2 Efficiency in equilibrium

2.1. Compute the PoA and the PoS under the egalitarian and the utilitarian utility of the following game:

		Player 2	
		A	B
Player 1	A	6,6	2,7
	B	7,2	0,0

2.2. Consider the sending from s to t game as seen in class. Analyze (compute or provide bounds for) the PoA and the PoS for the following social cost/utility functions:

- $C(s) = \begin{cases} \sum_{i \in N} u_i(s) & \text{if there is a path from } s \text{ to } t \text{ in } G[s] \\ n^2 & \text{otherwise} \end{cases}.$
- $U(s) = \max_{i \in N} u_i(s)$.
- $U(s) = \sum_{i \in N} u_i(s)$.

2.3. In the **cover game** the players are the vertices in an undirected graph $G = (V, E)$ on a set of n vertices. The goal of the game is to select a set of vertices X that covers a lot of edges. An edge is covered by a set X if at least one of its ends points belongs to X.

Formally, the set of actions allowed to player i is $A_i = \{0, 1\}$. Those players playing 1 will form the set. Let $s = (s_1, \dots, s_n)$, $s_i \in \{0, 1\}$, be an strategy profile, and let $X(s) = \{i \mid s_i = 1\}$.

The cost function for player $i \in V$ is defined as follows

$$c_i(s) = s_i + |\{(i, j) \in E \mid i, j \notin X(s)\}|.$$

Assuming that the social cost of a strategy profile is defined as

$$c(s) = |\{(i, j) \in E \mid i, j \notin X(s)\}|.$$

What can you say about PoA and PoS?

2.4. The *cooperation* game is defined as follows. There is a group of n people and a task to be performed. To perform correctly the task requires that exactly k persons cooperate. Each player can decide whether to cooperate (1) or not (0). The utility of a strategy profile $x \in \{1, 0\}^i$ for player i is defined as

$$u_i(x) = \begin{cases} 1 & \text{the task is performed and } x_i = 1. \\ 0 & \text{otherwise} \end{cases}$$

Consider as social utility the value 1 if the task is performed and 0 otherwise. What can you say about PoA and PoS?

2.5. The *matching* game is played in a bipartite graph $G = (V_1, V_2, E)$ in which edges connect only vertices in V_1 to vertices in V_2. The players are the vertices in the graph that is $V_1 \cup V_2$. Each player has to select one of its neighbors. Player i gets utility 1 when the selection is mutual (player i selects j and player j selects i) otherwise he gets 0.

Compute the PoA and the PoS when the social utility is the number of matched pairs.

2.6. Consider a set of n players that must be partitioned into two groups. However, there is a set of bad pairings and the two players in such a pair do not want to be in the same group. Moreover, each player is free to choose which of the two groups to be in. We can model this by a graph $G = (V, E)$ where each player i is a vertex. There is an edge (i, j) if i and j form a bad pair. The private objective of player i is to maximize the number of its neighbors that are in the other group.

Assuming that the social utility is the number of edges among the two groups, what can you say about PoA and PoS?

2.7. Assume that we have fixed a finite set K of k colors. Consider a graph $G = (V, E)$ with a labeling function $\ell : V \to 2^K \setminus \emptyset$ and define an associated *coloring game* $\Gamma(G, \ell)$ as follows

- the players are $V(G)$,
- the set of strategies for player v is $\ell(v)$,
- the payoff function of player v is $u_v(s) = |\{u \in N(v) \mid s_u = s_v\}|$.

Assuming that the social utility is $|\{(u, v) \in E \mid s_u = s_v\}|$, what can you say about the PoA and the PoS.