③ \quad ④}{\langle\rangle Nt(s(m)), \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \exists bNb\oplus\exists g(CNg/CNg) \Rightarrow Sf}\ \oplus L$$

$$\dfrac{\langle\rangle Nt(s(m)), \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \exists bNb\oplus\exists g(CNg/CNg) \Rightarrow Sf}{\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \exists bNb\oplus\exists g(CNg/CNg) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\ \backslash R$$

$$\dfrac{\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \exists bNb\oplus\exists g(CNg/CNg) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}{\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg)) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))}\ /R$$

$$\dfrac{\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg)) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)) \qquad [\blacksquare Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}{[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \boxed{((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf)} \Rightarrow Sf}\ \backslash L$$

$$\dfrac{[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \boxed{((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf)} \Rightarrow Sf}{[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [\boxed{[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))))\backslash(\langle\rangle Nt(s(m))\backslash Sf))}] \Rightarrow Sf}\ []^{-1}L$$

$$\dfrac{[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [\boxed{[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))))\backslash(\langle\rangle Nt(s(m))\backslash Sf))}] \Rightarrow Sf}{[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\boxed{[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))))\backslash(\langle\rangle Nt(s(m))\backslash Sf))}]] \Rightarrow Sf}\ []^{-1}L$$

②

$$\dfrac{② \qquad [\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\boxed{[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))))\backslash(\langle\rangle Nt(s(m))\backslash Sf))}]] \Rightarrow Sf}{\begin{array}{c}[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\blacksquare\forall gNt(s(g)),\\ \boxed{\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))}]] \Rightarrow Sf\end{array}}\ \backslash L$$

①

$$\dfrac{① \qquad \begin{array}{c}[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\blacksquare\forall gNt(s(g)),\\ \boxed{\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))}]] \Rightarrow Sf\end{array}}{\begin{array}{c}[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\blacksquare\forall gNt(s(g)),\\ \boxed{(\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf)))/\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))},\\ \Box\forall n(CNn/CNn)]] \Rightarrow Sf\end{array}}\ /L$$

$$\dfrac{\begin{array}{c}[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\blacksquare\forall gNt(s(g)),\\ \boxed{(\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf)))/\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))},\\ \Box\forall n(CNn/CNn)]] \Rightarrow Sf\end{array}}{\begin{array}{c}[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\blacksquare\forall gNt(s(g)),\\ \boxed{\forall a((\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf)))},\\ \Box\forall n(CNn/CNn)]] \Rightarrow Sf\end{array}}\ \forall L$$

$$\dfrac{\begin{array}{c}[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\blacksquare\forall gNt(s(g)),\\ \boxed{\forall a((\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf)))},\\ \Box\forall n(CNn/CNn)]] \Rightarrow Sf\end{array}}{\begin{array}{c}[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\blacksquare\forall gNt(s(g)),\\ \boxed{\forall f\forall a((\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf)))},\\ \Box\forall n(CNn/CNn)]] \Rightarrow Sf\end{array}}\ \forall L$$

(

$$\begin{array}{c}[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), [[\blacksquare\forall gNt(s(g)),\\ \boxed{\blacksquare\forall f\forall a((\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Na\backslash Sf)))},\\ \Box\forall n(CNn/CNn)]] \Rightarrow Sf\end{array}$$

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Appendix: Cut Elimination

We prove this by induction on the complexity (d,h) of top-most instances of *Cut*, where d is the size[5] of the cut formula and h is the length of the derivation the last rule of which is the Cut rule. There are four cases to consider: Cut with axiom in the minor premise, Cut with axiom in the major premise, principal Cuts, and permutation conversions. In each case, the complexity of the Cut is reduced. In order to save space, we will not be exhaustive showing all the cases because many follow the same pattern. In particular, for any synchronous logical rule there are always four cases to consider corresponding to the polarity of the subformulas. Here, and in the following, we will show only one representative example. Concerning continuous and discontinuous formulas, we will show only the discontinuous cases (discontinuous connectives are less known than the continuous ones of the plain Lambek Calculus). For the continuous instances, the reader has only to interchange the meta-linguistic wrap $|_k$ with the meta-linguistic concatenation $','$, $\odot_k$ with $\bullet$, $\uparrow_k$ with $/$ and $\downarrow_k$ with $\backslash$. The units cases (principal case and permutation conversion cases) are completely trivial.

Proof. - *Id* cases:

$$(20)\quad \frac{\overrightarrow{P} \Longrightarrow_w \boxed{P} \qquad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w B \diamond \text{foc}}{\Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w B \diamond \text{foc}}\, p\text{-}Cut_1 \quad \rightsquigarrow \quad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w B \diamond \text{foc}$$

$$\frac{\Delta \Longrightarrow_w N \diamond \text{foc} \qquad \overrightarrow{\boxed{N}} \Longrightarrow_w N}{\Delta\langle \overrightarrow{N} \rangle \Longrightarrow_w B \diamond \text{foc}}\, p\text{-}Cut_2 \quad \rightsquigarrow \quad \Delta\langle \overrightarrow{N} \rangle \Longrightarrow_w B \diamond \text{foc}$$

The attentive reader may have wondered whether the following *Id* case could arise:

$$(21)\quad \frac{\boxed{\overrightarrow{Q}} \Rightarrow Q \qquad \Gamma\langle \overrightarrow{Q} \rangle \Rightarrow A}{\Gamma\langle \boxed{\text{Q}} \rangle \Rightarrow A}\, n\text{-}Cut_i$$

If Q were a primitive type q, and Γ were not the empty context, we would have then a Cut-free underivable sequent. For example, if the right premise of the Cut rule in (21) were the derivable sequent $q, q\backslash s \Rightarrow s$, we would have then as conclusion:

$$(22)\quad \boxed{\text{q}}, q\backslash s \Rightarrow s$$

Since the primitive type q in the antecedent is focalised, there is no possibility of applying the $\backslash$ left rule, which is a synchronous rule that needs that its active formula to be focalised. Principal cases:

• *foc* cases:

$$(23)\quad \frac{\dfrac{\Delta \Longrightarrow_w \boxed{P}}{\Delta \Longrightarrow_w P}\, foc \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w A \diamond \text{foc}}{\Gamma\langle \Delta \rangle \Longrightarrow_w A \diamond \text{foc}}\, n\text{-}Cut_1 \quad \rightsquigarrow \quad \frac{\Delta \Longrightarrow_w \boxed{P} \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w A \diamond \text{foc}}{\Gamma\langle \Delta \rangle \Longrightarrow_w A \diamond \text{foc}}\, p\text{-}Cut_1$$

[5]The size of $|A|$ is the number of connectives appearing in A.

(24)
$$\frac{\Delta\Longrightarrow_w N \qquad \dfrac{\Delta\langle\boxed{N}\rangle\Longrightarrow_w A}{\Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w A}\,foc}{\Gamma\langle\Delta\rangle\Longrightarrow_w A}\,n\text{-}Cut_2 \quad\rightsquigarrow\quad \frac{\Delta\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w A}{\Gamma\langle\Delta\rangle\Longrightarrow_w A}\,p\text{-}Cut_2$$

• logical connectives:

(25)
$$\frac{\dfrac{\Delta|_k\overrightarrow{P_1}\Longrightarrow_w P_2\diamond \text{foc}}{\Delta\Longrightarrow_w P_2\uparrow_k P_1\diamond \text{foc}}\uparrow_k R \qquad \dfrac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \Gamma_2\langle\overrightarrow{P_2}\rangle\Longrightarrow_w A}{\Gamma_2\langle\overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k\Gamma_1\rangle\Longrightarrow_w A}\uparrow_k L}{\Gamma_2\langle\Delta|_k\Gamma_1\rangle\Longrightarrow_w A\diamond \text{foc}}\,p\text{-}Cut_2 \quad\rightsquigarrow$$

$$\frac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \dfrac{\Delta|_k\overrightarrow{P_1}\Longrightarrow P_2\diamond \text{foc} \qquad \Gamma_2\langle\overrightarrow{P_2}\rangle\Longrightarrow_w A}{\Gamma_2\langle\Delta|_k\overrightarrow{P_1}\rangle\Longrightarrow_w A\diamond \text{foc}}\,n\text{-}Cut_1}{\Gamma_2\langle\Delta|_k\Gamma_1\rangle\Longrightarrow_w A\diamond \text{foc}}\,p\text{-}Cut_1$$

The case of $\downarrow_k$ is entirely similar to the $\uparrow_k$ case.
The case of $\downarrow_k$ is entirely similar to the $\uparrow_k$ case.

(26)
$$\frac{\dfrac{\Delta_1\Longrightarrow_w\boxed{P} \qquad \Delta_2\Longrightarrow_w N}{\Delta_1|_k\Delta_2\Longrightarrow_w\boxed{P\odot_k N}}\odot_k R \qquad \dfrac{\Gamma\langle\overrightarrow{P}|_k\overrightarrow{N}\rangle\Longrightarrow_w A\diamond \text{foc}}{\Gamma\langle\overrightarrow{P\odot_k N}\rangle\Longrightarrow_w A\diamond \text{foc}}\odot_k L}{\Gamma\langle\Delta_1|_k\Delta_2\rangle\Longrightarrow_w A\diamond \text{foc}}\,p\text{-}Cut_1 \quad\rightsquigarrow$$

$$\frac{\Delta_2\Longrightarrow_w N \qquad \dfrac{\Delta_1\Longrightarrow\boxed{P} \qquad \Gamma\langle\overrightarrow{P}|_k\overrightarrow{N}\rangle\Longrightarrow_w A\diamond \text{foc}}{\Gamma\langle\Delta_1|_k\overrightarrow{N}\rangle\Longrightarrow_w A\diamond \text{foc}}\,p\text{-}Cut_1}{\Gamma\langle\Delta_1|_k\Delta_2\rangle\Longrightarrow_w A\diamond \text{foc}}\,n\text{-}Cut_2$$

(27)
$$\frac{\dfrac{\Delta\Longrightarrow_w Q\diamond \text{foc} \qquad \Delta\Longrightarrow_w A\diamond \text{foc}}{\Delta\Longrightarrow_w Q\&A\diamond \text{foc}}\&R \qquad \dfrac{\Gamma\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w B}{\Gamma\langle\overrightarrow{\boxed{Q\&A}}\Longrightarrow_w B}\&L}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond \text{foc}}\,p\text{-}Cut_2 \quad\rightsquigarrow$$

$$\frac{\Delta\Longrightarrow Q\diamond \text{foc} \qquad \Gamma\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w B}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond \text{foc}}\,p\text{-}Cut_2$$

(28)
$$\frac{\dfrac{\Delta\Longrightarrow_w M\diamond \text{foc} \qquad \Delta\Longrightarrow_w A\diamond \text{foc}}{\Delta\Longrightarrow_w M\&A\diamond \text{foc}}\&R \qquad \dfrac{\Gamma\langle\overrightarrow{M}\rangle\Longrightarrow_w B}{\Gamma\langle\overrightarrow{\boxed{M\&A}}\rangle\Longrightarrow_w B}\&L}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond \text{foc}}\,p\text{-}Cut_2 \quad\rightsquigarrow$$

$$\frac{\Delta\Longrightarrow M\diamond \text{foc} \qquad \Gamma\langle\overrightarrow{M}\rangle\Longrightarrow_w B}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond \text{foc}}\,n\text{-}Cut_1$$

- Left commutative *p-Cut* conversions:

(29)
$$\frac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w N}{\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w N}\,foc \qquad \Gamma\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{Q}\rangle\rangle\Longrightarrow_w C}\,p\text{-}Cut_2 \qquad \rightsquigarrow$$

$$\frac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{\boxed{Q}}\rangle\rangle\Longrightarrow_w C}\,p\text{-}Cut_2}{\Gamma\langle\Delta\langle\overrightarrow{Q}\rangle\rangle\Longrightarrow_w C}\,foc$$

(30)
$$\frac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w\boxed{P}}{\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w\boxed{P}}\,\odot_k L \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w C\Diamond\text{foc}}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\Diamond\text{foc}}\,p\text{-}Cut_1 \qquad \rightsquigarrow$$

$$\frac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w\boxed{P} \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w C\Diamond\text{foc}}{\Gamma\langle\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\Diamond\text{foc}}\,p\text{-}Cut_1}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\Diamond\text{foc}}\,\odot_k L$$

(31)
$$\frac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w N\Diamond\text{foc}}{\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w N\Diamond\text{foc}}\,\odot_k L \qquad \Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\Diamond\text{foc}}\,p\text{-}Cut_2 \qquad \rightsquigarrow$$

$$\frac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w N\Diamond\text{foc} \qquad \Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\Diamond\text{foc}}\,p\text{-}Cut_2}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\Diamond\text{foc}}\,\odot_k L$$

(32)
$$\frac{\dfrac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \Gamma_2\langle\overrightarrow{\boxed{N_1}}\rangle\Longrightarrow_w N}{\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\Longrightarrow_w N}\,\uparrow_k L \qquad \Theta\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w C}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\rangle\Longrightarrow_w C}\,p\text{-}Cut_2 \qquad \rightsquigarrow$$

$$\frac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \dfrac{\Gamma_1\langle\overrightarrow{\boxed{N_1}}\rangle\Longrightarrow_w N \qquad \Theta\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w C}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1}}\rangle\rangle\Longrightarrow_w C}\,p\text{-}Cut_2}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\rangle\Longrightarrow_w C}\,\uparrow_k L$$

(33)
$$\frac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w\boxed{P} \qquad \Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w\boxed{P}}{\Gamma\langle\overrightarrow{A\oplus B}\rangle\Longrightarrow_w\boxed{P}}\,\oplus L \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C\Diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\Diamond\text{foc}}\,p\text{-}Cut_1 \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w\boxed{P}\qquad\Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_1\qquad\dfrac{\Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w\boxed{P}\qquad\Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_1}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,\oplus L$$

(34) $$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w N\diamond\text{foc}\qquad\Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w N\diamond\text{foc}}{\Gamma\langle\overrightarrow{A\oplus B}\rangle\Longrightarrow_w N\diamond\text{foc}}\,\oplus L\qquad\Delta\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_2\quad\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w N\diamond\text{foc}\qquad\Delta\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\langle\overrightarrow{A}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_2\qquad\dfrac{\Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w N\diamond\text{foc}\qquad\Delta\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\langle\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_2}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,\oplus L$$

- Right commutative *p-Cut* conversions (unordered multiple distinguished occurrences are separated by semicolons):

(35) $$\dfrac{\Delta\Longrightarrow_w\boxed{P}\qquad\dfrac{\Gamma\langle\overrightarrow{P};\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w C}{\Gamma\langle\overrightarrow{P};\overrightarrow{Q}\rangle\Longrightarrow_w C}\,foc}{\Gamma\langle\Delta;\overrightarrow{Q}\rangle\Longrightarrow_w C}\,p\text{-}Cut_1\quad\rightsquigarrow\quad\dfrac{\dfrac{\Delta\Longrightarrow_w\boxed{P}\qquad\Gamma\langle\overrightarrow{P};\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta;\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w C}\,p\text{-}Cut_1}{\Gamma\langle\Delta;\overrightarrow{Q}\rangle\Longrightarrow_w C}\,foc$$

(36) $$\dfrac{\Delta\Longrightarrow_w\boxed{P_1}\qquad\dfrac{\Gamma\langle\overrightarrow{P_1}\rangle\Longrightarrow_w\boxed{P_2}}{\Gamma\langle\overrightarrow{P_1}\rangle\Longrightarrow_w P_2}\,foc}{\Gamma\langle\Delta\rangle\Longrightarrow_w P_2}\,p\text{-}Cut_1\quad\rightsquigarrow\quad\dfrac{\dfrac{\Delta\Longrightarrow_w\boxed{P_1}\qquad\Gamma\langle\overrightarrow{P_1}\rangle\Longrightarrow_w\boxed{P_2}}{\Gamma\langle\Delta\rangle\Longrightarrow_w\boxed{P_2}}\,p\text{-}Cut_1}{\Gamma\langle\Delta\rangle\Longrightarrow_w P_2}\,foc$$

(37) $$\dfrac{\Delta\Longrightarrow_w\boxed{P}\qquad\dfrac{\Gamma\langle\overrightarrow{P}\rangle|_k\overrightarrow{A}\Longrightarrow_w B\diamond\text{foc}}{\Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w B\uparrow_k A\diamond\text{foc}}\,\uparrow_k R}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\uparrow_k A\diamond\text{foc}}\,p\text{-}Cut_1\quad\rightsquigarrow\quad\dfrac{\dfrac{\Delta\Longrightarrow_w\boxed{P}\qquad\Gamma\langle\overrightarrow{P}\rangle|_k\overrightarrow{A}\Longrightarrow_w B\diamond\text{foc}}{\Gamma\langle\Delta\rangle|_k\overrightarrow{A}\Longrightarrow_w B\diamond\text{foc}}\,p\text{-}Cut_1}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\uparrow_k A\diamond\text{foc}}\,\uparrow_k R$$

(38) $$\dfrac{\Delta\Longrightarrow_w N\diamond\text{foc}\qquad\dfrac{\Gamma\langle\boxed{\overrightarrow{N}}\rangle|_k\overrightarrow{A}\Longrightarrow_w B}{\Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w B\uparrow_k A}\,\uparrow_k R}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\uparrow_k A\diamond\text{foc}}\,p\text{-}Cut_2\quad\rightsquigarrow\quad\dfrac{\dfrac{\Delta\Longrightarrow_w N\diamond\text{foc}\qquad\Gamma\langle\boxed{\overrightarrow{N}}\rangle|_k\overrightarrow{A}\Longrightarrow_w B}{\Gamma\langle\Delta\rangle|_k\overrightarrow{A}\Longrightarrow_w B\diamond\text{foc}}\,p\text{-}Cut_2}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\uparrow_k A\diamond\text{foc}}\,\uparrow_k R$$

(39) $$\dfrac{\Delta\Longrightarrow_w\boxed{P}\qquad\dfrac{\Gamma\langle\overrightarrow{P};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Gamma\langle\overrightarrow{P};\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond\text{foc}}\,\odot_k L}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_1\quad\rightsquigarrow\quad\dfrac{\dfrac{\Delta\Longrightarrow_w\boxed{P}\qquad\Gamma\langle\overrightarrow{P};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Gamma\langle\Delta;\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_1}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond\text{foc}}\,\odot_k L$$

(40) $$\dfrac{\Delta\Longrightarrow_w N\diamond\text{foc}\qquad\dfrac{\Gamma\langle\overrightarrow{\boxed{N}};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C}{\Gamma\langle\overrightarrow{\boxed{N}};\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C}\,\odot_k L}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_2\quad\rightsquigarrow\quad\dfrac{\dfrac{\Delta\Longrightarrow_w N\diamond\text{foc}\qquad\Gamma\langle\overrightarrow{\boxed{N}};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta;\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_2}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond\text{foc}}\,\odot_k L$$

(41) $$\dfrac{\Delta\Longrightarrow_w\boxed{P}\qquad\dfrac{\Gamma\Longrightarrow_w\boxed{P_1}\qquad\Theta\langle\overrightarrow{P_2};\overrightarrow{P}\rangle\Longrightarrow_w C}{\Theta\langle\overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k\Gamma;\overrightarrow{P}\rangle\Longrightarrow_w C}\,\uparrow_k L}{\Theta\langle\overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k\Gamma;\Delta\rangle\Longrightarrow_w C}\,p\text{-}Cut_1\quad\rightsquigarrow$$

$$\dfrac{\Gamma \Longrightarrow_w \boxed{P_1} \qquad \dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \Theta\langle \overrightarrow{P_2}; \overrightarrow{P} \rangle \Longrightarrow_w C}{\Theta\langle \overrightarrow{P_2}; \Delta \rangle \Longrightarrow_w C}\, p\text{-}Cut_1}{\Theta\langle \overrightarrow{\boxed{P_2 \uparrow_k P_1}} |_k \Gamma; \Delta \rangle \Longrightarrow_w C}\, \uparrow_k L$$

(42)
$$\dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \dfrac{\Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w A \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w B \diamond \text{foc}}{\Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w A\&B \diamond \text{foc}}\, \&R}{\Gamma\langle \Delta \rangle \Longrightarrow_w A\&B \diamond \text{foc}}\, p\text{-}Cut_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w A \diamond \text{foc}}{\Gamma\langle \Delta \rangle \Longrightarrow_w A \diamond \text{foc}}\, p\text{-}Cut_1 \qquad \dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w B \diamond \text{foc}}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \diamond \text{foc}}\, p\text{-}Cut_1}{\Gamma\langle \Delta \rangle \Longrightarrow_w A\&B \diamond \text{foc}}\, \&R$$

(43)
$$\dfrac{\Delta \Longrightarrow_w N \diamond \text{foc} \qquad \dfrac{\Gamma\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w A \qquad \Gamma\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w B}{\Gamma\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w A\&B}\, \&R}{\Gamma\langle \Delta \rangle \Longrightarrow_w A\&B \diamond \text{foc}}\, p\text{-}Cut_2 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta \Longrightarrow_w N \diamond \text{foc} \qquad \Gamma\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w A}{\Gamma\langle \Delta \rangle \Longrightarrow_w A \diamond \text{foc}}\, p\text{-}Cut_2 \qquad \dfrac{\Delta \Longrightarrow_w N \diamond \text{foc} \qquad \Gamma\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w B}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \diamond \text{foc}}\, p\text{-}Cut_2}{\Gamma\langle \Delta \rangle \Longrightarrow_w A\&B \diamond \text{foc}}\, \&R$$

- Left commutative *n-Cut* conversions:

(44)
$$\dfrac{\dfrac{\Delta\langle \overrightarrow{\boxed{Q}} \rangle \Longrightarrow_w P}{\Delta\langle \overrightarrow{Q} \rangle \Longrightarrow_w P}\, foc \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w C}{\Gamma\langle \Delta\langle \overrightarrow{Q} \rangle \rangle \Longrightarrow_w C}\, n\text{-}Cut_1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta\overrightarrow{\boxed{Q}} \Longrightarrow_w P \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w C}{\Gamma\langle \Delta\langle \overrightarrow{\boxed{Q}} \rangle \rangle \Longrightarrow_w C}\, n\text{-}Cut_1}{\Gamma\langle \Delta\langle \overrightarrow{Q} \rangle \rangle \Longrightarrow_w C}\, foc$$

(45)
$$\dfrac{\dfrac{\Delta\langle \overrightarrow{A} |_k \overrightarrow{B} \rangle \Longrightarrow_w P \diamond \text{foc}}{\Delta\langle \overrightarrow{A \odot_k B} \rangle \Longrightarrow_w P \diamond \text{foc}}\, \odot_k L \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w C}{\Gamma\langle \Delta\langle \overrightarrow{A \odot_k B} \rangle \rangle \Longrightarrow_w C \diamond \text{foc}}\, n\text{-}Cut_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\langle \overrightarrow{A} |_k \overrightarrow{B} \rangle \Longrightarrow_w P \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w C}{\Gamma\langle \Delta\langle \overrightarrow{A} |_k \overrightarrow{B} \rangle \rangle \Longrightarrow_w C \diamond \text{foc}}\, n\text{-}Cut_1}{\Gamma\langle \Delta\langle \overrightarrow{A \odot_k B} \rangle \rangle \Longrightarrow_w C \diamond \text{foc}}\, \odot_k L$$

(46)
$$\dfrac{\dfrac{\Delta\langle \overrightarrow{A} |_k \overrightarrow{B} \rangle \Longrightarrow_w N}{\Delta\langle \overrightarrow{A \odot_k B} \rangle \Longrightarrow_w N}\, \odot_k L \qquad \Gamma\langle \overrightarrow{N} \rangle \Longrightarrow_w C \diamond \text{foc}}{\Gamma\langle \Delta\langle \overrightarrow{A \odot_k B} \rangle \rangle \Longrightarrow_w C \diamond \text{foc}}\, n\text{-}Cut_2 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Gamma\langle\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,\odot_k L$$

(47)
$$\dfrac{\dfrac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \Gamma_2\langle\overrightarrow{\boxed{N_1}}\rangle\Longrightarrow_w P}{\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\Longrightarrow_w P}\,\uparrow_k L \qquad \Theta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\rangle\Longrightarrow_w C}\,n\text{-}Cut_1 \quad\rightsquigarrow$$

$$\dfrac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \dfrac{\Gamma_1\langle\overrightarrow{\boxed{N_1}}\rangle\Longrightarrow_w P \qquad \Theta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1}}\rangle\rangle\Longrightarrow_w C}\,n\text{-}Cut_1}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\rangle\Longrightarrow_w C}\,\uparrow_k L$$

(48)
$$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w P\diamond\text{foc} \qquad \Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w P\diamond\text{foc}}{\Gamma\langle\overrightarrow{A\oplus B}\rangle\Longrightarrow_w P\diamond\text{foc}}\,\oplus L \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_1 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w P\diamond\text{foc} \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\langle\overrightarrow{A}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_1 \qquad \dfrac{\Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w P\diamond\text{foc} \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\langle\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_1}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,\oplus L$$

(49)
$$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w N}{\Gamma\langle\overrightarrow{A\oplus B}\rangle\Longrightarrow_w N}\,\oplus L \qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w N \qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2 \qquad \dfrac{\Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w N \qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,\oplus L$$

- Right commutative *n-Cut* conversions:

(50)
$$\dfrac{\Delta\Longrightarrow_w N \qquad \dfrac{\Gamma\langle\overrightarrow{N};\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w C}{\Gamma\langle\overrightarrow{N};\overrightarrow{Q}\rangle\Longrightarrow_w C}\,foc}{\Gamma\langle\Delta;\overrightarrow{Q}\rangle\Longrightarrow_w C}\,n\text{-}Cut_2 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Delta\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N};\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta;\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w C}\,n\text{-}Cut_2}{\Gamma\langle\Delta;\overrightarrow{Q}\rangle\Longrightarrow_w C}\,foc$$

(51)
$$\dfrac{\Delta\Longrightarrow_w N \qquad \dfrac{\Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w\boxed{P}}{\Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w P}\,foc}{\Gamma\langle\Delta\rangle\Longrightarrow_w P}\,n\text{-}Cut_2 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Delta\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w\boxed{P}}{\Gamma\langle\Delta\rangle\Longrightarrow_w\boxed{P}}\,n\text{-}Cut_2}{\Gamma\langle\Delta\rangle\Longrightarrow_w P}\,foc$$

(52)
$$\dfrac{\Delta\Longrightarrow_w P\diamond \text{foc} \qquad \dfrac{\Gamma\langle\overrightarrow{P}\rangle|_k\overrightarrow{A}\Longrightarrow_w B}{\Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w B\uparrow_k A}\uparrow_k R}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\uparrow_k A\diamond \text{foc}}\; n\text{-}Cut_1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Delta\Longrightarrow_w P\diamond \text{foc} \qquad \Gamma\langle\overrightarrow{P}\rangle|_k\overrightarrow{A}\Longrightarrow_w B}{\Gamma\langle\Delta\rangle|_k\overrightarrow{A}\Longrightarrow_w B\diamond \text{foc}}\; n\text{-}Cut_1}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\uparrow_k A\diamond \text{foc}}\uparrow_k R$$

(53)
$$\dfrac{\Delta\Longrightarrow_w N \qquad \dfrac{\Gamma\langle\overrightarrow{P}\rangle|_k\overrightarrow{A}\Longrightarrow_w B\diamond \text{foc}}{\Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w B\uparrow_k A\diamond \text{foc}}\uparrow_k R}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\uparrow_k A\diamond \text{foc}}\; n\text{-}Cut_2 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Delta\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{P}\rangle|_k\overrightarrow{A}\Longrightarrow_w B\diamond \text{foc}}{\Gamma\langle\Delta\rangle|_k\overrightarrow{A}\Longrightarrow_w B\diamond \text{foc}}\; n\text{-}Cut_2}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\uparrow_k A\diamond \text{foc}}\uparrow_k R$$

(54)
$$\dfrac{\Delta\Longrightarrow_w P\diamond \text{foc} \qquad \dfrac{\Gamma\langle\overrightarrow{P};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C}{\Gamma\langle\overrightarrow{P};\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C}\odot_k L}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \text{foc}}\; n\text{-}Cut_1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Delta\Longrightarrow_w P\diamond \text{foc} \qquad \Gamma\langle\overrightarrow{P};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta;\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond \text{foc}}\; n\text{-}Cut_1}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \text{foc}}\odot_k L$$

(55)
$$\dfrac{\Delta\Longrightarrow_w N \qquad \dfrac{\Gamma\langle\overrightarrow{N};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond \text{foc}}{\Gamma\langle\overrightarrow{N};\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \text{foc}}\odot_k L}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \text{foc}}\; n\text{-}Cut_2 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Delta\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond \text{foc}}{\Gamma\langle\Delta;\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond \text{foc}}\; n\text{-}Cut_2}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \text{foc}}\odot_k L$$

(56)
$$\dfrac{\Delta\Longrightarrow_w N \qquad \dfrac{\Gamma\Longrightarrow_w \boxed{P_1} \qquad \Theta\langle\overrightarrow{P_2};\overrightarrow{N}\rangle\Longrightarrow_w C}{\Theta\langle\boxed{\overrightarrow{P_2\uparrow_k P_1}}|_k\Gamma;\overrightarrow{N}\rangle\Longrightarrow_w C}\uparrow_k L}{\Theta\langle\boxed{\overrightarrow{P_2\uparrow_k P_1}}|_k\Gamma;\Delta\rangle\Longrightarrow_w C}\; n\text{-}Cut_2 \quad\rightsquigarrow$$

$$\dfrac{\Gamma\Longrightarrow_w \boxed{P_1} \qquad \dfrac{\Delta\Longrightarrow_w N \qquad \Theta\langle\overrightarrow{P_2};\overrightarrow{N}\rangle\Longrightarrow_w C}{\Theta\langle\overrightarrow{P_2};\Delta\rangle\Longrightarrow_w C}\; n\text{-}Cut_2}{\Theta\langle\boxed{\overrightarrow{P_2\uparrow_k P_1}}|_k\Gamma;\Delta\rangle\Longrightarrow_w C}\uparrow_k L$$

(57)
$$\dfrac{\Delta\Longrightarrow_w P\diamond \text{foc} \qquad \dfrac{\Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w A \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w B}{\Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w A\&B}\&R}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\&B\diamond \text{foc}}\; n\text{-}Cut_1 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\Longrightarrow_w P \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w A}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\diamond \text{foc}}\; n\text{-}Cut_1 \qquad \dfrac{\Delta\Longrightarrow_w P\diamond \text{foc} \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w B}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond \text{foc}}\; n\text{-}Cut_1}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\&B\diamond \text{foc}}\&R$$

(58)
$$\dfrac{\Delta\Longrightarrow_w N \qquad \dfrac{\Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w A\diamond \text{foc} \qquad \Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w B\diamond \text{foc}}{\Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w A\&B\diamond \text{foc}}\&R}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\&B\diamond \text{foc}}\; n\text{-}Cut_2 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w A\diamond \text{foc}}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\diamond \text{foc}}\; n\text{-}Cut_2 \qquad \dfrac{\Delta\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w B\diamond \text{foc}}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond \text{foc}}\; n\text{-}Cut_2}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\&B\diamond \text{foc}}\&R$$

This completes the proof. □