Computational aspects of finding Nash Equilibria for 2-player games

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Spring 2023

1. Linear Algebra formulation
2. Zero-sum games
3. The complexity of finding a NE
4. An exact algorithm to compute NE
5. NE algorithms

Nash equilibrium

Consider a 2-player game $\Gamma = (A_1, A_2, u_1, u_2)$.
Let $X = \Delta(A_1)$ and $Y = \Delta(A_2)$.

($\Delta(A)$ is the set of probability distributions over A)

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A Nash equilibrium is a mixed strategy profile $\sigma = (x, y) \in X \times Y$ such that, for every $x' \in X$, $y' \in Y$, it holds

$$U_1(x, y) \geq U_1(x', y) \text{ and } U_2(x, y) \geq U_2(x, y')$$

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A best response can be computed in polynomial time for 2-player games with rational utilities.

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- In terms of matrices we have $C = -R$.

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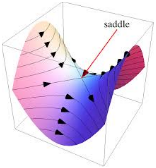

i.e., (x^*, y^*) is a saddle point
of the function $x^T R\, y$ defined over $X \times Y$.

Minimax inequality

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For any function $\Phi : X \times Y :\to \mathbb{R}$, *we have*

$$\sup_{x\in X} \inf_{y\in Y} \Phi(x,y) \leq \inf_{y\in Y} \sup_{x\in X} \Phi(x,y).$$

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Taking the supremum over $x' \in X$ on the left hand-side we get the inequality. $\square$

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We refer to $\inf_{y\in Y} \sup_{x\in X} x^T R\, y$ as the value of the game.

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$$v\mathbf{1}_n \geq Ry, y \in Y.$$

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- LP can be solved efficiently, thus there is a polynomial time algorithm for computing NE for zero-sum games.

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Polynomial Parity Argument on Directed Graphs

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- Such problems are defined by an implicitly defined directed graph G and an unbalanced node u of G
  and the objective is finding another unbalanced node.
- Usually G is huge but implicitly defined as the graphs defining solutions in local search algorithms.

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  has complete problems.
- But not a clear complexity cut.

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  Every directed graph with in/outdegree 1 and a source, has a sink.

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- We know that
  Every directed graph with in/outdegree 1 and a source, has a sink.
- Which guarantees that
  the End-of-Line problem has always a solution.

End-of-Line: graph representation

- G is given implicitly by a circuit C
- C provides a predecessor and successor pair for each given vertex in G, i.e. $C(u) = (v, w)$.
- A special label indicates that a node has no predecessor/successor.

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- C. Daskalakis, P-W. Goldberg, C.H. Papadimitriou: The complexity of computing a Nash equilibrium. SIAM J. Comput. 39(1): 195-259 (2009) first version STOC 2006
- X. Chen, X. Deng, S-H. Teng: Settling the complexity of computing two-player Nash equilibria. J. ACM 56(3) (2009) first version FOCS 2006

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NE characterization

Theorem

In a strategic game in which each player has finitely many actions a mixed strategy profile σ^* is a NE iff, for each player i,

- *the expected payoff, given σ_{-i}, to every action in the support of σ_i^* is the same*
- *the expected payoff, given σ_{-i}, to every action not in the support of σ_i^* is at most the expected payoff on an action in the support of σ_i^*.*

NE conditions given support

Let $A \subseteq \{1, \dots n\}$ and $B \subseteq \{1, \dots m\}$.
The conditions for having a NE on this particular support can be written as follows:

$$\max \lambda_1 + \lambda_2$$

Subject to:

$$\begin{aligned}
[R\,y]_i &= \lambda_1, \text{ for } i \in A \\
[R\,y]_i &\le \lambda_1, \text{ for } i \notin A \\
{}_j[C\,x] &= \lambda_2, \text{ for } j \in B \\
{}_j[C\,x] &\le \lambda_2, \text{ for } j \notin B
\end{aligned}$$

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  Solve the set of simultaneous equations using linear programming.
- This is an exact exponential time algorithm as the number of supports can be exponential.
- The same algorithm can be applied to a multiplayer game. We would be able to compute a NE on rationals if such a NE exists.

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NE algorithms

- Lemke-Howson (1964) algorithm defines a polytope based on best response conditions and membership to the support and uses ideas similar to Simplex with a ad-hoc pivoting rule.
  (See slides by Philippe Bich)

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- Mixed-Integer Programming formulations [Sandholm, Gilpin and Conitzer, AAAI-05]