Pure Nash Equilibria
certainty versus succinctness

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1. Complexity framework

2. Complexity analysis

3. Other succinct representations

4. Concluding remarks
Natural problems related to PNE

*Is Nash* (IsN)
Given a game \( \Gamma \) and a strategy profile \( a \), decide whether \( a \) is a Nash equilibrium of \( \Gamma \).

*Exists Pure Nash* (EPN)
Given a strategic game \( \Gamma \), decide whether \( \Gamma \) has a Pure Nash equilibrium.

*Pure Nash with Guarantees* (PNGRANT)
Given a strategic game \( \Gamma \) and a value \( v \), decide whether there is a pure Nash equilibrium in which the first player gets payoff \( v \) or higher.
How to represent a game?

- We are interested in fixing the representation of a game as an input to a program.
- It is natural to consider different levels of succinctness.
- In the most generic model some components of the game have to be represented by a TM, for example the utility functions.
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We only consider rational valued utility functions
The convention guarantees a correct and unique game definition from its description
Strategic games in explicit form.

- A game is given by a tuple

\[ \Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle. \]

- It has $n$ players,
- For each player $i$, $A_i$ is given explicitly by listing its elements.
- $T$ is a table with an entry for each strategy profile $s$ and player $i$.
- So, $u_i(s) = T(s, i)$. 
Strategic games in general form.

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- It has n players,
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- The description of their pay-off is given by \( \langle M, 1^t \rangle \).
- So, for each strategy profile \( s \) and player \( i \),
  \[ u_i(s) = M(s, i) \] stopping after \( t \) steps.
Implicit form

Strategic games in implicit form.

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Forms of representation

**Strategic games in explicit form.** A game is described by a tuple \( \Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle \).

**Strategic games in general form.** A game is described by a tuple \( \Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle \).

**Strategic games in implicit form.** A game is described by a tuple \( \Gamma = \langle 1^n, 1^m, M, 1^t \rangle \).
What is the most suitable level of succinctness?

- Prisoners’ dilemma?
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  - Explicit
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- Prisoners’ dilemma?  
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**IsPN** Given a game $\Gamma$ and a strategy profile $s$, is $s$ is a PNE?.
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The condition $u_i(s) \geq u_i(s_{-i}, a_i)$ can be checked in polynomial time given $\Gamma, s,$ and $a_i$.

Thus the problem is in coNP.
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  Is this classification tight?
IsPN implicit form: Hardness

A coNP complete problem?
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**SAT**: Given a boolean formula $F$ in CNF form, determine whether $F$ is satisfiable.

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Is an NP complete problem. So, its complement is coNP-complete.

We have to associate to $F$ a game $\Gamma$ and a strategy profile $s$ so that:

- $F$ is not satisfiable iff $s$ is a PNE of $\Gamma$
- and show that a description of $\Gamma$ in implicit form and of $s$ can be obtained in time polynomial in $|F|$. 

Given a CNF formula $F$ on $n$ variables consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0, 1\}^{n+1}$
- $u_1(0x) = 0$, for any $x \in \{0, 1\}^n$
- $u_1(1x) = F(x)$, for any $x \in \{0, 1\}^n$
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Consider the strategy $a_1 = 0^{n+1}$. 
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Thus $\Gamma(F), 0^{n+1}$ verify the first requirement.
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- $t = (n + |F|)^2$. 
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The time required to obtain $\langle 1^n, 1^m, M, 1^t \rangle$, given $F$, is...
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The time required to obtain $\langle 1^n, 1^m, M, 1^t \rangle$, given $F$, is polynomial in $|F|$.
IsPN implicit form

Theorem

The IsPN problem for strategic games in implicit form is coNP-complete.
Solving the EPN

**EPN** Given a game $\Gamma$ does it have a PNE?
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- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
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  - In the case that $n$ is constant, in $\text{P}$.
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  In the case that \( n \) is constant, in P.
- Given \( \Gamma = \langle 1^n, 1^m, M, 1^t \rangle \) the cost is exponential. A better classification? in \( \Sigma_2^P \).
The EPN problem for strategic games in general form is NP-complete.

We provide a reduction from SAT. Let $F$ be a CNF formula.

- $F \rightarrow \Gamma(F) = \langle 1^n, \{0, 1\} \ldots \{0, 1\}, M^F, 1^{(n+|F|)^2} \rangle$ where
- $n$ is the number of variables in $F$ and
- $M^F$ is a TM that on input $(a, i)$, evaluates $F$ on assignment $a$ and afterwards it implements the utility function of the $i$-th player. According to the following definition:
EPN: general form

\[ u_1(a) = \begin{cases} 
5 & \text{if } F(a) = 1, \\
4 & \text{if } F(a) = 0 \land a_1 = 0 \land a_2 = 1, \\
3 & \text{if } F(a) = 0 \land a_1 = 1 \land a_2 = 1, \\
2 & \text{if } F(a) = 0 \land a_1 = 1 \land a_2 = 0, \\
1 & \text{if } F(a) = 0 \land a_1 = 0 \land a_2 = 0, 
\end{cases} \]

\[ u_2(a) = \begin{cases} 
5 & \text{if } F(a) = 1, \\
4 & \text{if } F(a) = 0 \land a_1 = 0 \land a_2 = 0, \\
3 & \text{if } F(a) = 0 \land a_1 = 0 \land a_2 = 1, \\
2 & \text{if } F(a) = 0 \land a_1 = 1 \land a_2 = 1, \\
1 & \text{if } F(a) = 0 \land a_1 = 1 \land a_2 = 0. 
\end{cases} \]

And, for any \( j > 2 \)

\[ u_j(a) = \begin{cases} 
5 & \text{if } F(a) = 1, \\
1 & \text{otherwise.} 
\end{cases} \]
We have that

- Given a description of $F$, $\Gamma(F)$ is computable in polynomial time.
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  Similar arguments as before.
Reduction correctness

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- Given a description of $F$, $\Gamma(F)$ is computable in polynomial time.
- Similar arguments as before.
- $F$ is satisfiable iff $\Gamma(F)$ has a PNE?
Reduction trick

Look at the two player strategic game that can be played by the first and second players:

<table>
<thead>
<tr>
<th></th>
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<tr>
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<tr>
<td>1</td>
<td>2,1</td>
<td>3,2</td>
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PNE?
Reduction trick

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PNE?
None
Reduction correctness

- $F$ is a yes instance of SAT.
$F$ is a yes instance of SAT.
There is a satisfying assignment $x$. So $u_i(x) = 5$, for any $i$. Such a strategy profile is a PNE.
Reduction correctness

- $F$ is a yes instance of SAT.
  There is a satisfying assignment $x$. So $u_i(x) = 5$, for any $i$. Such a strategy profile is a PNE.
- $F$ is a no instance of SAT.
Reduction correctness

- **$F$** is a yes instance of SAT.
  There is a satisfying assignment $x$. So $u_i(x) = 5$, for any $i$.
  Such a strategy profile is a PNE.

- **$F$** is a no instance of SAT.
  For any strategy profile the payoff of players $j > 2$ is always 1.
  So they cannot change strategy and improve payoff.
  However, players 1 and 2 are engaged in a game with no PNE
  so one of them can change strategy and increase its payoff.
  Therefore $\Gamma(F)$ has no PNE
Let $L \subseteq \Sigma^*$ be a language.
$L \in \Sigma_2^p$ if and only if there is a polynomially decidable relation $R$ and a polynomial $p$ such that

$$L = \{x \mid \exists z |z| \leq p(|x|) \forall y |y| \leq p(|x|) \langle x, y, z \rangle \in R \}.$$
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\[
L = \{ x \mid \exists z |z| \leq p(|x|) \forall y |y| \leq p(|x|) \langle x, y, z \rangle \in R \}.
\]

**Q2SAT**
Given \( \Phi = \exists \alpha_1, \ldots, \alpha_{n_1} \forall \beta_1, \ldots, \beta_{n_2} F \) where \( F \) is a Boolean formula over the boolean variables \( \alpha_1, \ldots, \alpha_{n_1}, \beta_1, \ldots, \beta_{n_2} \), decide whether \( \Phi \) is valid.

**Q2SAT** is \( \Sigma_2^p \)-complete.
The EPN problem for strategic games in implicit form is $\Sigma^P_2$-complete.

Let's provide a reduction from $Q2SAT$. 
For each $\Phi = \exists \alpha_1, \ldots, \alpha_{n_1} \forall \beta_1, \ldots \beta_{n_2} F$
we define a game $\Gamma(\Phi)$ as follows.
There are four players:
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There are four players:

- Player 1, the *existential player*, assigns truth values to the boolean variables $\alpha_1, \ldots, \alpha_{n_1}$ and $A_1 = \{0, 1\}^{n_1}$ and $a_1 = (\alpha_1, \ldots \alpha_{n_1}) \in A_1$. 
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- **Player 2**, the *universal player*, assigns truth values to the boolean variables $\beta_1, \ldots, \beta_{n_2}$ and $A_2 = \{0, 1\}^{n_2}$ and $a_2 = (\beta_1, \ldots, \beta_{n_2}) \in A_2$. 
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- Players 3 and 4 avoid entering into a Nash equilibrium when the actions played by players 1 and 2 do not satisfy $F$. Their set of actions are $A_3 = A_4 = \{0, 1\}$.
Let us denote by $F(a_1, a_2)$ the truth value of $F$ under the assignment given by $a_1$ and $a_2$.

$$u_1(a_1, a_2, a_3, a_4) = \begin{cases} 1 & \text{if } F(a_1, a_2) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_2(a_1, a_2, a_3, a_4) = \begin{cases} 1 & \text{if } F(a_1, a_2) = 0, \\ 0 & \text{otherwise.} \end{cases}$$
$u_3(a_1, a_2, a_3, a_4) = \begin{cases} 
5 & \text{if } F(a_1, a_2) = 1, \\
4 & \text{if } F(a_1, a_2) = 0 \land a_3 = 0 \land a_4 = 1, \\
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\end{cases}$

$u_4(a_1, a_2, a_3, a_4) = \begin{cases} 
5 & \text{if } F(a_1, a_2) = 1, \\
3 & \text{if } F(a_1, a_2) = 0 \land a_3 = 0 \land a_4 = 1, \\
2 & \text{if } F(a_1, a_2) = 0 \land a_3 = 1 \land a_4 = 1, \\
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4 & \text{if } F(a_1, a_2) = 0 \land a_3 = 0 \land a_4 = 0. 
\end{cases}$
Let us assume that $\Phi = \exists \alpha_1, \ldots, \alpha_n \forall \beta_1, \ldots, \beta_m F$, where $F$ is a Boolean formula over the boolean variables $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$, is true.

Then there exists $\alpha \in \{0, 1\}^n$ such that for all $\beta \in \{0, 1\}^m$, $F(\alpha, \beta) = 1$.

This means that if player 1 plays action $\alpha$, for each $\beta \in \{0, 1\}^m$, $a_3, a_4 \in \{0, 1\}$, no player has incentive to change strategy.
Let us assume that $\Phi$ is not valid. It means that for any $\alpha \in \{0, 1\}^n$ there exists $\beta \in \{0, 1\}^m$ such that $F(\alpha, \beta) = 0$. Let $(\alpha, \beta, a, b)$ be a strategy profile. We have two cases.
Let us assume that $\Phi$ is not valid.

It means that for any $\alpha \in \{0, 1\}^n$ there exists $\beta \in \{0, 1\}^m$ such that $F(\alpha, \beta) = 0$.

Let $(\alpha, \beta, a, b)$ be a strategy profile. We have two cases.

Case 1: $F(\alpha, \beta) = 0$, in this case players 3 an 4 engage in a no PNE game.
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Case 1: $F(\alpha, \beta) = 0$, in this case players 3 and 4 engage in a no PNE game.

Case 2: $F(\alpha, \beta) = 1$, since $\Phi$ is not valid, there exists $\beta' \in \{0, 1\}^m$ such that $F(\alpha, \beta') = 0$. Therefore player 2 has an incentive to change strategy $\beta$ by $\beta'$. 

\[ \text{AGT-MIRI} \]
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Therefore, the strategy profile is not a PNE.
**PNGrant problem**

**PNGrant** Given a strategic game $\Gamma$ and a value $\nu$, decide whether there is a PNE $s$ so the $u_1(s) \geq \nu$.

**Theorem**

The PNGrant problem can be solved in polynomial time for strategic games given in explicit form but it is NP-complete for strategic games given in general form. It is $\Sigma^p_2$-complete for strategic games given in implicit form.
**PNGrant problem**

PNGrant Given a strategic game \( \Gamma \) and a value \( v \), decide whether there is a PNE \( s \) so the \( u_1(s) \geq v \).

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Membership follows from the same arguments.
The PNGrant problem

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The PNGrant problem can be solved in polynomial time for strategic games given in explicit form but it is NP-complete for strategic games given in general form is $\Sigma_2^P$-complete for strategic games given in implicit form.

Membership follows from the same arguments. In all the reduction the utility for the first player in all PNE is constant, this provides the value of $v$ in each reduction.
1 Complexity framework
2 Complexity analysis
3 Other succinct representations
4 Concluding remarks
In a circuit game, players still control disjoint sets of variables, but each player’s payoff is given by a single boolean circuit.

The boolean circuit computes a rational value as the quotient of two integers.

Boolean circuit games are the special case of circuit games where each player controls a single boolean variable.
(Boolean) Circuit games

[Schoenebeck and Vadhan, EC 2006 - ACM TCT 2012]

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TMs can be simulated by circuits and vice versa.
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TMs can be simulated by circuits and viceversa
• Circuit games are equivalent to implicit form games
• Boolean circuit games are a subset of general form games.
In a formula game, players still control disjoint sets of variables, but each player’s payoff is given by a weighted combination of boolean formulas.

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(Boolean) weighted formula games

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- Boolean formula games are the special case of formula games where each player controls a single boolean variable.
- Formulas can be casted as circuits but not viceversa as the size might grow exponentially.
- Nevertheless the utility functions of the provided reductions can be easily described in this way. So the problems are equivalent from the complexity point of view.
Graphical games

[Gottlob, Greco and Scarcello, JAIR 2005]

- Graphical games are a representation of multiplayer games meant to capture and exploit locality or sparsity of direct influences.
- They are most appropriate for large population games in which the payoffs of each player are determined by the actions of only a small subpopulation.
- Players’ relationship is described by a graph and the payoff of a player depends only on the actions of its neighbors.
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- Players’ relationship is described by a graph and the payoff of a player depends only on the actions of its neighbors.
- Provide a complementary framework to analyze complexity based on the graph parameters: bounded degree, bounded treewidth, ...
Uniform families of games with polynomial time computable utilities

- We have analyzed the representations of the strategic games as potential inputs to a problem.
- However those representation forms do not capture completely the notion of games whose utility functions are computable in polynomial time as we expect to have a TM describing the game family and not a TM per game.
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However those representation forms do not capture completely the notion of games whose utility functions are computable in polynomial time as we expect to have a TM describing the game family and not a TM per game.

Even though in many papers studying the computational complexity of some specific games, it is assumed that the utilities are computable in polynomial time this assumption has different interpretations.
Uniform families of games with polynomial time computable utilities

- We adopt: games defined uniformly by polynomial time Turing machines.
Uniform families of games with polynomial time computable utilities

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- Let $M$ be a DTM and let us assume that an alphabet $\Sigma$ is fixed.

- We define uniformly families of games associated to $M$: 
Uniform families of games with polynomial time computable utilities

- We adopt: games defined uniformly by polynomial time Turing machines.

- Let $M$ be a DTM and let us assume that an alphabet $\Sigma$ is fixed.

- We define uniformly families of games associated to $M$:

- Observe that this approach does not make sense for the explicit form.
**M-implicit form strategic family** Each instance of the family specifies the number of players $n$ and their set of actions in an succinct way.

$$\left\{ \left\langle 1^n, 1^{m_1}, \ldots, 1^{m_n} \right\rangle \mid n, m_1, \ldots, m_n \in \mathbb{N} \right\}.$$

In the game described by $\left\langle 1^n, 1^{m_1}, \ldots, 1^{m_n} \right\rangle$, $A_i = \Sigma^{\leq m_i}$ and if $a$ is a strategy profile of such a game, and $1 \leq i \leq n$, then the utility of the $i$-th player on $a$ is defined as $u_i(a) = M(a, i)$. 
**General form**

**M-general form strategic family.** Each instance of the family describes a game by giving the number of players $n$ and explicitly listing the set of actions of each player.

$$\{\langle 1^n, A_1, \ldots, A_n \rangle \mid n, m \in \mathbb{N} \land \forall i \ A_i \subseteq \Sigma^* \}$$

*In the game described by $\langle 1^n, A_1, \ldots, A_n \rangle$, if $a$ is a strategy profile of such game, and $1 \leq i \leq n$, then the utility of the $i$-th player on $a$ is defined as $u_i(a) = M(a, i)$.***
Hence, given a family of games defined from a polynomial time DTM $M$, we can also pose the question of determining whether a game of this family has a Nash equilibrium.

**$M$-Exists Pure Nash ($M$-EPN)**

Given a game $\Gamma$, defined uniformly by $M$, decide whether $\Gamma$ has a Pure Nash equilibrium.

**$M$-Pure Nash equilibrium with guarantee ($M$-PNGrant)**

Given a game $\Gamma$, defined uniformly by $M$, a value $u$, and a player $i$, decide whether $\Gamma$ has a Pure Nash equilibrium in which player $i$ gets payoff $u$ or higher.
Theorem

- There exists a polynomial time DTM $M$ for which the M-EPN problem for games in the M-implicit form strategic family is $\Sigma^p_2$-complete.
- There exists a polynomial time DTM $M$ for which the M-EPN problem for games in the M-general form strategic family is NP-complete.
- There exists a polynomial time DTM $M$ for which the M-PNGrant problem for games in the M-implicit form strategic family is $\Sigma^p_2$-complete.
- There exists a polynomial time DTM $M$ for which the M-PNGrant problem for games in the M-general form strategic family is NP-complete.
We are considering uniform families of games.

The main difference with respect to the proofs of the analogous results in the previous section is that the DTM cannot be parameterized by the quantified boolean formula $\Phi$ or the CNF formula $F$. 
Uniformity vs non-uniformity

- We are considering uniform families of games.
- The main difference with respect to the proofs of the analogous results in the previous section is that the DTM cannot be parameterized by the quantified boolean formula $\Phi$ or the CNF formula $F$.
- Now these formulae should be part of the input of the machines.
- This requires an additional trick.
For any fixed polynomial time DTM $M$, the problem of deciding whether a game $\Gamma$ in $M$-implicit form has a PNE can be solved by an Alternating TM, with 2 alternations, existential and universal, in polynomial time. Hence $M$-SPN $\in \Sigma_2^P$. 
To prove hardness, we have to define first the polynomial time DTM $M$.

Let $M$ be the TM such that on input $(\Phi, a_1, a_2, a_3, a_4, i)$ being $\Phi = \exists \alpha_1, \ldots, \alpha_{n_1} \forall \beta_1, \ldots, \beta_{n_2} F$ an instance of the Q2SAT problem, $a_1 \in A_1 = \{0, 1\}^{n_1}$, $a_2 \in A_2 = \{0, 1\}^{n_2}$ and $a_3, a_4 \in \{0, 1\}$, computes the utilities defined as before.

$M$ works in polynomial time with respect to the input length.
For each $\Phi$ we define a game $\Gamma(\Phi)$ with five players. Players 1, 2, 3 and 4 are defined exactly equal to the four players of the game defined in the previous reduction.
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As we have shown, $\Phi$ is valid if and only if $\Gamma(\Phi)$ has a PNE, and the description of $\Gamma(\Phi)$ in implicit form can be obtained in polynomial time.
1. Complexity framework
2. Complexity analysis
3. Other succinct representations
4. Concluding remarks
We have analyzed some ways of describing games with polynomial time computable utilities: uniform and non-uniform models for strategic games.

We have concentrated on the study of two computational problems.

As expected complexity increases with succinctness.

There are many other game classes and problems of interest not covered in this talk.
Contents taken from a subset of the results in

- C. Alvarez, J. Gabarr’o, M. Serna
  Equilibria problems on games: Complexity versus succinctness
  J. of Comp. and Sys. Sci. 77:1172-1197, 2011
Further suggested reading (among many others)

- G. Gottlob, G. Greco, F. Scarcello
  Pure Nash equilibria: Hard and easy games
  J. Artificial Intelligence Res. 24:357–406, 2005

- J. Gabarro, A. Garcia, M. Serna
  The complexity of game isomorphism
References

- M. Mavronicolas, B. Monien, K. Wagner
  Weighted boolean formula games
  in: X. Deng, F. Graham (Eds.), WINE 2007,

- G.R. Schoenebeck, S. Vadhan
  The Computational Complexity of Nash Equilibria in Concisely
  Represented Games
  ACM Transactions on Computational Theory, 4(2) article 4, 2012