An introduction to Computational Social Choice

Spring 2024

AGT-MIR	

Cooperative Game Theory

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2 Some properties of voting rules

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Social Choice Theory

- Mathematical theory for aggregating individual preferences into collective decisions
- Originated in ancient Greece. Formal foundations:
 - 18th Century (Condorcet and Borda)
 - 19th Century: Charles Dodgson (a.k.a. Lewis Carroll)
 - 20th Century: Nobel prizes to Arrow and Sen
- Objective: Methods to select a collective outcome based on (possibly different) individual preferences.

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Social Choice Theory

- Set of voters $N = \{1, \ldots, n\}$
- Set of alternatives $A = \{1, \dots, m\}$
- Voter *i* has a preference ranking over alternatives \succ_i
- Preference ranking \succ is the collection of all voters' rankings

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Ply	1	2	3
	а	С	b
	b	а	с
	с	b	а

• Social choice function

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Social choice function

- Takes as input a preference profile \succ
- Returns an alternative $a \in A$

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Social welfare function

- Takes as input a preference profile \succ
- Returns a societal preference on $A \succ_s$

Social choice function

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Social welfare function

- Takes as input a preference profile \succ
- Returns a societal preference on $A \succ_s$
- voting rule = social choice function

Plurality

Voting rules: Plurality

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- Each voter awards one point to her top alternative
- Alternative with the most point wins

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Plurality

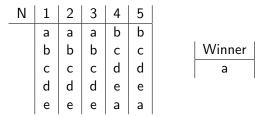
Voting rules: Plurality

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Ν	1	2	3	4	5	
	а	а	а	b	b	
	b	b	b	с	с	
	с	с	с	d	d	
	d	d	d	е	е	
	e	e	е	а	а	

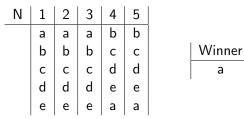
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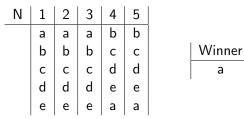
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- Most frequently used voting rule
- Many political elections use plurality

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Problems?

Voting rules: Borda

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- Each voter awards m k points to its rank k alternative
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Voting rules: Borda

- Each voter awards m k points to its rank k alternative
- Alternative with the most point wins

Ν	1		3			pnts
	а	а	а	b	b	4
	b	b	a b c d	с	с	3
	с	с	с	d	d	2
	d	d	d	е	е	1
	е		е	а	а	0

Voting rules: Borda

- Each voter awards m k points to its rank k alternative
- Alternative with the most point wins

Ν	1	2	3	4	5	pnts	Total
	а	а	а	b	b	4	a: 12
	b	b	b	с	с	3	b: 17
	с	с	с	d	d	2	c: 12
	d	d	d	e	e	1	d: 7
	е	е	е	а	а	0	e: 2

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Ν	1	2	3	4	5	pnts	Total		
	а	а	а	b	b	4	a: 12		
	b	b	b	с	с	3	b: 17	Winner	
	с	с	с	d	d	2	c: 12	b	
	d	d	d	е	е	1	d: 7		
	e	е	е	а	а	0	e: 2		

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	а	а	а	b	b	4	a: 12	-		
	b	b	b	с	с	3	b: 17		Winner	
	с	с	с	d	d	2	c: 12		b	
	d	d	d	e	e	1	d: 7			
	e	e	e	а	а	0	e: 2			

• Proposed in the 18th century by chevalier de Borda

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	b	b	b	с	с	3	b: 17	Winner
	с	с	с	d	d	2	c: 12	b
	d	d	d	e	e	1	d: 7	
	e	е	e	а	а	0	e: 2	

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- Used for elections to the national assembly of Slovenia

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	с	с	с	d	d	2	c: 12	b
	d	d	d	e	e	1	d: 7	
	e	e	e	а	а	0	e: 2	

- Proposed in the 18th century by chevalier de Borda
- Used for elections to the national assembly of Slovenia
- A modified Borda Count is used in the Eurovision Song Context, points to the top 10 songs with 12, 10, 8,9,...,1 points

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- Each voter awards 1 point to its first *k*-ranked alternatives and 0 to the others
- Alternative with the most point wins

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Ν	1	2	3	4	5	
	а	а	а	b	b	ĺ
	b	b	b	с	с	
	с	с	с	d	d	
	d	d	d	е	е	
	e	e	e	а	а	

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NI	1		ว	4	E	k = 3
Ν	1	2	3	4	5	Total
	а	а	а	b	b	
	b	b	b	с	с	a: 3
	с	с	с	d	d	b: 5
	Ι.					c: 5
	d	d	d	e	e	d: 2
	e	e	e	а	а	

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Approval

Voting rules: k-approval

- Each voter awards 1 point to its first k-ranked alternatives and 0 to the others
- Alternative with the most point wins

N I	1		2	4		k = 3
IN	1	2	3	4	5	Total
	а	а	а	b	b	
	b	b	b	с	с	a: 3
				d	d	b: 5
	C	C	C	u	u	c: 5
	d	d	d	e	e	d: 2
	e	e	e	а	а	u. 2
						e: 0

- Each voter awards 1 point to its first *k*-ranked alternatives and 0 to the others
- Alternative with the most point wins

Ν	1	2	3	4	5	$\begin{vmatrix} k = 3 \\ Tata \end{vmatrix}$	
	a b c d e	a b c d e	a b c d e	b c d e a	b c d e a	Total a: 3 b: 5 c: 5 d: 2	Winner b or c
					-	e: 0	

- Approval voting was used for papal conclaves between 1294 and 1621.
- Used to select potential consensus candidates for an election.

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Voting rules: Positional Scoring Rules

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Voting rules: Positional Scoring Rules

- Defined by a score vertor $s = (s_1, \ldots, s_m)$
- Each voter awards sk points to its rank k alternative
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Voting rules: Positional Scoring Rules

- Defined by a score vertor $s = (s_1, \ldots, s_m)$
- Each voter awards sk points to its rank k alternative
- Alternative with the most point wins
- The family include many rules
 - Plurality *s* = (1, 0, ..., 0)
 - Borda s = (m 1, m 2, ..., 0)
 - *k*-aproval s = (1, ..., 1, 0, ..., 0)
 - Veto *s* = (0,...,0,1)
 - . . .

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Voting rules: Plurality with runoff

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- First round: two alternatives with the highest plurality scores survive
- Second round: between these two alternatives, select the one that majority of voters prefer

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Ν	1	2	3	4	5	
	а	а	а	b	b	
	b	b	b	с	с	
	с	с	с	d	d	
	d	d	d	е	е	
	e	e	e	а	а	

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- First round: two alternatives with the highest plurality scores survive
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Ν	1	2	3	4	5	
	а	а	а	b	b	1st round
	b	b	b	с	с	Winners
	с	с	с	d	d	
	d	d	d	e	е	a, b
	e	e	e	a	а	

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- First round: two alternatives with the highest plurality scores survive
- Second round: between these two alternatives, select the one that majority of voters prefer

Ν			3			
	а	а	а	b	b	
	b	b	b	с	с	
	с	с	с	d	d	
	d	d	d	e	e	
	e	e	a b c d e	а	а	

1st round	2nd round
Winners	Winner
a, b	а

Voting rules: Plurality with runoff

- First round: two alternatives with the highest plurality scores survive
- Second round: between these two alternatives, select the one that majority of voters prefer

Ν	1	2	3	4	5	
	а	а	а	b	b	1st r
	b	b	b	с	с	Win
	с	с	с	d	d	
	d	d	d	e	e	a,
	e	e	e	а	а	

- Similar to the French presidential election system
 - Problem: vote division
 - Happened in the 2002 French presidential election

• Single Transferable Vote (STV): Plurality with multiple rounds

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- m-1 rounds.
- In each round, the alternative with the least plurality votes is eliminated.
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Ν		2						
	а	С	d	b	b	а	с	а
	b	c b	b	с	с	b	b	b
	с	а	с	d	d	d	е	е
	d	d	а	e	e	с	d	d
	e	e	е	а	a	e	а	с

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- *m* − 1 rounds.
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Ν	1	2	3	4	5	6	7	8		Loser
	а	с	d	b	b	а	с	а		Lobel
	b	b	b	b c d e a	с	b	b	b	R1	е
	с	а	с	d	d	d	e	e		
	d	d	а	e	e	с	d	d		
	e	e	e	a	a	e	а	с		

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Ν	1	2	3	4	5	6	7	8		loser
	а	с	d	b	b	а	с	а	1	LOSCI
	b	b	b	с	с	b	b	b	R1 R2	e d
	с	а	с	d e	d	d	е	e	R2	a
	d	d	а	e	e	с	d	d		
	e	e	e	a	a	e	а	с		

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	а	с	d	b	b	а	с	а		LOSCI
	b	b	b	с		b	b	b	R1	e
									R2	d
	С	а	С	d	d	d	e	e	R3	6
	d	d	а	e	e	C	d	d	1/2	C
	е	e	e	а	а	e	а	с		

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	а	с	d	b	b	а	с	а		LUSEI
	b	b	b	с	с	b	b	h	R1	е
					-			U	R2	d
	С	а	С	d	d	d	e	e	D3	6
	d	d	a	e	e	с	d	d		C
					2		2		R4	а
	e	e	e	a	a	e	а	С		

• Unhappiness: For a ranking σ on A.

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- Let $n_{a \succ b}$ be the number of voters who prefer a to b
- Player *i* is unhappy when $a \succ_{\sigma} b$ but $b \succ_i a$.
- For $(a \succ_{\sigma} b)$, σ makes $n_{b \succ a}$ players unhappy

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- Define the total unhappiness of σ as

$$\mathcal{K}(\sigma) = \sum_{a \succ_{\sigma} b} n_{b \succ a}$$

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• Select the ranking σ^* with minimum total unhappiness.

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- Define the total unhappiness of σ as

$$\mathcal{K}(\sigma) = \sum_{\mathsf{a}\succ_{\sigma}\mathsf{b}} \mathsf{n}_{\mathsf{b}\succ\mathsf{a}}$$

- Select the ranking σ^* with minimum total unhappiness.
- Social choice: The top alternative in σ^*

• x beats y in a pairwise election if a strict majority of voters prefer x to y.

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- x beats y in a pairwise election if a strict majority of voters prefer x to y.
- Copeland
 - Score(x) = #alternatives x beats in pairwise elections
 - elect x* with the maximum score

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- Maximin
 - $Score(x) = min_y n_{x \succ y}$
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Which rule to use?

- We just introduced infinitely many rules
- How do we know which is the "right" rule to use? Axioms, Characterization theorems, Impossibility Theorems
- Impossibility versus Computational hardness

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2 Some properties of voting rules



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Condorcet winner

• Recall: x beats y in a pairwise election if a strict majority of voters prefer x to y.

Condorcet winner

Recall: x beats y in a pairwise election if a strict majority of voters prefer x to y.
 The majority preference prefers x to y

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Condorcet

Condorcet winner

- Recall: x beats y in a pairwise election if a strict majority of voters prefer x to y.
 The majority preference prefers x to y
- A Condorcet winner is an alternative that beats every other alternative in pairwise election

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Condorcet winner

- Recall: x beats y in a pairwise election if a strict majority of voters prefer x to y.
 The majority preference prefers x to y
- A Condorcet winner is an alternative that beats every other alternative in pairwise election
- A Condorcet paradox happens when the majority preference has a cycle.

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Condorcet Paradox: Example

Condorcet Paradox: Example

Ν	1	2	3	Majority Pref
	а	с	b	$a \succ b$
	b	а	с	$b \succ c$
	с	b	а	$c \succ a$

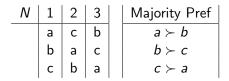
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Condorcet Paradox: Example



Also known as Dodgson's Paradox (Alice in Wonderland by Charles L. Dodgson alias Lewis Carroll)

Condorcet consistency

• If a Condorcet winner exists, it is unique.

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- A voting rule is Condorcet consistent if it always selects the Condorcet winner if one exists.

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- Among rules we just saw
 - All positional scoring rules (plurality, Borda, . . .), plurality with runoff. STV. are NOT Condorcet consistent.
 - Kemeny, Copeland, Maximin ARE Condorcet consistent.

Condorcet

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- Among rules we just saw
 - All positional scoring rules (plurality, Borda, . . .), plurality with runoff. STV. are NOT Condorcet consistent.
 - Kemeny, Copeland, Maximin ARE Condorcet consistent.
 - What is the complexity of Existence of Condorcet winner, obtaining the Condorcet winner

Strategy-proofness

- A voting rule is strategy-proof if there exists no profile where some voter can obtain a preferred outcome by changing her preferences.
- Which voting rules are strategy-proof?
- Do they have good properties?
- When they are not, can the manipulation be computed easily?

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E-manipulation: Given a set *C* of candidates, a set *V* of nonmanipulative voters, a set *S* of manipulative voters, with $S \cap V = \emptyset$, and a candidate $c \in C$. Is there a way to set the preference lists of the voters in *S* such that, under election system E, *c* is the (a) winner?

E-manipulation: Given a set *C* of candidates, a set *V* of nonmanipulative voters, a set *S* of manipulative voters, with $S \cap V = \emptyset$, and a candidate $c \in C$. Is there a way to set the preference lists of the voters in *S* such that, under election system E, *c* is the (a) winner? **E-Bribery**: Given a set *C* of candidates, a set *V* of voters, a candidate $c \in C$, and a nonnegative integer *k*. Is there a way to set the preference lists of at most *k* voters such that, under election system E, *c* is the (a) winner?

E-Control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is the (a) winner?

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E-Control under additive candidates: Given a set *C* of candidates, a pool *D* of potential additional candidates, a candidate $c \in C$, and a set of voters *V* with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, *c* is the (a) winner?

E-Destructive control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is not a winner?

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