Mathematical Logic and Linguistics
Slides 4

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On tree-based hypersequent syntax

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- A logical metaphor: $\mathbf{NL_{Assc}}$ vs **L**.
- Absorbing structural rules in the Lambek calculus **L**.

The $\mathbf{NL}_{\mathbf{Assc}}$-$\mathbf{L}$ metaphor

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Consider the following set of structural terms:

$$\textbf{StructTerm} \quad ::= \quad \mathcal{F} \,|\, \mathbb{I} \,|\, (\textbf{StructTerm} \circ \textbf{StructTerm})$$

In fact, **StructTerm** is a free groupoid generated by $\mathcal{F}$ with a distinguished structural constant.

The $\mathbf{NL_{Assc}}$-$\mathbf{L}$ metaphor

The $\mathbf{NL_{Assc}}$-L metaphor

Let us consider the *non-associative Lambek calculus* **NL**:

$$\frac{}{A \rightarrow A, \text{ where } A \in Pr}\ Id \qquad \frac{T \rightarrow A \qquad S[A] \rightarrow B}{S[T] \rightarrow B}\ Cut$$

$$\frac{T \rightarrow A \qquad S[B] \rightarrow C}{S[(B/A \circ T)] \rightarrow C}\ /L \qquad \frac{(T \circ A) \Rightarrow B}{T \Rightarrow B/A}\ /R$$

$$\frac{T \rightarrow A \qquad S[B] \rightarrow C}{S[(T \circ A \backslash B)] \rightarrow C}\ \backslash L \qquad \frac{(A \circ T) \Rightarrow B}{T \Rightarrow A \backslash A}\ \backslash R$$

$$\frac{T[(A \circ B)] \rightarrow C}{T[(A \bullet B)] \rightarrow C}\ \bullet L \qquad \frac{T \rightarrow A \qquad S \rightarrow B}{(T \circ S) \rightarrow A \bullet B}\ \bullet R$$

NL continued

$$\frac{T[\mathbb{I}] \rightarrow A}{T[I] \rightarrow A}\ IL \qquad \frac{}{\mathbb{I} \rightarrow I}\ IR$$

$$\frac{T[S \circ \mathbb{I}] \rightarrow A}{T[S] \rightarrow A}\ Unit_1 \qquad \frac{T[\mathbb{I} \circ S] \rightarrow A}{T[S] \rightarrow A}\ Unit_2$$

$$\frac{T[S] \rightarrow A}{T[S \circ \mathbb{I}] \rightarrow A}\ Unit_3 \qquad \frac{T[S] \rightarrow A}{T[\mathbb{I} \circ S] \rightarrow A}\ Unit_4$$

NL_{Assc}

$\mathbf{NL_{Assc}}$

$$\mathbf{NL_{Assc}} \triangleq \mathbf{NL} + \mathbf{Associativity}$$

$$\frac{T[(S \circ (K \circ L))] \to A}{T[((S \circ K) \circ L)] \to A}\, Assc_1 \qquad \frac{T[((S \circ K) \circ L)] \to A}{T[(S \circ (K \circ L))] \to A}\, Assc_2$$

The equational class of monoids

The equational class of monoids

$$\begin{aligned} (x+y)+z &\approx x+(y+z) \\ x+(y+z) &\approx (x+y)+z \\ x+0 &\approx x \\ &\approx 0+x \end{aligned}$$

The set of Lambek configurations $\mathcal{O}_L$ is the free monoid generated by the set of types $\mathcal{F}_L$.

Faithful embedding between $\mathbf{NL}_{\mathbf{Assc}}$ and $\mathbf{L}$

Faithful embedding between $\mathbf{NL}_{\mathbf{Assc}}$ and **L**

We consider the following embedding translation from $\mathbf{NL}_{\mathbf{Assc}}$ to **L**:

$$\begin{array}{rcl} (\cdot)^{\sharp} : \mathbf{NL}_{\mathbf{Assc}} = (\mathcal{F}, \mathbf{StructTerm}, \rightarrow) & \longrightarrow & \mathbf{L} = (\mathcal{F}, \mathcal{O}_L, \Rightarrow) \\ T \rightarrow A & \mapsto & (T)^{\sharp} \Rightarrow (A)^{\sharp} \end{array}$$

$(\cdot)^{\sharp}$ is such that:

$$A^{\sharp} = A \text{ if } A \text{ is a type}$$
$$(T_1 \circ T_2)^{\sharp} = T_1^{\sharp}, T_2^{\sharp}$$
$$\mathbb{I}^{\sharp} = \Lambda$$

$(\cdot)^{\sharp}$ satisfies:

$$(T[S])^{\sharp} = T^{\sharp}(S^{\sharp})$$

On $(\cdot)^\sharp$

On $(\cdot)^\sharp$

$(\cdot)^\sharp$ is faithful, i.e.:

- If $T \rightarrow A$ then $T^\sharp \Rightarrow A$.
- Conversely, for any T_Δ such that $(T_\Delta)^\sharp = \Delta$ and $\Delta \Rightarrow A$, then $T_\Delta \rightarrow A$.
- $(\cdot)^\sharp$ absorbs the structural rules. If $T \in$ **StructTerm** and $T \leftrightarrow^* S$, then:

  $$T^\sharp = S^\sharp$$

  Where $\leftrightarrow^*$ is the reflexive, symmetric and transitive closure of $\leftrightarrow$, where $\leftrightarrow$ is the result applying a single structural rule to a (structural) term.

Summary of the metaphor

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Slogan:

- **L** is free of structural rules.
- In fact, **L** absorbs the structural rules of $\mathbf{NL}_{\mathbf{Assc}}$, which correspond to the equations defining the class of monoids.
- The set of $\mathcal{F}_L$ is the free monoid generated by the set of Lambek types.

From the metaphor $\mathbf{NL_{Assc}/L}$ to ?/**hD**

From the metaphor $\mathbf{NL}_{\mathbf{Assc}}$/**L** to ?/**hD**

- **hD** is free of structural rules.
- Does **hD** absorb the structural rules of a (ω-sorted) multimodal calculus?
- YES!
- This absorbed structural rules correspond to sorted equations of a certain ω-sorted equational class.

The equational class of displacements algebras $\mathcal{DA}$

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Continuous associativity

$$x + (y + z) \approx (x + y) + z$$

Discontinuous associativity

$$x \times_i (y \times_j z) \approx (x \times_i y) \times_{i+j-1} z$$
$$(x \times_i y) \times_j z \approx x \times_i (y \times_{j-i+1} z) \text{ if } i \leq j \leq 1 + s(y) - 1$$

Mixed permutation

$$(x \times_i y) \times_j z \approx (x \times_{j-S(y)+1} z) \times_i y \text{ if } j > i + s(y) - 1$$
$$(x \times_i z) \times_j y \approx (x \times_j y) \times_{i+S(y)-1} z \text{ if } j < i$$

Mixed associativity

$$(x + y) \times_i z \approx (x \times_i z) + y \text{ if } 1 \leq i \leq s(x)$$
$$(x + y) \times_i z \approx x + (y \times_{i-s(x)} z) \text{ if } x + 1 \leq i \leq s(x) + s(y)$$

Continuous unit and discontinuous unit

$$0 + x \approx x \approx x + 0 \text{ and } 1 \times_1 x \approx x \approx x \times_i 1$$

The equational class of displacements algebras $\mathcal{DA}$

The equational class of displacements algebras $\mathcal{DA}$

- The class of standard displacement algebras (DAs) is properly contained in $\mathcal{DA}$.
- The set of hyperconfigurations $\mathcal{O}_D$ is the free DA algebra with the set of ω-sorted generators $\mathcal{F}_D$. I.e.:

  (2) **Theorem** (*Freeness of $\mathcal{O}_D$*)

$$FDA(\mathcal{F}_D) \quad = \quad \mathcal{O}_D$$

$$\begin{array}{lll} \textbf{StructTerm} & ::= & \mathcal{F} \,|\, \mathbb{I} \,|\, (\textbf{StructTerm} \circ \textbf{StructTerm}) \,| \\ & ::= & \mathbb{J} \,|\, (\textbf{StructTerm} \circ_i \textbf{StructTerm}) \end{array}$$

StructTerm is ω-sorted, i.e. $\textbf{StructTerm} = \bigcup_{i \in \omega} \textbf{StructTerm}_i$.

The ω-sorted multimodal displacement calculus **mD**

The ω-sorted multimodal displacement calculus **mD**

The logical rules:

$$A \to A \ \text{Id} \qquad \frac{S \to A \qquad T[A] \to B}{T[S] \to B}\ Cut$$

$$\frac{T[\mathbb{I}] \to A}{T[I] \to A}\ IL \qquad \frac{}{\mathbb{I} \Rightarrow I}\ IR$$

$$\frac{T[\mathbb{J}] \to A}{T[J] \to A}\ JL \qquad \frac{}{\mathbb{J} \Rightarrow J}\ JR$$

$$\frac{X \to A \qquad Y[B] \to C}{Y[X \circ A \backslash B] \to C}\ \backslash L \qquad \frac{A \circ X \to B}{X \to A \backslash B}\ \backslash R$$

$$\frac{X \to A \qquad Y[B] \to C}{Y[B/A \circ X] \to C}\ /L \qquad \frac{X \circ A \to B}{X \to B/A}\ /R$$

$$\frac{X \to A \qquad Y[B] \to C}{Y[B \uparrow_i A \circ_i X] \to C}\ \uparrow_i L \qquad \frac{X \circ_i A \to B}{X \to B \uparrow_i A}\ \uparrow_i R$$

mD continued

mD continued

More logical rules:

$$\frac{X \rightarrow A \qquad Y[B] \rightarrow C}{Y[X \circ_i A \downarrow_i B] \rightarrow C} \downarrow_i L \qquad \frac{A \circ_i X \rightarrow B}{X \rightarrow A \downarrow_i B} \downarrow_i R$$

$$\frac{X[A \circ B] \rightarrow C}{X[A \bullet B] \rightarrow C} \bullet L \qquad \frac{X \rightarrow A \qquad Y \rightarrow B}{X \circ Y \rightarrow A \bullet B} \bullet R$$

$$\frac{X[A \circ_i B] \rightarrow C}{X[A \odot_i B] \rightarrow C} \odot_i L \qquad \frac{X \rightarrow A \qquad Y \rightarrow B}{X \circ_i Y \rightarrow A \odot_i B} \odot_i R$$

mD continued

mD continued

Some useful stuff on terms:

(4) **Definition** (*Wrapping and Permutable Terms*)

Given the term $(T_1 \circ_i T_2) \circ_j T_3$, we say that:

(P1) $T_2 \prec_{T_1} T_3$ iff $i + t_2 - 1 < j$.
(P2) $T_3 \prec_{T_1} T_2$ iff $j < i$.
(O) $T_2 \between_{T_1} T_3$ iff $i \leq j \leq i + t_2 - 1$.

mD continued

mD continued

The structural rules:

Continuous unit:

$$\frac{T[X]\to A}{T[\mathbb{I}\circ X]\to A} \qquad \frac{T[\mathbb{I}\circ X]\to A}{T[X]\to A} \qquad \frac{T[X]\to A}{T[X\circ\mathbb{I}]\to A} \qquad \frac{T[X\circ\mathbb{I}]\to A}{T[X]\to A}$$

Discontinuous unit:

$$\frac{T[X]\to A}{T[\mathbb{J}\circ_1 X]\to A} \qquad \frac{T[\mathbb{J}\circ_1 X]\to A}{T[X]\to A} \qquad \frac{T[X]\to A}{T[X\circ_i\mathbb{J}]\to A} \qquad \frac{T[X\circ_i\mathbb{J}]\to A}{T[X]\to A}$$

mD continued

mD continued

More structural rules:

Continuous associativity

$$\frac{X[(T_1 \circ T_2) \circ T_3] \rightarrow D}{X[T_1 \circ (T_2 \circ T_3)] \rightarrow D}\, Assc_c \qquad \frac{X[T_1 \circ (T_2 \circ T_3)] \rightarrow D}{X[(T_1 \circ T_2) \circ T_3] \rightarrow D}\, Assc_c$$

Discontinuous associativity $T_2 \between_{T_1} T_3$

$$\frac{S[T_1 \circ_i (T_2 \circ_j T_3)] \rightarrow C}{S[(T_1 \circ_i T_2) \circ_{i+j-1} T_3)] \rightarrow C}\, \mathbf{Assc_d 1} \qquad \frac{S[(T_1 \circ_i T_2) \circ_j T_3] \rightarrow C}{S[T_1 \circ_i (T_2 \circ_{j-i+1} T_3)] \rightarrow C}\, \mathbf{Assc_d 2}$$

Mixed permutation 1 case $T_2 \prec_{T_1} T_3$

$$\frac{S[(T_1 \circ_i T_2) \circ_j T_3] \rightarrow C}{S[(T_1 \circ_{j-S(T_2)+1} T_3) \circ_i T_2] \rightarrow C}\, \mathbf{MixPerm1} \qquad \frac{S[(T_1 \circ_i T_3) \circ_j T_2] \rightarrow C}{S[(T_1 \circ_j T_2) \circ_{i+S(T_2)-1} T_3] \rightarrow C}\, \mathbf{MixPerm1}$$

mD continued

mD continued

More structural rules:

Mixed permutation 2 case $T_3 \prec_{T_1} T_2$

$$\frac{S[(T_1 \circ_i T_2) \circ_j T_3] \to C}{S[(T_1 \circ_j T_3) \circ_{i+S(T_3)-1} T_2] \to C}\ \textbf{MixPerm2} \qquad \frac{S[(T_1 \circ_i T_3) \circ_j T_2] \to C}{S[(T_1 \circ_{j-S(T_3)+1} T_2) \circ_i T_3] \to C}\ \textbf{MixPerm2}$$

Mixed associativity I

$$\frac{R[(T \circ S) \circ_i K] \to A}{R[(T \circ_i K) \circ S] \to A}$$

Mixed associativity II

$$\frac{R[(T \circ S) \circ_i K] \to A}{R[(T \circ (S \circ_{i-s(T)} K] \to A}$$

mD vs hD

mD vs hD

Let us define the following map between sequent calculi: We consider the following embedding translation from **mD** to **hD**: We consider the following embedding translation from **mD** to **hD**:

$$
\begin{array}{ccc}
(\cdot)^\sharp : \mathbf{mD} = (\mathcal{F}, \mathbf{StructTerm}, \rightarrow) & \longrightarrow & \mathbf{hD} = (\mathcal{F}, \mathcal{O}, \Rightarrow) \\
T \rightarrow A & \mapsto & (T)^\sharp \Rightarrow (A)^\sharp
\end{array}
$$

$(\cdot)^\sharp$ is such that:

$$
\begin{array}{l}
A^\sharp = \overrightarrow{A} \text{ if } A \text{ is of sort strictly greater than 0} \\
A^\sharp = A \text{ if A is of sort 0} \\
(T_1 \circ T_2)^\sharp = T_1^\sharp, T_2^\sharp \\
(T_1 \circ_i T_2)^\sharp = T_1^\sharp |_i T_2^\sharp \\
\mathbb{I}^\sharp = \Lambda \\
\mathbb{J}^\sharp = 1
\end{array}
$$

Mutually recursive definition of hyperconfigurations

Mutually recursive definition of hyperconfigurations

$$
\begin{array}{rcl}
\mathcal{O} & ::= & \Lambda \\
\mathcal{O} & ::= & A,\mathcal{O} \text{ for } s(A) = 0 \\
\mathcal{O} & ::= & 1,\mathcal{O} \\
\mathcal{O} & ::= & A\{\underbrace{\mathcal{O} : \ldots : \mathcal{O}}_{a \text{ times}}\},\mathcal{O}
\end{array}
$$

On the morphism $(\cdot)^\sharp$

(5) **Lemma** ($(\cdot)^\sharp$ *is an Epimorphism*)

For every $\Delta \in \mathcal{O}$ there exists a structural term[1] T_Δ such that:

$$(T_\Delta)^\sharp = \Delta$$

Proof. This can be proved by induction on the structure of hyperconfigurations. We define recursively T_Δ such that $(T_\Delta)^\sharp = \Delta$:

- Case $\Delta = \Lambda$ (the empty tree): $T_\Delta = \mathbb{I}$.
- Case where $\Delta = A, \Gamma$: $T_\Delta = A \circ T_\Gamma$, where by induction hypothesis (i.h.) $(T_\Gamma)^\sharp = \Gamma$.
- Case where $\Delta = 1, \Gamma$: $T_\Delta = \mathbb{J} \circ T_\Gamma$, where by i.h. $(T_\Gamma)^\sharp = \Gamma$.
- Case $\Delta = \overrightarrow{A} \otimes \langle \Delta_1, \cdots, \Delta_a \rangle, \Delta_{a+1}$. By i.h. we have:

$$(T_{\Delta_i})^\sharp = \Delta_i \text{ for } 1 \leq i \leq a+1$$

$$T_\Delta = (A \circ_1 T_{\Delta_1}) \circ T_{\Delta_2} \text{ if } a = 1$$
$$T_\Delta = ((\cdots((A \circ_1 T_{\Delta_1}) \circ_{1+d_1} T_{\Delta_2}) \cdots) \circ_{1+d_1+\cdots+d_{a-1}} T_{\Delta_a}) \circ T_{\Delta_{a+1}} \text{ if } a > 1$$

□

[1] In fact there exists an infinite set of such structural terms.

mD vs hD

(6) **Theorem** (*Faithfulness of* $(\cdot)^\sharp$ *Embedding Translation*)

Let $A,\ X$ and Δ be respectively a type, a structural term and a hyperconfiguration. The following statements hold:

i) If $\vdash_{\mathbf{mD}} X \rightarrow A$ then $\vdash_{\mathbf{hD}} (X)^\sharp \Rightarrow A$
ii) For any X such that $(X)^\sharp = \Delta$, if $\vdash_{\mathbf{hD}} \Delta \Rightarrow A$ then $\vdash_{\mathbf{mD}} X \rightarrow A$

hD absorbs the structural rules

hD absorbs the structural rules

Again, as before with $\mathbf{NL_{Assc}}/\mathbf{L}$, the embedding translation mapping satisfies:

$$(R[T])^\sharp = R^\sharp \langle T^\sharp \rangle$$

Since $\mathcal{O}_D$ is the free algebra of DAs over $\mathcal{F}_D$, $(\cdot)^\sharp$ absorbs the structural rules of **mD**.