Mathematical Logic and Linguistics
Slides 12

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Algebraic interpretation for **DA**

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- Displacement algebras (DAs) versus standard DAs (SDA).
- Phase semantics for **DA**: soundness and completeness.
- Powerset DAs over SDAs: soundness and completeness for the so-called *implicative* fragment.
- Extending phase semantics to the other connectives of **DL**.

DAs vs SDAs

General DAs have the following axiomatisation:

Continuous associativity

$x + (y + z) \approx (x + y) + z$

Discontinuous associativity

$x \times_i (y \times_j z) \approx (x \times_i y) \times_{i+j-1} z$
$(x \times_i y) \times_j z \approx x \times_i (y \times_{j-i+1} z)$ if $i \leq j \leq 1 + s(y) - 1$

Mixed permutation

$(x \times_i y) \times_j z \approx (x \times_{j-S(y)+1} z) \times_i y$ if $j > i + s(y) - 1$
$(x \times_i z) \times_j y \approx (x \times_j y) \times_{i+S(y)-1} z$ if $j < i$

Mixed associativity

$(x + y) \times_i z \approx (x \times_i y) + z$ if $1 \leq i \leq s(x)$
$(x + y) \times_i z \approx x + (y \times_{i-s(x)} z)$ if $x + 1 \leq i \leq s(x) + s(y)$

Continuous unit and discontinuous unit

$0 + x \approx x \approx x + 0$ and $1 \times_1 x \approx x \approx x \times_i 1$

Figure: Equational theory for **DA**

SDAs

D is model-theoretically motivated, and the key to its conception is the class of standard displacement algebras. Some definitions are needed. Let $\mathbf{M} = (M, +, 0, 1)$ be a free monoid where 1 is a distinguished element of the set of generators X of **M**. We call such an algebra a *separated monoid*. Given an element $a \in M$, we can associate to it a number, called its *sort* as follows:

$$\begin{array}{lll} s(1) & = & 1 \\ s(a) & = & 0 \text{ if } a \in X \text{ and } a \neq 1 \\ s(w_1 + w_2) & = & s(w_1) + s(w_2) \end{array} \tag{1}$$

This induction is well-defined, for **M** is free, and 1 is a (distinguished) generator. The sort function $s(\cdot)$ in a separated monoid simply counts the number of separators an element contains.

SDAs (Continued)

Definition

(**Sort Domains**)
Where $\mathbf{M} = (M, +, 0, 1)$ is a separated monoid, the *sort domains* M_i of sort i are defined as follows:

$$M_i \quad = \quad \{a \in M : s(a) = i\}, i \geq 0$$

It is readily seen that for every $i, j \geq 0$, $M_i \cap M_j = \emptyset$ iff $i \neq j$.

SDAs (Continued)

Definition

(**Standard Displacement Algebra**)
The *standard displacement algebra* (or standard DA) defined by a separated monoid $(M, +, 0, 1)$ is the ω-sorted algebra with the ω-sorted signature $\Sigma_D = (+, \{\times_i\}_{i>0}, 0, 1\})$ with sort functionality $((i, j \to i + j)_{i,j \geq 0}, (i, j \to i + j - 1)_{i>0, j \geq 0}, 0, 1)$:

$$(\{M_i\}_{i \geq 0}, +, \{\times_i\}_{i>0}, 0, 1)$$

where:

operation	is such that
$+ : M_i \times M_j \to M_{i+j}$	as in the separated monoid
$\times_k : M_i \times M_j \to M_{i+j-1}$	$\times_k(s, t)$ is the result of replacing the k-th separator in s by t

Phase semantics for **DA**

Consider **DA** augmented with the $(\top_i)_i$ such that $S(\top_i) = i$:

(2) $$\frac{}{\Delta \Rightarrow \top_i}\ \top_i R$$

Rule (2) has the side-condition requiring that Δ is any configuration of sort i.

Phase semantics for **DA**

Definition

A displacement phase space $\mathbf{P} = (\mathbf{A}, \mathbf{Facts})$ is a structure such that:

1. $\mathbf{A}$ is a DA.
2. $\mathbf{Facts} = (\mathbf{Facts}_i)_i \subseteq sb(A)$ is a set of subsets such that $\mathbf{Facts}_i \cap \mathbf{Facts}_j = \emptyset$ iff $i \neq j$, and:
   a) For every $F \in \mathbf{Facts}_i$, $F \subseteq A_i$.
   b) $\mathbf{Facts}$ is closed by intersections of arbitrary families of same-sort subsets.
   c) $A_i \in \mathbf{Facts}_i$.
   d) For all $F \in \mathbf{Facts}_i$, and for all $x \in A_j$:

$$
\begin{array}{lcl}
x \backslash F & \in & \mathbf{Facts}_{i-j} \\
F / x & \in & \mathbf{Facts}_{i-j} \\
F \uparrow_k x & \in & \mathbf{Facts}_{i-j+1} \\
x \downarrow_k F & \in & \mathbf{Facts}_{i-j+1}
\end{array}
$$

Facts is also called (an ω-sorted) *closure system*. When $F \in \mathbf{Facts}$ we say that F is a *fact* or a *closed* subset.

Where F, G denote subsets of A of sort i, we define the closure operator:

$$(3) \quad cl(G) \quad \triangleq \quad \bigcap\{F \in \mathbf{Facts}_i : G \subseteq F\}$$

$cl(\cdot)$ is well-defined for in definition (3) it is required that $A_i \in \mathbf{Facts}_i$.

It is readily seen that:

- $cl(F)$ is the least closed set such that $F \subseteq cl(F)$.
- $cl(\cdot)$ is extensive, i.e.: $G \subseteq cl(G)$.
- $cl(\cdot)$ is monotone, i.e.: if $G_1 \subseteq G_2$ then $cl(G_1) \subseteq cl(G_2)$.
- $cl(\cdot)$ is idempotent, i.e.: $cl^2(G) = cl(G)$.

Some notation

We sometimes notate $\overline{G}$ instead of $cl(G)$, for a give same-sort subset G of A. We define the following operators:

- $F \circ G \triangleq \{f + g : f \in F \text{ and } g \in G\}$
- $F \circ_i G \triangleq \{f \times_i g : f \in F \text{ and } g \in G\}$
- $f \circ G \triangleq \{f\} \circ G$ and $F \circ g \triangleq F \circ \{g\}$
- $f \circ_i G \triangleq \{f\} \circ_i G$ and $F \circ_i g \triangleq F \circ_i \{g\}$
- $G//F \triangleq \{h : \forall f \in F, h + f \in G\}$
- $G \uparrow\uparrow_i F \triangleq \{h : \forall f \in F, h \times_i f \in G\}$
- $G//f \triangleq G//\{f\}$
- $G \uparrow\uparrow_i f \triangleq G \uparrow\uparrow_i \{f\}$

Recovering a closure system from a closure operator

Given $cl(\cdot)$ we put:

(4) $G \in \textbf{Facts}$ iff $cl(G) = G$

The indexed subuniverses $\overline{A_i} \subseteq A_i$, whence $A_i \in \textbf{Facts}_i$. Closure by arbitrary intersections of families of same-sort closed subsets holds for:

$\bigcap_i F_i \subseteq F_i$ where $F_i \in \textbf{Facts}$ i.e.,$F_i = cl(F_i)$, and $i \in I$
$cl(\bigcap_i F_i) \subseteq \bigcap_i F_i$ but $\bigcap_i F_i \subseteq cl(\bigcap_i F_i)$ whence:
$\bigcap_i F_i \in \textbf{Facts}$

In order to recover 2.d) we need additional properties (see later).

Properties on $cl(\cdot)$

- $F \circ G \subseteq H$ iff $F \subseteq H//G$ iff $G \subseteq F\backslash\backslash H$.
- $F \circ_i G \subseteq H$ iff $F \subseteq H \uparrow\uparrow_i G$ iff $G \subseteq F \downarrow\downarrow_i H$.
- By construction, $cl(F)$ is the least closed subset such that $F \subseteq cl(F)$. Hence:
- If $A \subseteq F$ and $cl(F) = F$ then $cl(A) \subseteq cl(F)$.

More properties on $cl(\cdot)$

- If A is closed, then:

  $A//F, F\backslash\backslash A, A\uparrow\uparrow_i F,$ and $F\downarrow\downarrow_i A$ are closed.

  Proof: $A\uparrow\uparrow_i F = \bigcap_{x\in F} A\uparrow\uparrow_i x,$ whence $A\uparrow\uparrow_i F$ is closed. □

  Similar for the other implicative operations.

- $cl(F)\circ cl(G) \subseteq cl(F\circ G)$. Similarly, $cl(F)\circ_i cl(G) \subseteq cl(F\circ_i G)(\star)$.
- Hence, $cl(cl(F)\circ cl(G)) \subseteq cl(F\circ G)$, and $cl(cl(F)\circ_i cl(G)) \subseteq cl(F\circ_i G)$

More properties on $cl(\cdot)$ (continued)

- Proof of ($\star$): $cl(F)\circ_i cl(G) \subseteq cl(F\circ G)$.

Proof.

$F\circ_i G \subseteq cl(F\circ_i G)$. By residuation, $F \subseteq cl(F\circ_i G)\uparrow\uparrow_i G$. $cl(F\circ_i G)\uparrow\uparrow_i G$ is a closed subset (see previous proof). Hence, $cl(F) \subseteq cl(F\circ_i G)\uparrow\uparrow_i G$. Applying again residuation, we have $cl(F)\circ_i G \subseteq cl(F\circ G)$. We repeat the process with G, obtaining $cl(G) \subseteq cl(F)\downarrow\downarrow_i cl(F\circ_i G)$. It follows that:

$$cl(F)\circ_i cl(G) \subseteq cl(F\circ_i G)$$

□

- It follows that: $cl(cl(F)\circ_i cl(G)) \subseteq cl(F\circ_i G)$. The inclusion trivially holds, whence we have equality:

$$cl(cl(F)\circ_i cl(G)) = cl(F\circ_i G)$$

Closed operations between closed subsets

Given F, G closed sets:

- $F\overline{\circ}G \triangleq cl(F \circ G)$.
- $F\overline{\circ_i}G \triangleq cl(F \circ_i G)$.
- $\overline{T_i} \triangleq A_i$, where A_i are the same-sort subuniverses.
- $F\overline{\&}G \triangleq F \cap G$. In general we write $F \cap G$.
- $F\overline{\cup}G \triangleq cl(F \cup G)$.
- $G\overline{\uparrow\uparrow_i}F \triangleq G\uparrow\uparrow_i F$. In general we write $\uparrow\uparrow_i$Similar for the other implications.
- $\overline{\mathbb{I}} \triangleq cl(\{0\})$.
- $\overline{\mathbb{J}} \triangleq cl(\{J\})$.

Closed operations between closed subsets and algebraic interpretations

Let A, B be arbitrary types. Given a valuation $v : \mathrm{Pr} \to \mathcal{F}$:

- $v(p)$ is closed subset of A_i where p is primitive of sort i.
  We extend recursively (we drop the notation $\hat{v}$) v:
- $v(B{\uparrow_i}A) \triangleq v(B){\uparrow\uparrow_i}v(A)$. Similar for the other implications.
- $v(A \bullet B) \triangleq v(A)\overline{\circ}v(B)$.
- $v(A \odot_i B) \triangleq v(A)\overline{\circ_i}v(B)$.
- $v(A \oplus B) \triangleq v(A)\overline{\cup}v(B)$.
- $v(I) \triangleq \overline{\mathbb{I}}$.
- $v(J) \triangleq \overline{\mathbb{J}}$.
- $v(\top_i) \triangleq A_i$.

It follows that for any type A, $v(A)$ is a closed subset.

Soundness of DA

Let us recall the categorical calculus. We have the faithful translation $(\cdot)^\sharp$: **cD** $\Rightarrow$ **hD**, such that For any Δ and A, we have:

$$\mathbf{cD} \vdash (\Delta)^{\bullet} \rightarrow A \text{ iff } \mathbf{hD} \vdash \Delta \Rightarrow A$$

From the above translation we get (almost) for free soundness of **DA** w.r.t. phase semantics. Residuation is obvious. Only postulates are to be checked whether they hold of a phase displacement model. Some postulates:

- **Continuous associativity**

$$A \bullet (B \bullet C) \rightarrow (A \bullet B) \bullet C \text{ and } (A \bullet B) \bullet C \rightarrow A \bullet (B \bullet C)$$

- **Mixed associativity** If we have that $B \between_A C$:

$$A \odot_i (B \odot_j C) \Rightarrow (A \odot_i B) \odot_{i+j-1} C \text{ and } (A \odot_i B) \odot_{i+j-1} C) \rightarrow A \odot_i (B \odot_j C)$$

- **Mixed permutation** If we have that $B \prec_A C$:

$$(A \odot_i B) \odot_j C \rightarrow (A \odot_{j-b+1} C) \odot_i B \text{ and } (A \odot_{j-b+1} C) \odot_i B \rightarrow (A \odot_i B) \odot_j C)$$

Soundness of **DA** (Continued)

Let us check mixed permutation, in the case $B \between_A C$. Remaining postulates are proved to hold similarly. Let v be a valuation in a phase displacement space:

$$
\begin{aligned}
v((A \odot_i B) \odot_j C) &= (v(A)\overline{\circ_i}v(B))\overline{\circ_j}v(C) \\
&= cl(cl(v(A)\circ_i v(B))\circ_j v(C)) \\
&= cl(cl(v(A)\circ_i v(B))\circ_j cl(v(C))) \text{ since } cl(v(C)) = v(C) \\
&= cl((v(A)\circ_i v(B))\circ_j v(C)) \\
&= cl((v(A) \odot_{j-b+1} v(C)) \odot_i v(B)) \\
&= (v(A)\overline{\circ_{j-b+1}}v(C))\overline{\circ_i}v(C) \\
&= v((A \odot_{j-b+1} C) \odot_i B) \quad \square
\end{aligned}
$$

Strong completeness in the sense of Okada

The notation $[A]$ for a given type A, corresponds to:

$$[A] \quad = \quad \{\Gamma : \Gamma \Rightarrow A \text{ without the Cut rule}\}$$

We write $\vdash^{-} \Gamma \Rightarrow A$ to indicate provability without Cut, or simply $\Gamma \Rightarrow A$ when it is from the context that we are considering Cut-free provability.

Strong completeness in the sense of Okada (Continued)

- Let us define the syntactic displacement phase space $\mathbf{P} = (\mathbf{M}, \mathbf{Facts})$ as follows:

  - $\mathbf{M} = (\mathcal{O}, conc, (interc_i)_{i>0}, \Lambda, 1)$ where $\mathcal{O}$ is the set of configurations and *conc* and $(interc_i)_{i>0}$ are respectively the concatenation and intercalation functions of configurations. In general, if Δ and Γ are configurations, we write Δ, Γ and $\Delta|_i\Gamma$ instead of $conc(\Delta, \Gamma)$ and $interc_i(\Delta, \Gamma)$.

  - $\mathbf{M}$ is the free DA over the set of types $\mathcal{F}$. **Facts** is defined as the least set closed by arbitrary intersections containing $\mathcal{O}_i$, and $[A]$ for every type A. In practice, by construction, for every fact F, there exists a collection $\mathcal{G}$ of types such that $F = \bigcap_{D \in \mathcal{G}} [D]$. If $F = \mathcal{O}_i$, then $\mathcal{G} = \{\top_i\}$. 2.d) from the definition of phase displacement spaces holds.

Strong completeness in the sense of Okada (Continued)

We define the canonical valuation v:

$$(5) \quad v(p) = [p]$$

Theorem (Truth lemma)

For any type A:

$$v(A) = [A]$$

Strong completeness in the sense of Okada (Continued)

Proof.

By induction on the structure of type A:

- If $A = p$ where p is a primitive type, we have by definition $v(A) = [A]$. Hence, $\overrightarrow{A} \in v(A) \subseteq [A]$.

- Suppose $A = B \odot_i C$.
$v(B) \circ_i v(C) = \{\Gamma_B |_i \Gamma_C : \Gamma_B \in v(B), \text{ and } \Gamma_C \in v(C)\}$. By i.h. $v(B) \subseteq [B]$ and $v(C) \subseteq [C]$. Hence, by application of $\odot_i L$ $v(B) \circ_i v(C) \subseteq [B \odot_i C]$. Hence, $cl(v(B) \circ_i v(C)) \subseteq [B \odot_i C]$. This proves $v(B \odot_i C) \subseteq [B \odot_i C]$.

On the other hand, $v(B) \overline{\circ_i} v(C) = \bigcap_{D \in \mathcal{G}} [D]$ for a certain $\mathcal{G}$. For every $D \in \mathcal{G}$, by i.h. $\overrightarrow{B} |_i \overrightarrow{C} \in [D]$. By application of $\odot_i L$, $\overrightarrow{B \odot_i C} \in [D]$. Hence, $\overrightarrow{B \odot_i C} \in v(B \odot_i C)$.

□

Strong completeness in the sense of Okada (proof of truth lemma continued)

- Suppose $A = C{\uparrow_i}B$. Let $\Gamma \in v(C){\uparrow\uparrow_i}v(B)$. By i.h., $\overrightarrow{B} \in v(B)$. We have $\Gamma|_i\overrightarrow{B} \Rightarrow v(C)$ and $v(C) \subseteq [C]$ by i.h. Hence, $\Gamma|_i\overrightarrow{B} \Rightarrow {}^{-}C$, and by application of ${\uparrow_i}R$, $\Gamma \Rightarrow C{\uparrow_i}B$, i.e., $\Gamma \in [C{\uparrow_i}B]$.

By i.h., $\overrightarrow{C} \in v(C)$. $v(C) = \bigcap_{D\in\mathcal{G}}[D]$ for some $\mathcal{G}$. Applying ${\uparrow_i}L$, we get $\overrightarrow{C{\uparrow_i}B}|_i\Gamma_B \in [D]$ for all $\Gamma_B \in [B]$. Hence, $\overrightarrow{C{\uparrow_i}B}\circ_i[B] \in [D]$ for all $D \in \mathcal{G}$. Therefore, $\overrightarrow{C{\uparrow_i}B}\circ_i[B] \subseteq v(C)$. We have that $\overrightarrow{C{\uparrow_i}B}\circ_i v(B) \subseteq \overrightarrow{C{\uparrow_i}B}\circ_i[B]$, since by i.h., $v(B) \subseteq [B]$. By applying residuation, $\overrightarrow{C{\uparrow_i}B} \in v(C){\uparrow\uparrow_i}v(B)$.

Strong completeness in the sense of Okada (proof of truth lemma continued)

- $A = B\&C$. Let $\Gamma \in v(B) \cap v(C)$. In particular $\Gamma \in v(B)$ and $\Gamma \in v(C)$. By i.h. $\Gamma \in [B]$ and $\Gamma \in [C]$. By application of $\&R$ we get $\Gamma \in [B\&C]$. This proves $v(B\&C) \subseteq [B\&C]$.

- $v(C) = \bigcap_{D \in \mathcal{G}}[D]$ for a certain $\mathcal{G}$. For every $D \in \mathcal{G}$, $\overrightarrow{C} \in [D]$. By applying $\&2L$ we get $\overrightarrow{C\&B} \in [D]$. Hence, $\overrightarrow{C\&B} \in v(C)$. By a similar reasoning, we have $\overrightarrow{C\&B} \in v(B)$. It follows that $\overrightarrow{C\&B} \in v(C\&B)$.

Strong completeness in the sense of Okada (proof of truth lemma continued)

- Case $A = B \oplus C$. By i.h. $v(B) \subseteq [B]$ and $v(C) \subseteq [C]$. Hence, $v(B) \cup v(C) \subseteq cl([B] \cup [C]) \subseteq [B \oplus C]$. The first inclusion is due to the monotony property and properties of cl. In fact, we have $[B] \cup [C] \subseteq [B \oplus C]$. For, $[B] \subseteq [B \oplus C]$ and $[C] \subseteq [B \oplus C]$ by $\oplus iR$ $(i = 1, 2)$. It follows that $cl(v(B) \cup v(C)) \subseteq [B \oplus C]$.

- On the other hand, $v(B \oplus C) = \bigcap_{D \in \mathcal{G}} [D]$ for a certain $\mathcal{G}$. By i.h $\overrightarrow{B} \in v(B)$. Hence, $\overrightarrow{B} \subseteq cl(v(B) \cup v(C))$. Similarly, $\overrightarrow{C} \subseteq cl(v(B) \cup v(C))$. Therefore, for any $D \in \mathcal{G}$, $\overrightarrow{B} \in [D]$ and $\overrightarrow{C} \in [D]$. By $\oplus L$ we get $\overrightarrow{B \oplus C} \in [D]$. It follows that $\overrightarrow{B \oplus C} \subseteq v(B \oplus C)$. □

Strong completeness in the sense of Okada

Theorem

(Strong Completeness à la Okada) Let $\Delta \Rightarrow A$ be such that for every $(\mathbf{P}, v)$, $(\mathbf{P}, v) \models \Delta \Rightarrow B$. It follows that $\Delta \Rightarrow {}^{-}B$.

Corollary (Cut admissibility)

The Cut rule is admissible.

□

Strong completeness in the sense of Okada (proof of the theorem)

Proof.

In particular, this sequent holds in the syntactic phase displacement model. By the previous lemma, for any A, $\overrightarrow{A} \in v(A)$. Hence $\Delta \in v(\Delta)$. Since $(\mathbf{P}, v) \models \Delta \in v(B)$, we have that $\Delta \in v(B)$. Again, by the previous lemma $\Delta \in [B]$. It follows that $\Delta \Rightarrow {}^{-}B$. $\square$