Displacement Logic for Grammar

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Lecture 2

From Natural Deduction to Sequents: Hypersequents

On the Need of a Sequent Calculus for the Displacement Calculus **D**

- **D** like **L** is a substructural logic. We have presented a labelled natural deduction for **D**.
- But we want **D** to be the core logic of what we call *Displacement Logic*, which we denote **DL**.
- For example, we want to incorporate into **DL**, polymorphism (additive connectives from Linear Logic), and first-order quantifiers (not yet incorporated dependent types!).

On the Need of a Sequent Calculus for the Displacement Calculus **D**

- Furthermore, the connectives we have considered preserve the *linearity* status of **DL**. But we want more. We want to account for non-linear phenomena which are ubiquitous in natural language. The idea is to add to the linear component of **DL** a pair of exponential-like modalities which control the structural rules of *contraction* and *expansion* respectively.
- For all this new kind of logical machinery we need a sequent calculus. Why? a) Natural deduction is not well-suited to these new connectives, b) sequent calculus is symmetric (in its rules), c) a simpler result of normalization for **DL** (Haupsatz) is proved in sequent calculus, and d) a fast proof-search is easier (using focalised proof search).

The $\mathbf{NL_{Assc}}$-L metaphor

- Let **NL** be the non-associative Lambek calculus. We call its sequent calculus **NL** . Consider the substructural logic **NL** with the structural rule of associativity:
  $\mathbf{NL_{Assc}} =_{def} \mathbf{NL} + \mathbf{Associativity}$.
- A logical metaphor: $\mathbf{NL_{Assc}}$ vs **L**.

The $\mathbf{NL}_{\mathbf{Assc}}$-L metaphor

- The sequent calculus (Lambek 1958) for **L** is free of structural rules. What kind of data-structure is used for the antecedents of **L** sequents? (Possibly empty[1]) lists of types. But lists of items of a given set X have a name in universal algebra. Lists are the free monoid (up to isomorphism) generated by a set of generators X.
- The natural algebraic semantics for **L** is the class of residuated monoids.
- We can consider then a multimodal calculus with a set of structural postulates (or rules) which is *equivalent* to **L**. We already have it: $\mathbf{NL}_{\mathbf{Assc}}$. It is a story which is folklore, but it worth going into the details.

[1] Recall we are considering the Lambek calculus with empty antecedent.

The Metaphor in Details

Let $\mathcal{O}_{\mathbf{L}}$ be the set of configurations of **L** (we denote it $\mathcal{O}$). $\mathcal{O}$ is simply, as we know, a possibly non-empty list of types, where we denote the set of Lambek types as $\mathcal{F}$.

The sequent calculus of **L** is free of structural rules (the associativity is built-in).

The Metaphor in Details

Which kind of data-structure do we have for **NL**? Binary trees.
Consider the following set of structural terms:

$$\textbf{StructTerm} \quad ::= \quad \mathcal{F} \,|\, \mathbb{I} \,|\, (\textbf{StructTerm} \circ \textbf{StructTerm})$$

In fact, **StructTerm** is the free unital *groupoid* generated by $\mathcal{F}$ with a distinguished structural 0-ary term constructor (or constant) related to the product unit of **NL** (and $\textbf{NL}_{\textbf{Assc}}$).

The Metaphor in Details

Let us consider the *non-associative Lambek sequent calculus* **NL**:

$$\frac{}{A \to A, \text{ where } A \in Pr}\ Id \qquad \frac{T \to A \qquad S[A] \to B}{S[T] \to B}\ Cut$$

$$\frac{T \to A \qquad S[B] \to C}{S[(B/A \circ T)] \to C}\ /L \qquad \frac{(T \circ A) \Rightarrow B}{T \Rightarrow B/A}\ /R$$

$$\frac{T \to A \qquad S[B] \to C}{S[(T \circ A \backslash B)] \to C}\ \backslash L \qquad \frac{(A \circ T) \Rightarrow B}{T \Rightarrow A \backslash A}\ \backslash R$$

$$\frac{T[(A \circ B)] \to C}{T[(A \bullet B)] \to C}\ \bullet L \qquad \frac{T \to A \qquad S \to B}{(T \circ S) \to A \bullet B}\ \bullet R$$

NL continued

$$\frac{T[\mathbb{I}] \rightarrow A}{T[I] \rightarrow A}\ IL \qquad \frac{}{\mathbb{I} \rightarrow I}\ IR$$

$$\frac{T[S \circ \mathbb{I}] \rightarrow A}{T[S] \rightarrow A}\ Unit_1 \qquad \frac{T[\mathbb{I} \circ S] \rightarrow A}{T[S] \rightarrow A}\ Unit_2$$

$$\frac{T[S] \rightarrow A}{T[S \circ \mathbb{I}] \rightarrow A}\ Unit_3 \qquad \frac{T[S] \rightarrow A}{T[\mathbb{I} \circ S] \rightarrow A}\ Unit_4$$

NL_{Assc}

$\mathbf{NL_{Assc}}$

$$\mathbf{NL_{Assc}} \quad \triangleq \quad \mathbf{NL} + \mathbf{Associativity}$$

$$\frac{T[(S\circ(K\circ L))]\to A}{T[((S\circ K)\circ L)]\to A}\,Assc_1 \qquad \frac{T[(S\circ K)\circ L)]\to A}{T[(S\circ(K\circ L))]\to A}\,Assc_2$$

The variety (equational class) of monoids

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$$
\begin{aligned}
(x+y)+z &\approx x+(y+z)\\
x+(y+z) &\approx (x+y)+z\\
x+0 &\approx x\\
&\approx 0+x
\end{aligned}
$$

As we have seen, the data-structure for **L**-sequents $\mathcal{O}$ is the free monoid generated by the set of types $\mathcal{F}_L$, i.e., lists of Lambek types. In any variety $\mathbb{V}$ we can build the relatively free algebra $FA_{\mathbb{V}}(X)$ w.r.t. a set of generators.

Faithful embedding between $\mathbf{NL}_{\mathbf{Assc}}$ and $\mathbf{L}$

Faithful embedding between $\mathbf{NL_{Assc}}$ and $\mathbf{L}$

We consider the following embedding translation from $\mathbf{NL_{Assc}}$ to $\mathbf{L}$:

$$\begin{array}{rclcl} (\cdot)^\sharp : \mathbf{NL_{Assc}} = (\mathcal{F}, \mathbf{StructTerm}, \rightarrow) & \longrightarrow & \mathbf{L} = (\mathcal{F}, \mathcal{O}_L, \Rightarrow) \\ T \rightarrow A & \mapsto & (T)^\sharp \Rightarrow (A)^\sharp \end{array}$$

$(\cdot)^\sharp$ is such that:

$$\begin{array}{l} A^\sharp = A \text{ if } A \text{ is a type} \\ (T_1 \circ T_2)^\sharp = T_1^\sharp, T_2^\sharp \\ \mathbb{I}^\sharp = \Lambda \end{array}$$

$(\cdot)^\sharp$ satisfies:

$$(T[S])^\sharp = T^\sharp(S^\sharp)$$

On $(\cdot)^\sharp$

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$(\cdot)^\sharp$ is faithful, i.e.:

- If $T \rightarrow A$ then $T^\sharp \Rightarrow A$.
- Conversely, for any T_Δ such that $(T_\Delta)^\sharp = \Delta$ and $\mathbf{L} \vdash \Delta \Rightarrow A$, then $\mathbf{NL_{Assc}} \vdash T_\Delta \rightarrow A$.
- $(\cdot)^\sharp$ absorbs the structural rules. If $T \in$ **StructTerm** and $T \leftrightarrow^* S$, then:

  $$T^\sharp = S^\sharp$$

  Where $\leftrightarrow^*$ is the reflexive, symmetric and transitive closure of $\leftrightarrow$, compatible with operations, and closed by substitutions. $\leftrightarrow$ is the result of applying a single structural rule to a (structural) term, and here the structural rules are the rules of associativity.

Summary of the metaphor

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Slogan:

- **L** is free of structural rules.
- In fact, **L** absorbs the structural rules of $\mathbf{NL}_{\mathbf{Assc}}$, which correspond to the equations defining the variety of monoids.
- The data-structure corresponding to the antecedent of an **L**-sequent is precisely isomorphic to the free algebra of monoids, i.e. to lists of types.
- We have a faithful embedding translation betwee $\mathbf{NL}_{\mathbf{Assc}}$ and **L**.

(Hyper)sequents

Configurations $\mathcal{O}$ are defined by the following, where Λ is the empty string, and the metalinguistic separator 1 marks holes:

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$$\begin{array}{lcl} \mathcal{O} & ::= & \Lambda \mid \mathcal{T}, \mathcal{O} \\ \mathcal{T} & ::= & 1 \mid \mathcal{F}_0 \mid \mathcal{F}_{i>0}\{\underbrace{\mathcal{O} : \ldots : \mathcal{O}}_{i\ \mathcal{O}\text{'s}}\} \end{array}$$

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Sequents Σ are defined by:

$$\mathcal{O} \Rightarrow \mathcal{F} \text{ such that } s\mathcal{O} = s\mathcal{F}$$

Sequent calculus

Sequent calculus

The figure $\overrightarrow{A}$ of a type A is defined by:

Sequent calculus

The figure $\overrightarrow{A}$ of a type A is defined by:

$$\overrightarrow{A} = \begin{cases} A & \text{if } sA = 0 \\ A\{\underbrace{1 : \ldots : 1}_{sA\ 1\text{'s}}\} & \text{if } sA > 0 \end{cases}$$

Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Gamma \otimes \langle \Delta_1, \ldots, \Delta_i \rangle$ is the result of replacing the successive holes in Γ by $\Delta_1, \ldots, \Delta_i$ respectively.

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Intuitively, a prosodic inhabitant of the fold $\Gamma \otimes \langle \Delta_1, \ldots, \Delta_i \rangle$ is: $\gamma_0 + \delta_1 + \gamma_1 + \cdots + \gamma_{i-1} + \delta_i + \gamma_i$, where $\gamma_0 + 1 + \gamma_1 + \cdots + \gamma_{i-1} + 1 + \gamma_i$ is a prosodic inhabitant of Γ and δ_k is a prosodic inhabitant of Δ_k for $1 \leq k \leq i$.

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Where Γ is of sort i, the notation $\Delta\langle \Gamma \rangle$ abbreviates $\Delta_0(\Gamma \otimes \langle \Delta_1, \dots, \Delta_i \rangle)$, i.e. a context configuration Δ (which is externally Δ_0 and internally $\Delta_1, \dots, \Delta_i$) with a potentially discontinuous distinguished subconfiguration Γ.

Continuous logical rules

$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{C/B},\Gamma\rangle \Rightarrow D}/L \qquad \frac{\Gamma,\overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C/B}/R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\Gamma,\overrightarrow{A\backslash C}\rangle \Rightarrow D}\backslash L \qquad \frac{\overrightarrow{A},\Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\backslash R$$

$$\frac{\Delta\langle\overrightarrow{A},\overrightarrow{B}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{A\bullet B}\rangle \Rightarrow D}\bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1,\Gamma_2 \Rightarrow A\bullet B}\bullet R$$

$$\frac{\Delta\langle\Lambda\rangle \Rightarrow A}{\Delta\langle\overrightarrow{I}\rangle \Rightarrow A}IL \qquad \frac{}{\Lambda \Rightarrow I}IR$$

Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth metalinguistic wrap $\Delta\,|_k\,\Gamma$, $1 \leq k \leq i$, is given by:

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i.e. $\Delta\,|_k\,\Gamma$ is the configuration resulting from replacing by Γ the kth hole in Δ.

Discontinuous logical rules

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$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle \vec{C} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{C{\uparrow_k}B} \,|_k\, \Gamma \rangle \Rightarrow D}\ {\uparrow_k}L \qquad\qquad \frac{\Gamma \,|_k\, \vec{B} \Rightarrow C}{\Gamma \Rightarrow C{\uparrow_k}B}\ {\uparrow_k}R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle \vec{C} \rangle \Rightarrow D}{\Delta\langle \Gamma \,|_k\, \overrightarrow{A{\downarrow_k}C} \rangle \Rightarrow D}\ {\downarrow_k}L \qquad\qquad \frac{\vec{A} \,|_k\, \Gamma \Rightarrow C}{\Gamma \Rightarrow A{\downarrow_k}C}\ {\downarrow_k}R$$

$$\frac{\Delta\langle \vec{A} \,|_k\, \vec{B} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{A{\odot_k}B} \rangle \Rightarrow D}\ {\odot_k}L \qquad\qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1 \,|_k\, \Gamma_2 \Rightarrow A{\odot_k}B}\ {\odot_k}R$$

$$\frac{\Delta\langle 1 \rangle \Rightarrow A}{\Delta\langle \vec{J} \rangle \Rightarrow A}\ JL \qquad\qquad \frac{}{1 \Rightarrow J}\ JR$$

Sequent Calculus: Hypersequents

From the metaphor $\mathbf{NL_{Assc}/L}$ to ?/$\mathbf{hD}$

From the metaphor $\mathbf{NL_{Assc}}$/**L** to ?/**hD**

- **hD** is free of structural rules.
- Does **hD** absorb the structural rules of a (ω-sorted) multimodal calculus?
- YES!
- These absorbed structural rules correspond to the sorted equational theory of a certain ω-sorted variety.
- Finally, the data-structure $\mathcal{O}_{\mathbf{D}}$, as we will see, corresponds to the relatively free algebra of this sorted variety.

The (Sorted) Variety of displacements algebras $\mathbb{DA}$

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Continuous associativity

$$x + (y + z) \approx (x + y) + z$$

Discontinuous associativity

$$x \times_i (y \times_j z) \approx (x \times_i y) \times_{i+j-1} z$$
$$(x \times_i y) \times_j z \approx x \times_i (y \times_{j-i+1} z) \text{ if } i \leq j \leq 1 + s(y) - 1$$

Mixed permutation

$$(x \times_i y) \times_j z \approx (x \times_{j-S(y)+1} z) \times_i y \text{ if } j > i + s(y) - 1$$
$$(x \times_i z) \times_j y \approx (x \times_j y) \times_{i+S(y)-1} z \text{ if } j < i$$

Mixed associativity

$$(x + y) \times_i z \approx (x \times_i z) + y \text{ if } 1 \leq i \leq s(x)$$
$$(x + y) \times_i z \approx x + (y \times_{i-s(x)} z) \text{ if } x + 1 \leq i \leq s(x) + s(y)$$

Continuous unit and discontinuous unit

$$0 + x \approx x \approx x + 0 \text{ and } 1 \times_1 x \approx x \approx x \times_i 1$$

The (Sorted) Variety of displacements algebras $\mathbb{DA}$

The (Sorted) Variety of displacements algebras $\mathbb{DA}$

- The class of standard displacement algebras (DAs) is properly contained in $\mathcal{DA}$.
- The set of hyperconfigurations $\mathcal{O}_{\mathbf{D}}$ is the relatively free DA of the variety $\mathbb{DA}$ with the set of ω-sorted generators $\mathcal{F}_D$. I.e.:

(2) **Theorem** (*Freeness of $\mathcal{O}_{\mathbf{D}}$*)

$$FA_{\mathbb{DA}}(\mathcal{F}) \quad = \quad \mathcal{O}_{\mathbf{D}}$$

Proof. Via the equivalence theorem (see Valentín 2012). □

The ω-sorted multimodal displacement calculus **mD**

$$
\begin{array}{lll}
\textbf{StructTerm} & ::= & \mathcal{F} \,|\, \mathbb{I} \,|\, (\textbf{StructTerm} \circ \textbf{StructTerm}) \,| \\
 & ::= & \mathbb{J} \,|\, (\textbf{StructTerm} \circ_i \textbf{StructTerm})
\end{array}
$$

StructTerm is ω-sorted, i.e. **StructTerm** $= \bigcup_{i \in \omega}$ **StructTerm**$_i$.

The ω-sorted multimodal displacement calculus **mD**

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The logical rules:

$$A \to A \ \text{Id} \qquad \frac{S \to A \qquad T[A] \to B}{T[S] \to B}\ Cut$$

$$\frac{T[\mathbb{I}] \to A}{T[I] \to A}\ IL \qquad \frac{}{\mathbb{I} \Rightarrow I}\ IR$$

$$\frac{T[\mathbb{J}] \to A}{T[J] \to A}\ JL \qquad \frac{}{\mathbb{J} \Rightarrow J}\ JR$$

$$\frac{X \to A \qquad Y[B] \to C}{Y[X \circ A\backslash B] \to C}\ \backslash L \qquad \frac{A \circ X \to B}{X \to A\backslash B}\ \backslash R$$

$$\frac{X \to A \qquad Y[B] \to C}{Y[B/A \circ X] \to C}\ /L \qquad \frac{X \circ A \to B}{X \to B/A}\ /R$$

$$\frac{X \to A \qquad Y[B] \to C}{Y[B \uparrow_i A \circ_i X] \to C}\ \uparrow_i L \qquad \frac{X \circ_i A \to B}{X \to B \uparrow_i A}\ \uparrow_i R$$

mD continued

mD continued

More logical rules:

$$\frac{X \to A \qquad Y[B] \to C}{Y[X \circ_i A \downarrow_i B] \to C} \downarrow_i L \qquad \frac{A \circ_i X \to B}{X \to A \downarrow_i B} \downarrow_i R$$

$$\frac{X[A \circ B] \to C}{X[A \bullet B] \to C} \bullet L \qquad \frac{X \to A \qquad Y \to B}{X \circ Y \to A \bullet B} \bullet R$$

$$\frac{X[A \circ_i B] \to C}{X[A \odot_i B] \to C} \odot_i L \qquad \frac{X \to A \qquad Y \to B}{X \circ_i Y \to A \odot_i B} \odot_i R$$

mD continued

mD continued

Some useful stuff on terms:

(5) **Definition** (*Wrapping and Permutable Terms*)

Given the structural term term $(T_1 \circ_i T_2) \circ_j T_3$, we say that:

(P1) $T_2 \prec_{T_1} T_3$ iff $i + t_2 - 1 < j$.
(P2) $T_3 \prec_{T_1} T_2$ iff $j < i$.
(O) $T_2 \between_{T_1} T_3$ iff $i \leq j \leq i + t_2 - 1$.

Intuitively, (P1) holds when in the approximate algebraic interpretation is such that:

(6)

$$
\begin{array}{lcll}
[[(T_1 \circ_i T_2) \circ_j T_3]] & = & \cdots + [[T_2]] + \cdots + [[T_3]] + \cdots & (P1) \\
[[(T_1 \circ_i T_2) \circ_j T_3]] & = & \cdots + [[T_3]] + \cdots + [[T_2]] + \cdots & (P2) \\
[[(T_1 \circ_i T_2) \circ_j T_3]] & = & \cdots + \beta_0 + \cdots + \beta_k + [[T_3]] + \beta_{k+1+} \cdots & (O)
\end{array}
$$

Where $\beta_0 + 1 + \cdots + \beta_k + 1 + \beta_{k+1} + \cdots + \beta_{s(T_2)} \in [[T_2]]$

mD continued

mD continued

The structural rules:

Continuous unit:

$$\frac{T[X]\rightarrow A}{T[\mathbb{I}\circ X]\rightarrow A} \qquad \frac{T[\mathbb{I}\circ X]\rightarrow A}{T[X]\rightarrow A} \qquad \frac{T[X]\rightarrow A}{T[X\circ\mathbb{I}]\rightarrow A} \qquad \frac{T[X\circ\mathbb{I}]\rightarrow A}{T[X]\rightarrow A}$$

Discontinuous unit:

$$\frac{T[X]\rightarrow A}{T[\mathbb{J}\circ_1 X]\rightarrow A} \qquad \frac{T[\mathbb{J}\circ_1 X]\rightarrow A}{T[X]\rightarrow A} \qquad \frac{T[X]\rightarrow A}{T[X\circ_i\mathbb{J}]\rightarrow A} \qquad \frac{T[X\circ_i\mathbb{J}]\rightarrow A}{T[X]\rightarrow A}$$

mD continued

mD continued

More structural rules:

Continuous associativity

$$\frac{X[(T_1 \circ T_2) \circ T_3] \to D}{X[T_1 \circ (T_2 \circ T_3)] \to D}\ \mathit{Assc}_c \qquad \frac{X[T_1 \circ (T_2 \circ T_3)] \to D}{X[(T_1 \circ T_2) \circ T_3] \to D}\ \mathit{Assc}_c$$

Discontinuous associativity $T_2 \between_{T_1} T_3$

$$\frac{S[T_1 \circ_i (T_2 \circ_j T_3)] \to C}{S[(T_1 \circ_i T_2) \circ_{i+j-1} T_3)] \to C}\ \mathbf{Assc_d 1} \qquad \frac{S[(T_1 \circ_i T_2) \circ_j T_3] \to C}{S[T_1 \circ_i (T_2 \circ_{j-i+1} T_3)] \to C}\ \mathbf{Assc_d 2}$$

Mixed permutation 1 case $T_2 \prec_{T_1} T_3$

$$\frac{S[(T_1 \circ_i T_2) \circ_j T_3] \to C}{S[(T_1 \circ_{j-S(T_2)+1} T_3) \circ_i T_2] \to C}\ \mathbf{MixPerm1} \qquad \frac{S[(T_1 \circ_i T_3) \circ_j T_2] \to C}{S[(T_1 \circ_j T_2) \circ_{i+S(T_2)-1} T_3] \to C}\ \mathbf{MixPerm1}$$

mD continued

mD continued

More structural rules:

Mixed permutation 2 case $T_3 \prec_{T_1} T_2$

$$\frac{S[(T_1 \circ_i T_2) \circ_j T_3] \to C}{S[(T_1 \circ_j T_3) \circ_{i+S(T_3)-1} T_2] \to C}\ \textbf{MixPerm2} \qquad \frac{S[(T_1 \circ_i T_3) \circ_j T_2] \to C}{S[(T_1 \circ_{j-S(T_3)+1} T_2) \circ_i T_3] \to C}\ \textbf{MixPerm2}$$

Mixed associativity I

$$\frac{R[(T \circ S) \circ_i K] \to A}{R[(T \circ_i K) \circ S] \to A}$$

Mixed associativity II

$$\frac{R[(T \circ S) \circ_i K] \to A}{R[(T \circ (S \circ_{i-s(T)} K] \to A}$$

mD vs hD

mD vs hD

Let us define the following map between sequent calculi: We consider the following embedding translation from **mD** to **hD**: We consider the following embedding translation from **mD** to **hD**:

$$\begin{array}{rcl} (\cdot)^\sharp : \mathbf{mD} = (\mathcal{F}, \mathbf{StructTerm}, \rightarrow) & \longrightarrow & \mathbf{hD} = (\mathcal{F}, \mathcal{O}, \Rightarrow) \\ T \rightarrow A & \mapsto & (T)^\sharp \Rightarrow (A)^\sharp \end{array}$$

$(\cdot)^\sharp$ is such that:

$$\begin{array}{l} A^\sharp = \overrightarrow{A} \text{ if } A \text{ is of sort strictly greater than 0} \\ A^\sharp = A \text{ if A is of sort 0} \\ (T_1 \circ T_2)^\sharp = T_1^\sharp, T_2^\sharp \\ (T_1 \circ_i T_2)^\sharp = T_1^\sharp |_i T_2^\sharp \\ \mathbb{I}^\sharp = \Lambda \\ \mathbb{J}^\sharp = 1 \end{array}$$

On the morphism $(\cdot)^\sharp$

(7) **Lemma** ($(\cdot)^\sharp$ *is an Epimorphism*)

For every $\Delta \in \mathcal{O}$ there exists a structural term[2] T_Δ such that:

$$(T_\Delta)^\sharp = \Delta$$

Proof. This can be proved by induction on the structure of hyperconfigurations. We define recursively T_Δ such that $(T_\Delta)^\sharp = \Delta$:

- Case $\Delta = \Lambda$ (the empty tree): $T_\Delta = \mathbb{I}$.
- Case where $\Delta = A, \Gamma$: $T_\Delta = A \circ T_\Gamma$, where by induction hypothesis (i.h.) $(T_\Gamma)^\sharp = \Gamma$.
- Case where $\Delta = 1, \Gamma$: $T_\Delta = \mathbb{J} \circ T_\Gamma$, where by i.h. $(T_\Gamma)^\sharp = \Gamma$.
- Case $\Delta = \overrightarrow{A} \otimes \langle \Delta_1, \cdots, \Delta_a \rangle, \Delta_{a+1}$. By i.h. we have:

$$(T_{\Delta_i})^\sharp = \Delta_i \text{ for } 1 \leq i \leq a+1$$

$$T_\Delta = (A \circ_1 T_{\Delta_1}) \circ T_{\Delta_2} \text{ if } a = 1$$
$$T_\Delta = ((\cdots((A \circ_1 T_{\Delta_1}) \circ_{1+d_1} T_{\Delta_2}) \cdots) \circ_{1+d_1+\cdots+d_{a-1}} T_{\Delta_a}) \circ T_{\Delta_{a+1}} \text{ if } a > 1$$

□

[2] In fact there exists an infinite set of such structural terms.

mD vs hD

(8) **Theorem** (*Faithfulness of* $(\cdot)^\sharp$ *Embedding Translation*)

Let A, X and Δ be respectively a type, a structural term and a hyperconfiguration. The following statements hold:

i) If $\vdash_{\mathbf{mD}} X \to A$ then $\vdash_{\mathbf{hD}} (X)^\sharp \Rightarrow A$
ii) For any X such that $(X)^\sharp = \Delta$, if $\vdash_{\mathbf{hD}} \Delta \Rightarrow A$ then $\vdash_{\mathbf{mD}} X \to A$

hD absorbs the structural rules

hD absorbs the structural rules

Again, as before with $\mathbf{NL_{Assc}}/\mathbf{L}$, the embedding translation mapping satisfies:

$$(R[T])^\sharp = R^\sharp \langle T^\sharp \rangle$$

Since $\mathcal{O}$ is the free algebra of DAs over $\mathcal{F}$, $(\cdot)^\sharp$ absorbs the structural rules of **mD**. I.e., if $T \leftrightarrow^* S$ (where $\leftrightarrow^*$ is defined as before, but w.r.t. to **D**), then $(R[T])^\sharp = (R[S])^\sharp$.

The Categorical Calculus of Displacement Calculus

(9)

$A \to A$ Axiom

$A{\bullet}B \to C$ iff $A \to C/B$ iff $B \to A\backslash C$ $\quad$ Res_{cont}

$A \odot_i B \to C$ iff $A \to C{\uparrow_i}B$ iff $B \to A{\downarrow_i}C$ $\quad$ Res_{disc}

$A \bullet I \leftrightarrow A \leftrightarrow I \bullet A \quad A \odot_i J \leftrightarrow A \leftrightarrow J \odot_1 A$

$(A \bullet B) \bullet C \leftrightarrow A \bullet (B \bullet C)$ $\quad$ Continuous associativity

$A \odot_i (B \odot_j C) \leftrightarrow (A \odot_i B) \odot_{i+j-1} C$ $\quad$ Discontinuous associativity

$(A \odot_i B) \odot_j C \leftrightarrow A \odot_i (B \odot_{j-i+1} C)$, if $i \leq j \leq 1 + s(B) - 1$

$(A \odot_i B) \odot_j C \leftrightarrow (A \odot_{j-s(B)+1} C) \odot_i B$, if $j > i + s(B) - 1$ Mixed permutation

$(A \odot_i C) \odot_j B \leftrightarrow (A \odot_j B) \odot_{i+s(B)-1} C$, if $j < i$

$(A \bullet B) \odot_i C \leftrightarrow (A \odot_i C) \bullet B$, if $1 \leq i \leq S(A)$ Mixed associativity

$(A \bullet B) \odot_i C \leftrightarrow A \bullet (B \odot_{i-s(C)} C)$, if $s(A) + 1 \leq i \leq s(A) + s(B)$

From $A \to B$ and $B \to C$ we have $A \to C$ $\quad$ Transitivity

The Categorical Calculus of Displacement Calculus

From **mD** we have its equivalent categorical calculus **cD**. **cD** is the logic of pure residuation for the continuous and discontinuous multiplicatives with the set of postulates corresponding to the axioms of the variety of DAs.

From **cD** we directly build the corresponding Lindenbaum-Tarski algebra, which is the very first step to the study of algebraic semantics of **D**.