1 Strategic games

1.1. The *cooperation* game is defined as follows. There is a group N of n people and a task to be performed. To perform correctly the task requires that exactly k persons cooperate. Each player can decide whether to cooperate (1) or not (0). The utility of a strategy profile $x \in \{1,0\}^i$ for player i is defined as

$$u_i(x) = \begin{cases} 1 & \text{the task is performed and } x_i = 1. \\ 0 & \text{otherwise} \end{cases}$$

- Provide a formal characterization of the best response sets, for player $i \in N$ and strategy profile x.
- Provide a formal characterization of the strategy profiles that are pure Nash equilibrium of the cooperation game.
- Analyze the computational complexity of the problems related to pure Nash equilibria for this family of games.

1.2. The *weak cooperation* game is defined as follows. There is a group N of n people and a task to be performed. To perform correctly the task requires that at least k persons cooperate. Each player can decide whether to cooperate (1) or not (0). The utility of a strategy profile $x \in \{1,0\}^i$ for player i is defined as

$$u_i(x) = \begin{cases} 1 & \text{the task is performed and } x_i = 1. \\ 0 & \text{otherwise} \end{cases}$$

- Provide a formal characterization of the best response sets for player $i \in N$ and strategy profile x.
- Provide a formal characterization of the strategy profiles that are pure Nash equilibrium of this game.
- Analyze the computational complexity of the problems related to pure Nash equilibria for this family of games.

1.3. The *split cooperation* game is defined as follows. There is a group N of n people and a task to be performed. To perform correctly the task requires that at least k persons cooperate. Each player can decide whether to cooperate (1) or not (0). The utility of a strategy profile $x \in \{1,0\}^i$ for player i is defined as

$$u_i(x) = \begin{cases} \frac{k}{|x|_1} & \text{the task is performed and } x_i = 1. \\ 0 & \text{otherwise} \end{cases}$$

where $|x|_1 = |\{i \mid x_i = 1\}|$.

- Provide a formal characterization of the best response sets for player $i \in N$ and strategy profile x.
- Provide a formal characterization of the strategy profiles that are pure Nash equilibrium of this cooperation game.
- Analyze the computational complexity of the problems related to pure Nash equilibria for this family of games.

1.4. The *matching* game is played in a bipartite graph $G = (V_1, V_2, E)$ in which edges are connect only vertices V_1 to vertices in V_2. The players are the vertices in the graph that is $V_1 \cup V_2$. Each player has to select one of its neighbors. Player i gets utility 1 when the selection is mutual (player i selects j and player j selects i) otherwise he gets 0.

Provide a formal characterization of the strategy profiles that are pure Nash equilibrium of the matching game. Analyze the complexity of the problems related to pure Nash equilibria for this family of games.

1.5. In the **cover game** the players are the vertices in an undirected graph $G = (V, E)$ on a set of n vertices. The goal of the game is to select a set of vertices X that covers a lot of edges. An edge is covered by a set X if at least one of its ends points belongs to X.

Formally, the set of actions allowed to player i is $A_i = \{0, 1\}$. Those players playing 1 will form the set. Let $s = (s_1, \dots, s_n)$, $s_i \in \{0, 1\}$, be an strategy profile, and let $X(s) = \{i \mid s_i = 1\}$.

The cost function for player $i \in V$ is defined as follows

$$c_i(s) = s_i + |\{(i, j) \in E \mid i, j \notin X(s)\}|.$$

- Provide a formal characterization of the best response set for player $i \in V$.
- Does this game have always a pure Nash equilibrium? If not, provide an example of a game in the family with no PNE.
- Analyze the computational complexity of the problems related to pure Nash equilibria for this family of games.

1.6. Consider a set of n players that must be partitioned into two groups. However, there is a set of bad pairings and the two players in such a pair do not want to be in the same group. Moreover, each player is free to choose which of the two groups to be in. We can model this by a graph $G = (V, E)$ where each player i is a vertex. There is an edge (i, j) if i and j form a bad pair. The private objective of player i is to maximize the number of its neighbors that are in the other group.

Provide a formal characterization of the strategy profiles that are pure Nash equilibrium of this game. Analyze the complexity of the problems related to pure Nash equilibria for this family of games.