

Auctions: An introduction to mechanism design

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- 1 Auctions
- 2 Truth telling
- 3 Revenue
- 4 Selling one item
- 5 VCG mechanism
- 6 Sponsored search

Context

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How to sell items to potential buyers with private valuations.
- What is the **right** price for objects? groups of objects?
- Objectives:
 - Truth-telling
 - Efficiency: **social welfare**
 - Revenue: **maximize profit**
 - Envy-freeness :

Not all of them can be achieved at the same time.

Prices

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- How can the true preferences be revealed?
- At which cost?

Auction theory

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- **Aim:** Design and analyze the rules and properties of an auction.
- **Goal:** Design an auction so that **in equilibrium** we get the results we want.
- As in Game theory we rely on rationality.

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 - The set of possible **resource allocations**.
The number (or portion) of goods of each type including legal or other restrictions on how the goods may be allocated.
 - Rules for bidding and clearing.
 - A procedure to determine **who wins what** (allocation) and **how much pays** (payment) on the basis of the received information.

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- When?
- What?
- To whom?

Single item auctions

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- We analyze three mechanisms



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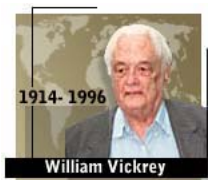
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Second price (SP) Auction

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- The bidders write down a **price** and send it to the auctioneer.
- The auctioneer awards the good to **the bidder with the highest bid**.
- The winner **pays the amount bid by the second-highest bidder**.
- Second price auctions are also known as **Vickrey auctions**, defined by William Vickrey in 1961. Vickrey won the Nobel prize in Economics in 1996.



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- The auctioneer awards the good to the bidder with the highest bid.
- Everyone pays the amount of their bid regardless of whether or not they get the good.

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Single item auction: model

- n bidders
- Each bidder has value v_i for the item **willingness to pay**.
Known only to him – **private value**.
- If Bidder i wins and pays p_i , his utility is $v_i - p_i$.
Her utility is 0 when she loses.

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- Bidders prefer losing than paying more than their value.**
- Bidders have to decide on a strategy to bid, a function applied to their valuation.

SP-Auctions: Equilibrium behaviour

Theorem

In SP-price auctions truth-telling is a dominant strategy.

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 - With the actual highest value, not just the highest bid.
 - Without assuming anything on the values.
- However the seller does not get maximum revenue.

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Hard to select an strategy without some information about the others. We continue the analysis on a Bayesian setting.

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FP auctions: Bayesian analysis

- How do people behave?
- They have **beliefs** on the valuations of the other players!
- As usual beliefs are modeled with probability distributions.
- Bidders do not know their opponent's values, i.e., there is **incomplete information**.
Each bidder's strategy must **maximize her expected payoff** accounting for the uncertainty about opponent values.

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- What is the equilibrium in this game of incomplete information?

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2 bidders uniform distribution

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$$\begin{aligned} \text{Prob}[b_1 > b_2] (v_1 - b_1) &= \\ &= \text{Prob}[b_1 > v_2/2] (v_1 - b_1) \\ &= 2b_1 (v_1 - b_1) \end{aligned}$$

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- maximizing for b_1 we have: $[2b_1 (v_1 - b_1)]' = 2v_1 - 4b_1 = 0$
which gives $b_1 = v_1/2$

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Theorem

In a FP auction with n bidders under the uniform values model, the strategy $b_i = \frac{n-1}{n} v_i$, for $1 \leq i \leq n$, is a Bayesian Nash equilibrium.

FP uniform values: Efficiency

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FP uniform values: Efficiency

- An auction is **efficient** if, in a Bayesian Nash equilibrium, the bidder with the highest value always wins.
- Thus, in the uniform value model FP is efficient.

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- Can we have a truthful auction giving maximum revenue?

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- When analyzing revenue take into account that when the item is not sold the seller gets a benefit of u .

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- With more intricate analyses, you can determine the optimal reserve price for a second-price auction with multiple bidders

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Buyer's utility

- Bidders have private values v_i for the item
- A winning bidder gets a utility of $u_i = v_i - p - i$
- A losing bidder pays nothing and gets $u_i = 0$

Seller's incentive

- **Maximize social welfare:** SP auctions
- **Maximize revenue:** SP auctions with reserve price
 - For example, reserve-pr $[0,100]$
 - Reserve price is independent of the number of players
 - Optimality assumes a technical assumption on the distributions.
- Revenue equivalence results

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Can this be achieved in other models?

- Moving from a specific example (single-item auctions) to a more general mechanism design setting.
- **Objective:** Design the right incentives such that the efficient outcome will be chosen.

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- They should decide:
 - Do they want to buy a TV together?
 - If so, how should they share the costs?
- Individual preferences have to be taken into consideration in deciding the efficient outcome.

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- You can maximize efficiency by:
 - Choosing the efficient outcome (given the bids) as allocation.
 - Each player pays his **social cost**.

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 - In a single item auction **when i wins the object** this payment is
2nd highest bid minus 0

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- VCG rules for k -item auctions:

- Highest k bids win.
- The winners pay the $(k + 1)$ st bid.

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- Highest k bids win.
- The winners pay the $(k + 1)$ st bid.

Here, again, truthfulness is a dominant strategy.

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 - But, total payment is $20 + 30 < 100!$
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 There is no mechanism that is both efficient and budget balanced.

- 1 Auctions
- 2 Truth telling
- 3 Revenue
- 4 Selling one item
- 5 VCG mechanism
- 6 Sponsored search**

Keyword Auctions/ Sponsored search

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 - Multiple positions, but advertisers submit only a single bid.
 - Search is highly targeted, and transaction oriented.

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- Late 1990s: Yahoo! and MSN adopt Overture, but mechanism proves unstable. Advertisers constantly change bids to avoid paying more than necessary.
- 2002: Google modifies keyword auction to have advertisers pay minimum amount necessary to maintain their position (GSP) - followed by Yahoo! and MSN.

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In the simplest model $\gamma_i = 1$.

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- If advertiser i is assigned to slot j at a price of p_i per click then her utility is

$$u_i = \alpha_j \gamma_i (v_i - p_i),$$

which is the number of clicks received times profit per click.

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$$\begin{aligned}
 SW(p, \pi, v, \gamma) &= \sum_{i=1}^n \alpha_{\pi^{-1}(i)} \gamma_i (v_i - p_i) + \sum_{j=1}^n \alpha_j \gamma_{\pi(j)} p_{\pi(j)} \\
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The social welfare is independent of the payments and the bids!

$$SW(\pi, v, \gamma)$$

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Exercise: what would be the prices in the VCG auction?

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- That is, the GSP mechanism assigns slots with higher click-through-rate to advertisers with higher effective bids.

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- In the case $\gamma_i = 1$, for each i ,

$$p_i = b_{\pi(k+1)}.$$

GSP:utility

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- $u_i(b, \gamma)$ is the utility derived by advertiser i from the GSP mechanism when advertisers bid according to b :

$$\begin{aligned} u_i(b, \gamma) &= \alpha_{\pi^{-1}(i)} \gamma_i (v_i - p_i) \\ &= \alpha_{\pi^{-1}(i)} [\gamma_i v_i - \gamma_{\pi(\pi^{-1}(i)+1)} b_{\pi(\pi^{-1}(i)+1)}]. \end{aligned}$$

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- If competing bids are 6 and 8, better to bid 10...
- It is not a dominant strategy to bid “truthfully”

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and, for $k \geq j$,

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- A player increasing his bid can only acquire a higher slot paying not the price the player in this slot is paying but its bid.

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- Is there an efficient NE?

GSP: Envy-free equilibrium

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Definition

Given a GSP with n players defined by click-through-rates

$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$, quality scores $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n$ and valuations v_1, \dots, v_n .

A bid vector b is an *envy-free* equilibrium if, for any pair j, k of players, player j would not prefer player k 's allocation and payments rather than their own.

Formally

$$\alpha_j(\gamma_{\pi(j)} v_{\pi(j)} - \gamma_{\pi(j+1)} b_{\pi(j+1)}) \geq \alpha_k(\gamma_{\pi(j)} v_{\pi(j)} - \gamma_{\pi(k+1)} b_{\pi(k+1)})$$

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This is a envy-free equilibrium!

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- Take

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The (pure) PoA of GSP in the full information setting is at most the golden ratio $\frac{1}{2}(1 + \sqrt{5}) \approx 1.618$

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- **Individual Rationality**: Each player has net non-negative utility from participating in the auction, i.e., $u_i \geq 0$.
- **Incentive compatibility** (a.k.a. truthfulness): It is a dominant strategy for each player to participate in the auction and report their true value.
- **Pareto-optimality**: An allocation π and payments p is **Pareto-optimal** if and only if there is no alternative allocation and payments where all players' utilities and the revenue of the auctioneer do not decrease, and at least one of them increases.