1 Strategic games

Definition A *fully mixed strategy* is a mixed strategy that assigns positive probability to all the possible actions.

1.7. Assume that we have fixed a finite set K of k colors. Consider a graph $G = (V, E)$ with a labeling function $\ell : V \to 2^K \setminus \{\emptyset\}$ and define an associated *coloring game* $\Gamma(G, \ell)$ as follows

- the players are $V(G)$,
- the set of strategies for player v is $\ell(v)$,
- the payoff function of player v is $u_v(s) = |\{u \in N(v) \mid s_u = s_v\}|$.

Provide a formal characterization of the strategy profiles that are pure Nash equilibrium of the coloring game game. Analyze the complexity of the problems related to pure Nash equilibria for this family of games.

1.8. Consider a network formation game where players are interested only in creating a connected network. In such a game we are given a connected graph G in which each edge e has a cost $c(e)$. Consider a game with one player per vertex in G. Player u can select any subset s_u of the edges incident with u in G. The cost for each player is ∞ if the subgraph resulting from the union of the selected edges is not connected, otherwise the cost for player u is the sum of the costs of the edges in s_u.

   (a) Can a best response be computed in polynomial time?

   (b) Provide a characterization of the PNE of the game. Can a PNE (if any) be computed in polynomial time?

1.9. Assume that we have a graph with edge and node non-negative weights, i.e. $G = (V, E, w, b)$ where $w : E \to \mathbb{Z}^+$ and $b : V \to \mathbb{Z}^+$. An *information gathering* game is defined on top of G as $\Gamma = (I, G)$ where $I \subseteq V$. The game has $m = |I|$ players. Player $i \in I$ can select any path in G starting at node i. The players use the selected path to gather the information hidden at the nodes of the graph. In order to traverse a path they have to pay the toll fees represented by the edge weights. The value of the information hidden in a node u is $b(u)$. However, the value of the information degrades proportionally to the number of players that discover such a piece of information.

- Provide a formal definition of the cost function for the information gathering game.
- Analyze the complexity of computing a best response.

1.10. Consider the *local diameter* network creation game in which we have a set V of n players. A player i can select strategically its set of neighbors from $V \setminus i$. For a given profile $s = (s_1, \ldots, s_n)$, let $G[s]$ be the undirected graph induced by the player's decisions. The cost for player $u \in V$ is defined by means of three parameters $\alpha, \beta, w \geq 0$ according to the following expression:

$$c_u(s) = \alpha|s_u| + w|\{v \mid d_{G[s]}(u, v) > \beta\}|.$$

We say that graph $G = (V, E)$ is a NE graph if there is a strategy profile s such that $G[s] = G$

Analyze under which parameter assignments the following graphs are NE graphs.

(a) S_n a star graph ($n-1$ vertices that form an independent set, all of them connected to the other vertex).

(b) K_n a complete graph.

(c) I_n an independen set.

(d) A tree with diameter β.

1.11. Consider the following strategic game:

		Player 2		
		R	S	T
Player 1	A	6,6	2,7	2,6
	B	7,2	3,4	0,0

Determine whether the strategy profile

$$(0.6, 0.4), (0.2, 0.4, 0.4)$$

is a Nash equilibrium.

1.12. Consider the two player's game described by the following bi-matrix.

		Player 2 A	Player 2 B
Player 1	C	1,1	4,2
Player 1	D	3,3	1,1
Player 1	E	2,2	2,3

Find all the NE for the game having a fully mixed strategy for each player or show that such Nash equilibria do not exists.

1.13. Consider the cover game defined in exercise 1.5. Is a fully mixed strategy profile a NE for the game?

1.14. Assume that we have a graph $G = (V, E)$ and two designated vertices s and t. To analyze the damage done to the network by edge failures the manager considers the following strategic game:

- The players are the edges of G.
- The set of actions for player $e \in E$ is either fail (1) or do not fail (0).
- A strategy profile s defines a graph $G(s) = (V, E_s)$, where $E_s = \{e, s_e = 0\}$.
- Let $D(s)$ be the maximum number of edge disjoint paths from s to t in $G(s)$.
- The cost function for player e, under strategy profile s, is

$$c_e(s) = s_e + D(s).$$

(a) Can a best response be computed in polynomial time?

(b) Show that network failure games have always a PNE.