Cooperative Game Theory: Solution concepts

Spring 2024

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- Notation: N, set of players, $C, S, X \subseteq N$ are coalitions.

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  N is the grand coalition.
- A partition of N is a splitting of all the players into disjoint coalitions.

Characteristic Function Games

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- usually it is assumed that v is
  - normalized: $v(\emptyset) = 0$,
  - non-negative: $v(C) \geq 0$, for any $C \subseteq N$, and
  - monotone: $v(C) \leq v(D)$, for any C, D such that $C \subseteq D$
- Example: $N = \{A, B, C\}$ and

$\mathcal{C}_N$	$\emptyset$	A	B	C	AB	AC	BC	ABC
v	0	12	0	0	18	18	18	24

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- The children have utility for ice-cream but do not care about money.
- The payoff of each group is the maximum quantity of ice-cream the members of the group can buy by pooling all their money.
- The ice-cream can be shared arbitrarily within the group.

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- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 750, v(\{C, P\}) = 750, v(\{M, P\}) = 500$
- $v(\{C, M, P\}) = 1000$

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  - $x_i \geq 0$, for all $i \in N$
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- $((\{1,2,3\},\{4,5\}),(3,3,3,3,1))$ is an outcome
- $((\{1,2,3\},\{4,5\}),(2,3,2,3,3))$ is NOT an outcome as transfers between coalitions are not allowed

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Notation: we denote $\sum_{i \in A} x_i$ by $x(A)$

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- To simplify the presentation we consider superadditive games.

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- A game $G = (N, v)$ is called superadditive if

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- Example: $v(C) = |C|^2$

$$v(C \cup D) = (|C| + |D|)^2 \geq |C|^2 + |D|^2 = v(C) + v(D)$$

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  i.e., an outcome is a vector $x = (x_1, \ldots, x_n)$ with $x(N) = v(N)$

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Ice-cream pots: $w = (500, 750, 100)$ and $p = (\$7, \$9, \$11)$

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(200, 200, 350) is not stable!

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  as $x(\{2,4\}) = 6$ and $v(\{2,4\}) = 7$
- no subgroup of players can deviate so that each member of the subgroup gets more.

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- $(750,\ 0,\ 0)$ is also in the core:
  Marcie and Pattie cannot get more on their own!

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  - We have $x_1, x_2, x_3 \geq 0$, $x_1 + x_2 + x_3 = 1$, and $x_i + x_j = 1$, for $i \neq j$
  - As, $x_1 + x_2 + x_3 \geq 1$, for some $i \in \{1, 2, 3\}$, $x_i \geq 1/3$.
  - Assume that $i = 1$, we have $x_2 + x_3 = 1 - x_1 \leq 1 - 1/3 \leq 1$!

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  - As, $x_1 + x_2 + x_3 \geq 1$, for some $i \in \{1, 2, 3\}$, $x_i \geq 1/3$.
  - Assume that $i = 1$, we have $x_2 + x_3 = 1 - x_1 \leq 1 - 1/3 \leq 1!$
- Thus the core of Γ is empty.

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  $v(C) = 1$ if $|C| > 1$ and $v(C) = 0$ otherwise
  - not superadditive: $v(\{1, 2\}) + v(\{3, 4\}) = 2 > v(\{1, 2, 3, 4\})$

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  - But $((\{1, 2\}, \{3, 4\}), (1/2, 1/2, 1/2, 1/2))$ is in the core

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- For the example, the least core is the $1/3$-core.

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- How do we divide payoffs in a fair way?

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- z1 $=$ (x1 $+$ y1)/2 $=$ 10, z2 $=$ (x2 $+$ y2)/2 $=$ 10
  the resulting outcome is fair!
- Can we generalize this idea?

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- The Shapley value of player i in a game $\Gamma = (N, v)$ with n players is

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- In the previous slide we have $\Phi_1 = \Phi_2 = 10$

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- Φ_i is i's average marginal contribution to the coalition of its predecessors, over all permutations
- Suppose that we choose a permutation of players uniformly at random, then Φ_i is the expected marginal contribution of player i to the coalition of his predecessors

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- Two players i and j are said to be symmetric in Γ if

$$v(C \cup \{i\}) = v(C \cup \{j\}), \text{ for any } C \subseteq N \setminus \{i, j\}$$

Shapley value: Axiomatic Characterization

Properties of the Shapley value:

- Efficiency: $\Phi_1 + ... + \Phi_n = v(N)$
- Dummy: if i is a dummy, $\Phi_i = 0$
- Symmetry: if i and j are symmetric, $\Phi_i = \Phi_j$
- Additivity: $\Phi_i(\Gamma_1 + \Gamma_2) = \Phi_i((\Gamma_1) + \Phi_i(\Gamma_2)$

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Theorem

The Shapley value is the only payoff distribution scheme that has properties (1) - (4)

$\Gamma = \Gamma_1 + \Gamma_2$ is the game (N, v) with $v(C) = v_1(C) + v_2(C)$

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The Banzhaf index of player i in game $\Gamma = (N, v)$ is

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Dummy player, symmetry, additivity, but not efficiency.

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  - Extensive list values of all coalitions
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    some undecidable questions might arise
- We are usually interested in algorithms whose running time is polynomial in n
- So what can we do? subclasses?

Checking Non-emptiness of the Core: Superadditive Games

- An outcome in the core of a superadditive game satisfies the following constraints:

$$x_i \geq 0 \text{ for all } i \in N$$
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- A linear feasibility program, with one constraint for each coalition: $2^n + n + 1$ constraints

Superadditive Games: Computing the Least Core

- Starting from the linear feasibility problem for the core

$$\min \epsilon$$

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- A minimization program, rather than a feasibility program

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  Convergence guaranteed by Law of Large Numbers

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- Observe that $v(\emptyset) = 0$ and $v(N) = w(E)$.

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- Weights can be exponential in n and still have polynomial size.

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$$v(C) = \begin{cases} 0 & if |C| \leq 1 \\ 1 & if |C| = 2 \\ 6 & if |C| = 3 \end{cases}$$

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  - By the first condition all self-loops must have weight 0.
  - By the second condition any pair of different vertices must be connected by an edge with weight 1. So G must be a triangle.

Completeness?

- Is this is a complete representation?
  All coalitional games can be represented as induced subgraph games?
  NO
  Consider the game $\Gamma = (N, v)$, where $n = \{1, 2, 3\}$ and

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- Assume that $\Gamma(G, w)$ realizes Γ.
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  - But then $v(\{1, 2, 3\}) = 3 \neq 6$.

Properties of valuations

- monotone if $v(C) \leq v(D)$ for $C \subseteq D \subseteq N$.
- superadditive if $v(C \cup D) \geq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
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- A game (N, v) is convex iff v is supermodular.
- Since we allow for negative edge weights, induced subgraph games are not necessarily monotone.
- However, when all edge weights are non-negative, induced subgraph games are convex.

Can the core be empty?

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The core of $\Gamma(N, v)$ is the set of all imputations x such that $v(S) \leq x(S)$, for each coalition $S \subseteq N$.

Can the core be empty?

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Theorem

If $\Gamma = (N, v)$ *is a convex game, then* Γ *has a non-empty core.*

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- Fix an arbitrary permutation π, and let x_i be the marginal contribution of i with respect to π.

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- Fix an arbitrary permutation π, and let x_i be the marginal contribution of i with respect to π.
- Let us show that $(x_1, ..., x_n)$ is in the core of Γ.
  - For $C \subseteq N$, we can assume that $C = \{i_1, \ldots, i_s\}$ where $\pi(i_1) < \cdots < \pi(i_s)$.
  - So,
    $v(C) = v(\{i_1\}) - v(\emptyset) + v(\{i_1, i_2\}) - v(\{i_1\}) + \cdots + v(C) - v(C \setminus \{i_s\})$.
  - By supermodularity we have,
    $v(\{i_1, \ldots, i_{j-1}, i_j\}) - v(\{i_1, \ldots, i_{j-1}\}) \leq v(\{1, \ldots, i_j\}) - v(\{1, \ldots, i_{j-1}\})$.
  - Therefore $v(C) \leq x(C)$ and $v(N) = x(N)$.

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  - Therefore $v(C) \leq x(C)$ and $v(N) = x(N)$.
- Observe that we have shown that the vector formed by the Shapley value is in the core of a convex game.

Computing the Shapley value

- For $C \subseteq N$, let $\delta_i(C) = v(C \cup \{i\}) - v(C)$
- The Shapley value of player i in a game $\Gamma = (N, v)$ with n players is

$$\Phi_i(\Gamma) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \delta_i(S_\pi(i))$$

Properties of the Shapley value:

- Efficiency: $\Phi_1 + \ldots + \Phi_n = v(N)$
- Dummy: if i is a dummy, $\Phi_i = 0$
- Symmetry: if i and j are symmetric, $\Phi_i = \Phi_j$
- Additivity: $\Phi_i(\Gamma_1 + \Gamma_2) = \Phi_i((\Gamma_1) + \Phi_i(\Gamma_2)$

Theorem

The Shapley value is the only payoff distribution scheme that has properties (1) - (4)

Computing the Shapley value

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Theorem

The Shapley value of player i in $\Gamma(G, w)$ is

$$\Phi(i) = \frac{1}{2} \sum_{(i,j) \in E} w_{i,j}.$$

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- When i is not incident to e_j, i is a dummy in Γ_j and $\Phi_i(\Gamma_j) = 0$.
- When $e_j = (i, \ell)$ for some $\ell \in N$, players i and ℓ are symmetric in Γ_j.
- Since the value of the grand coalition in Γ_j equals $w(i, \ell)$, by efficiency and symmetry we get $\Phi_i(\Gamma_j) = w(i, \ell)/2$.

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Theorem

The Shapley value of player i in $\Gamma(G, w)$, when w is positive, is

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Corollary

The Shapley values of induced subgraph games can be computed in polynomial time.
Checking if the core is non-empoty for positive induced subgraph games can be done in polynomial time

Complexity of core related problems

Theorem

The following problems are NP-hard:

- *Given* (G, w) *and an imputation* x*, is it not in the core of* $\Gamma(G, w)$*?*
- *Given* (G, w)*, is the vector of Shapley values of* $\Gamma(G, w)$ *not in the core of* $\Gamma(G, w)$*?*
- *Given* (G, w)*, is the core of* $\Gamma(G, w)$ *empty?*

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Theorem

Given (G, w), when all weights are non-negative, we can test in polynomial time

- *whether the core is non-empty.*
- *whether an imputation x is in the core of $\Gamma(G, w)$.*

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Given (G, w), when all weights are non-negative, we can test in polynomial time

- *whether the core is non-empty.*
- *whether an imputation x is in the core of $\Gamma(G, w)$.*

The first question is trivial as the vector of Shapley values belong to the core. The second problem can be solved by a reduction to MAX-FLOW.

MST Games

Minimum cost spanning tree games

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- The cost of a singleton coalition $\{i\}$ is $c(\{i\}) = w_{0,i}$.
- Observe that $v(\emptyset) = 0$ and $v(N) = w(T)$ where T is a MST of G.

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- The representation is succinct as long as the number of bits required to encode edge weights is polynomial in $|N|$: using an adjacency matrix to represent the graph requires only n^2 entries.

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  - By the first condition $w_{0,i} = 0$, for $i \in \{1, 2, 3\}$.
  - Thus, a coalition with $|C| = 2$ has a MST with zero cost and the second condition cannot be met.

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- MST games are not necessarily monotone. Consider a triangle on $V = \{0, 1, 2\}$ and weights $w_{0,1} = 1$, $w_{0,2} = 10$ and $w_{1,2} = 1$ $c(N) = 2$ and $c(\{1\}) = 1$ and $c(\{2\}) = 10$

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  $c(N) = 2$ and $c(\{1\}) = 1$ and $c(\{2\}) = 10$
- c is subadditive.

Can the core be empty?

Theorem

Consider a MST game $\Gamma(G, w)$. Let T^* be a MST of (G, w) obtained using Prim's algorithm. The vector $x = (x_1, \ldots, x_n)$ that allocates to player $i \in N$ the weight of the first edge i encounters on the (unique path) from v_i to v_0 in T^* belongs to the core of Γ.

Such an allocation is called standard core allocation

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- For j in S, let e_j be the first edge j encounters on the path from v_j to v_0 in T and let $y_j = w(e_j)$.

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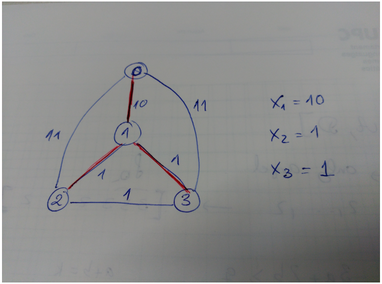
- The selected edge corresponds to the point in which Prim's algorithm connects the vertex to the component including v_0, i.e., it is a minimum weight edge in the allowed cut.
- Analyzing carefully both executions it can be shown that $x_j \leq y_j$ as the edges considered in one partition are a subset of the other.

How fair are standard core allocations?



- Most of the cost is charged to player 1.
- How to find more appropriate core allocations?

More appropriate core allocations?

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- Granot and Huberman [1984] prose the weak demand allocation and strong demand allocation procedures. Which rectify standard allocations by transfering cost (whenever possible) from one node to their children.
- Norde, Moretti and Tijs [2001] show how to find a population monotonic allocation scheme (PMAS), which is an allocation scheme that provides a core element for the game and all its subgames and which, moreover, satisfies a monotonicity condition in the sense that players have to pay less in larger coalitions.

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The proof follows by a reduction from EXACT COVER BY 3-SETS [Faigle et al., Int. J. Game Theory 1997]

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