Simple Games

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- A simple game can be described by a pair $(N, \mathcal{W})$:
  - N is a set of players,
  - $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotone set of winning coalitions, those coalitions X with $v(X) = 1$.
  - $\mathcal{L} = \mathcal{C}_N \backslash \mathcal{W}$ is the set of losing coalitions those coalitions X with $v(X) = 0$.

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- Members of $N = \{1, \ldots, n\}$ are called players or voters.

Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players N:

- *winning coalitions* $\mathcal{W}$.
- *losing coalitions* $\mathcal{L}$.
- *minimal winning coalitions* $\mathcal{W}^m$
  $\mathcal{W}^m = \{X \in \mathcal{W}; \forall Z \in \mathcal{W}, Z \not\subseteq X\}$
- *maximal losing coalitions* $\mathcal{L}^M$
  $\mathcal{L}^M = \{X \in \mathcal{L}; \forall Z \in \mathcal{L}, X \not\subseteq L\}$

This provides us with many representation forms for simple games.

Weighted voting games

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  $$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$
- WVG can be represented by a tuple of integers $(q; w_1, \ldots, w_n)$.
  as any weighted game admits such an integer realization,
  [Carreras and Freixas, Math. Soc.Sci., 1996]

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  Both are simple games
- A simple game Γ is a vector weighted voting game if there are WVGs $\Gamma_1, \dots, \Gamma_k$, for some $k \geq 1$, so that $\Gamma = \Gamma_1 \cap \dots \cap \Gamma_k$.

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  The game with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.
  - Assume it is given by $(q; w_1, w_2, w_3, w_4)$.
  - We have $w_1 + w_2 \geq q$ and $w_3 + w_4 \geq q$.
  - Thus $\max\{w_1, w_2\} \geq q/2$ and $\max\{w_3, w_4\} \geq q/2$,
  - So, $\max\{w_1, w_2\} + \max\{w_3, w_4\} \geq q$ which cannot be.

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    Any set that is not contained in C wins!
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    A winning coalition cannot be a subset of any losing coalition.
- The dimension of a simple games is the minimum number of WVGs that allows its representation as VWVG

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- The maximal losing coalitions are $\{\{1, 3\}, \{1, 4\}, \{2, 3\}\{2, 4\}\}$
- This gives four WVG, according to the previous construction

$$\Gamma = [1; 0, 1, 0, 1] \cap [1; 0, 1, 1, 0] \cap [1; 1, 0, 0, 1] \cap [1; 1, 0, 1, 0].$$

Input representations

- Simple Games
  $(N, \mathcal{W})$: extensive wining, $(N, \mathcal{W}^m)$: minimal wining
  $(N, \mathcal{L})$: extensive losing, $(N, \mathcal{L}^M)$ maximal losing
  (N, C): monotone circuit winning
  (N, F): monotone formula winning,
- Weighted voting games: $(q; w_1, \ldots, w_n)$
- Vector weighted voting games: $(q_1; w_1^1, \ldots, w_n^1), \ldots, (q_k; w_1^k, \ldots, w_n^k)$

All numbers are integers

Problems on simple games

In general we state a property P, for simple games, and consider the associated decision problem which has the form:

> *Name:* IsP
> *Input: A simple game/WVG/VWVG* Γ
> *Question: Does* Γ *satisfy property* P*?*

Four properties

A simple game $(N, \mathcal{W})$ is

- strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$.
- proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.
- a weighted voting game.
- a vector weighted voting game.

IsStrong: Simple Games

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Theorem

The IsStrong *problem, when Γ is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.*

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Theorem

The IsStrong *problem, when Γ is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.*

Proof

- First observe that, given a family of subsets F, we can check, for any set in F, whether its complement is not in F in polynomial time.
- Therefore, the IsStrong problem, when the input is given in explicit losing form is polynomial time solvable.

IsStrong: Simple Games loosing forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \ \wedge \ N \setminus S \in \mathcal{L}$$

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  which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in \mathcal{L}^M : S \subseteq L_1 \;\wedge\; N \setminus S \subseteq L_2$$

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- This can be checked in polynomial time, given $\mathcal{L}^M$.

IsStrong: explicit winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- Given $(N, \mathcal{W})$, for $i \in N$ consider the family $\mathcal{W}_i = \{X \setminus \{i\} \mid X \in \mathcal{W}\}$ and $R = \cup_{i \in N} \mathcal{W}_i$.
- All the coalitions in $R \setminus W$ are losing coalitions.
- Furthermore for a coalition $X \in \mathcal{L}^M$ and $i \notin X$, $X \cup \{i\} \in \mathcal{W}$.
- Thereofore, $\mathcal{L}^M \subseteq R \setminus W$ and $(R \setminus W)^M = \mathcal{L}^M$.
- Then, we compute $\mathcal{L}^M$ from $\mathcal{W}$ in polynomial time and then use the algoritm for the maximal losing form.

end proof

IsStrong: minimal winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The IsStrong *problem is coNP-complete when the input game is given in explicit minimal winning form.*

Proof

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Proof

- The property can be expressed as

$$\forall S \ [(S \in \mathcal{W}) \text{ or } (S \notin \mathcal{W} \text{ and } N \setminus S \in \mathcal{W})]$$

- Observe that the property $S \in \mathcal{W}$ can be checked in polynomial time given S and $\mathcal{W}^m$.
- Thus the problem belongs to coNP.

IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.
- An instance of the set splitting problem is a collection C of subsets of a finite set N. The question is whether it is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$.

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- We have to decide whether $P \subseteq N$ exists such that

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We associate to a set splitting instance (N, C) the simple game in explicit minimal winning form (N, C^m).

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- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in C contains a set in C^m.
- Therefore, (N, C) has a set splitting iff (N, C^m) is not strong.

end proof

IsProper: winning forms

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The IsProper *problem, when the game is given in explicit winning or losing form or in minimal winning form, can be solved in polynomial time.*

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Proof

- As before, given a family of subsets F, we can check, for any set in F, whether its complement is not in F in polynomial time.
  Taking into account the definitions, the IsProper problem is polynomial time solvable for the explicit forms

IsProper: winning forms

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- Which can be checked in polynomial time when $\mathcal{W}^m$ is given.

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IsProper: maximal losing form

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Theorem

The IsProper problem is coNP-complete when the input game is given in extensive maximal losing form.

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$$\exists S \subseteq N : S \notin \mathcal{L} \; \wedge \; N \setminus S \notin \mathcal{L}$$

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$$\exists S \subseteq N : \forall T_1, T_2 \in \mathcal{L}^M : S \not\subseteq T_1 \ \wedge \ N \setminus S \not\subseteq T_2$$

- Therefore IsProper belongs to coNP.

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To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

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- If a family C of subsets of N is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, \mathcal{W}^m)$, in minimal winning form, we construct the game $\Gamma' = (N, \{N \setminus L : L \in \mathcal{W}^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.

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- Which can be obtained in polynomial time.
- Besides, Γ is strong iff Γ' is proper.

end proof

Explicit forms

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit winning or losing form.*

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We can obtain $\mathcal{W}^m$ and $\mathcal{L}^M$ in polynomial time.
Once this is done we write, in polynomial time, the LP

$$
\begin{array}{lll}
\texttt{min } q & & \\
\texttt{subject to} & w(S) \geq q & \texttt{if } S \in W^m \\
& w(S) < q & \texttt{if } S \in L^M \\
& 0 \leq w_i & \texttt{for all } 1 \leq i \leq n \\
& 0 \leq q &
\end{array}
$$

IsWeighted: Minimal and Maximal

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

Proof

IsWeighted: Minimal and Maximal

Lemma

The ISWEIGHTED *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

Proof

- For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the i'th coordinate equal to 1 if and only if $i \in C$.

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Proof

- For $C \subseteq N$ we let $x_C \in \{0,1\}^n$ denote the vector with the i'th coordinate equal to 1 if and only if $i \in C$.
- In polynomial time we compute the boolean function Φ_{W^m} given by the DNF:

$$\Phi_{W^m}(x) = \bigvee_{S \in W^m} (\wedge_{i \in S} x_i)$$

IsWeighted: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

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By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

- It is well known that Φ_{W^m} is a threshold function iff the game given by (N, W^m) is weighted.
- Further Φ_{W^m} is monotonic (i.e. *positive*)
- But deciding whether a monotonic formula describes a threshold function can be solved in polynomial time.

IsWeighted: Maximal loosing

- we can prove a similar result given (N, L^M).
- The dual of game $\Gamma = (N, \mathcal{W})$ is the game $\Gamma^d = (N, \mathcal{W}^d)$ where $S \in \mathcal{W}^d$ iff $N \setminus S \notin \mathcal{W}$.
- Observe that Γ is weighted iff Γ^d is weighted.
- We can compute a monotone CNF formula describing the loosing coalitions of Γ. Negating this formula we get a DNF on negated variables. Replacing $\overline{x}_i$ by y_i we get a DNF describing $\mathcal{W}^d$.
- As the formula can be computed in polynomial time the result follows.

end proof

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- Any simple game is a vector weighted voting game.
- The dimension of a simple games is the minimum number of WVGs that allows its representation as VWVG

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Theorem

For a simple game $\Gamma = (N, \mathcal{L}^M)$, $dim(\Gamma) \leq |\mathcal{L}^M|$.

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- The intersection of the above games describes Γ, as a minimal winning coalition cannot be a subset of any losing coalition.

□

Hardness

Theorem

Let d_1 and d_2 two fixed integers with $1 \leq d_2 < d_1$. Then the problem of deciding whether the intersection of d_1 WVGs can be represented as the intersection of d_2 WVGs is NP-hard.

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The reduction is from the NP-complete problem

Name: SUBSET SUM

Input: $n + 1$ *integer values,* $x_1, \ldots, x_n$ *and* b

Question: Is there $S \subseteq \{1, \ldots, n\}$ *for which*

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- So, $\Gamma(I)$ is equivalent to the WVG in which $w(p_i) = x_i$, $1 \leq i \leq k$, $w(q_j) = w(q'_j) = 0$, $1 \leq j \leq d$, and $q = b + 1$

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- Let $\alpha \neq \beta$ such that $h(\alpha) = h(\beta) = \gamma$, consider H_γ
  - Let W be the total weight of players p_i, $1 \leq i \leq k$, and q_j, q'_j, $j \notin \{\alpha, \beta\}$
  - Let w_α, w_β, w'_α, w'_β the weight of players q_α, q_β, q'_α, q'_β.
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- The coalition with players p_i with $i \in J$ and $\{q_1, \dots, q_d\}$ wins in $\Gamma(I)$, so $W + w_\alpha + w_\beta \geq q^*$.
- The coalition with players p_i with $i \in J$ and $\{q'_1, \dots, q'_d\}$ wins in $\Gamma(I)$, so $W + w'_\alpha + w'_\beta \geq q^*$.
- So, $2W + w_\alpha + w_\beta + w'_\alpha + w'_\beta$ must be $> q^*$ and $\leq q^*$, we get a contradiction

End Proof

Hardness

Theorem

Computing the dimension of a VWVG is NP-hard

Theorem

The IsWeighted *problem is NP hard for VWVGs*

The core

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Proof.

- If i is a veto player in Γ, the payoff $x_i = 1$, $x_j = 0$, is in the core as any coalition containing i gets 1.
- If Γ has no veto player and x is in the core, $x(N) = 1$.
- If there is $i \in N$ with $x_i > 0$, so $x(N \setminus \{i\}) < 1$.
- But as i is not a veto player $v(N \setminus \{i\}) = 1$.

□

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