An introduction to Computational Social Choice

Spring 2024

1. Social Choice
2. Some properties of voting rules

Social Choice Theory

- Mathematical theory for aggregating individual preferences into collective decisions
- Originated in ancient Greece. Formal foundations:
  - 18th Century (Condorcet and Borda)
  - 19th Century: Charles Dodgson (a.k.a. Lewis Carroll)
  - 20th Century: Nobel prizes to Arrow and Sen
- Objective: Methods to select a collective outcome based on (possibly different) individual preferences.

Social Choice Theory

- Set of voters $N = \{1, \ldots, n\}$
- Set of alternatives $A = \{1, \ldots, m\}$
- Voter i has a preference ranking over alternatives $\succ_i$
- Preference ranking $\succ$ is the collection of all voters' rankings

Social Choice Theory

- Set of voters $N = \{1, \ldots, n\}$
- Set of alternatives $A = \{1, \ldots, m\}$
- Voter i has a preference ranking over alternatives $\succ_i$
- Preference ranking $\succ$ is the collection of all voters' rankings

Ply	1	2	3
	a	c	b
	b	a	c
	c	b	a

Social Choice Theory: Objective

- Social choice function

Social Choice Theory: Objective

- Social choice function
  - Takes as input a preference profile $\succ$
  - Returns an alternative $a \in A$

Social Choice Theory: Objective

- Social choice function
    - Takes as input a preference profile $\succ$
    - Returns an alternative $a \in A$
- Social welfare function
    - Takes as input a preference profile $\succ$
    - Returns a societal preference on $A \succ_s$

Social Choice Theory: Objective

- Social choice function
  - Takes as input a preference profile $\succ$
  - Returns an alternative $a \in A$
- Social welfare function
  - Takes as input a preference profile $\succ$
  - Returns a societal preference on A $\succ_s$
- voting rule = social choice function

Voting rules: Plurality

Voting rules: Plurality

- Each voter awards one point to her top alternative
- Alternative with the most point wins

Voting rules: Plurality

- Each voter awards one point to her top alternative
- Alternative with the most point wins

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

Voting rules: Plurality

- Each voter awards one point to her top alternative
- Alternative with the most point wins

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

Winner
a

Voting rules: Plurality

- Each voter awards one point to her top alternative
- Alternative with the most point wins

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

Winner
a

- Most frequently used voting rule
- Many political elections use plurality

Voting rules: Plurality

- Each voter awards one point to her top alternative
- Alternative with the most point wins

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

Winner
a

- Most frequently used voting rule
- Many political elections use plurality

Problems?

Voting rules: Borda

Voting rules: Borda

- Each voter awards $m - k$ points to its rank k alternative
- Alternative with the most point wins

Voting rules: Borda

- Each voter awards $m - k$ points to its rank k alternative
- Alternative with the most point wins

N	1	2	3	4	5	pnts
	a	a	a	b	b	4
	b	b	b	c	c	3
	c	c	c	d	d	2
	d	d	d	e	e	1
	e	e	e	a	a	0

Voting rules: Borda

- Each voter awards $m - k$ points to its rank k alternative
- Alternative with the most point wins

N	1	2	3	4	5	pnts
	a	a	a	b	b	4
	b	b	b	c	c	3
	c	c	c	d	d	2
	d	d	d	e	e	1
	e	e	e	a	a	0

Total
a: 12
b: 17
c: 12
d: 7
e: 2

Voting rules: Borda

- Each voter awards $m - k$ points to its rank k alternative
- Alternative with the most point wins

N	1	2	3	4	5	pnts
	a	a	a	b	b	4
	b	b	b	c	c	3
	c	c	c	d	d	2
	d	d	d	e	e	1
	e	e	e	a	a	0

Total
a: 12
b: 17
c: 12
d: 7
e: 2

Winner
b

Voting rules: Borda

- Each voter awards $m - k$ points to its rank k alternative
- Alternative with the most point wins

N	1	2	3	4	5	pnts
	a	a	a	b	b	4
	b	b	b	c	c	3
	c	c	c	d	d	2
	d	d	d	e	e	1
	e	e	e	a	a	0

Total
a: 12
b: 17
c: 12
d: 7
e: 2

Winner
b

- Proposed in the 18th century by chevalier de Borda

Voting rules: Borda

- Each voter awards $m - k$ points to its rank k alternative
- Alternative with the most point wins

N	1	2	3	4	5	pnts
	a	a	a	b	b	4
	b	b	b	c	c	3
	c	c	c	d	d	2
	d	d	d	e	e	1
	e	e	e	a	a	0

Total
a: 12
b: 17
c: 12
d: 7
e: 2

Winner
b

- Proposed in the 18th century by chevalier de Borda
- Used for elections to the national assembly of Slovenia

Voting rules: Borda

- Each voter awards $m - k$ points to its rank k alternative
- Alternative with the most point wins

N	1	2	3	4	5	pnts
	a	a	a	b	b	4
	b	b	b	c	c	3
	c	c	c	d	d	2
	d	d	d	e	e	1
	e	e	e	a	a	0

Total
a: 12
b: 17
c: 12
d: 7
e: 2

Winner
b

- Proposed in the 18th century by chevalier de Borda
- Used for elections to the national assembly of Slovenia
- A modified Borda Count is used in the Eurovision Song Context, points to the top 10 songs with 12, 10, 8,9,.. ,1 points

Voting rules: k-approval

Voting rules: k-approval

- Each voter awards 1 point to its first k-ranked alternatives and 0 to the others
- Alternative with the most point wins

Voting rules: *k*-approval

- Each voter awards 1 point to its first k-ranked alternatives and 0 to the others
- Alternative with the most point wins

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

Voting rules: *k*-approval

- Each voter awards 1 point to its first k-ranked alternatives and 0 to the others
- Alternative with the most point wins

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

$k = 3$ Total
a: 3
b: 5
c: 5
d: 2
e: 0

Voting rules: k-approval

- Each voter awards 1 point to its first k-ranked alternatives and 0 to the others
- Alternative with the most point wins

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

$k = 3$ Total
a: 3
b: 5
c: 5
d: 2
e: 0

Winner
b or c

Voting rules: k-approval

- Each voter awards 1 point to its first k-ranked alternatives and 0 to the others
- Alternative with the most point wins

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

$k = 3$ Total
a: 3
b: 5
c: 5
d: 2
e: 0

Winner
b or c

- Approval voting was used for papal conclaves between 1294 and 1621.
- Used to select potential consensus candidates for an election.

Voting rules: Positional Scoring Rules

Voting rules: Positional Scoring Rules

- Defined by a score vertor $s = (s_1, \ldots, s_m)$
- Each voter awards s_k points to its rank k alternative
- Alternative with the most point wins

Voting rules: Positional Scoring Rules

- Defined by a score vertor $s = (s_1, \dots, s_m)$
- Each voter awards s_k points to its rank k alternative
- Alternative with the most point wins
- The family include many rules
  - Plurality $s = (1, 0, \dots, 0)$
  - Borda $s = (m - 1, m - 2, \dots, 0)$
  - k-aproval $s = (1, \dots, 1, 0, \dots, 0)$
  - Veto $s = (0, \dots, 0, 1)$
  - ...

Voting rules: Plurality with runoff

Voting rules: Plurality with runoff

- First round: two alternatives with the highest plurality scores survive
- Second round: between these two alternatives, select the one that majority of voters prefer

Voting rules: Plurality with runoff

- First round: two alternatives with the highest plurality scores survive
- Second round: between these two alternatives, select the one that majority of voters prefer

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

Voting rules: Plurality with runoff

- First round: two alternatives with the highest plurality scores survive
- Second round: between these two alternatives, select the one that majority of voters prefer

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

1st round Winners
a, b

Voting rules: Plurality with runoff

- First round: two alternatives with the highest plurality scores survive
- Second round: between these two alternatives, select the one that majority of voters prefer

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

1st round Winners
a, b

2nd round Winner
a

Voting rules: Plurality with runoff

- First round: two alternatives with the highest plurality scores survive
- Second round: between these two alternatives, select the one that majority of voters prefer

N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

1st round Winners
a, b

2nd round Winner
a

- Similar to the French presidential election system
  - Problem: vote division
  - Happened in the 2002 French presidential election

Voting rules: STV

- Single Transferable Vote (STV): Plurality with multiple rounds

Voting rules: STV

- Single Transferable Vote (STV): Plurality with multiple rounds
- $m - 1$ rounds.
- In each round, the alternative with the least plurality votes is eliminated.
- The selected alternative is the standing one.

Voting rules: STV

- Single Transferable Vote (STV): Plurality with multiple rounds
- $m-1$ rounds.
- In each round, the alternative with the least plurality votes is eliminated.
- The selected alternative is the standing one.

N	1	2	3	4	5	6	7	8
	a	c	d	b	b	a	c	a
	b	b	b	c	c	b	b	b
	c	a	c	d	d	d	e	e
	d	d	a	e	e	c	d	d
	e	e	e	a	a	e	a	c

Voting rules: STV

- Single Transferable Vote (STV): Plurality with multiple rounds
- $m - 1$ rounds.
- In each round, the alternative with the least plurality votes is eliminated.
- The selected alternative is the standing one.

N	1	2	3	4	5	6	7	8
	a	c	d	b	b	a	c	a
	b	b	b	c	c	b	b	b
	c	a	c	d	d	d	e	e
	d	d	a	e	e	c	d	d
	e	e	e	a	a	e	a	c

	Loser
R1	e

Voting rules: STV

- Single Transferable Vote (STV): Plurality with multiple rounds
- $m - 1$ rounds.
- In each round, the alternative with the least plurality votes is eliminated.
- The selected alternative is the standing one.

N	1	2	3	4	5	6	7	8
	a	c	d	b	b	a	c	a
	b	b	b	c	c	b	b	b
	c	a	c	d	d	d	e	e
	d	d	a	e	e	c	d	d
	e	e	e	a	a	e	a	c

	Loser
R1	e
R2	d

Voting rules: STV

- Single Transferable Vote (STV): Plurality with multiple rounds
- $m - 1$ rounds.
- In each round, the alternative with the least plurality votes is eliminated.
- The selected alternative is the standing one.

N	1	2	3	4	5	6	7	8
	a	c	d	b	b	a	c	a
	b	b	b	c	c	b	b	b
	c	a	c	d	d	d	e	e
	d	d	a	e	e	c	d	d
	e	e	e	a	a	e	a	c

	Loser
R1	e
R2	d
R3	c

Voting rules: STV

- Single Transferable Vote (STV): Plurality with multiple rounds
- $m - 1$ rounds.
- In each round, the alternative with the least plurality votes is eliminated.
- The selected alternative is the standing one.

N	1	2	3	4	5	6	7	8
	a	c	d	b	b	a	c	a
	b	b	b	c	c	b	b	b
	c	a	c	d	d	d	e	e
	d	d	a	e	e	c	d	d
	e	e	e	a	a	e	a	c

	Loser
R1	e
R2	d
R3	c
R4	a

Social welfare function: Kemeny's Rule

Social welfare function: Kemeny's Rule

- Unhappiness: For a ranking σ on A.

Social welfare function: Kemeny's Rule

- Unhappiness: For a ranking σ on A.
    - Let $n_{a \succ b}$ be the number of voters who prefer a to b
    - Player i is unhappy when $a \succ_\sigma b$ but $b \succ_i a$.
    - For $(a \succ_\sigma b)$, σ makes $n_{b \succ a}$ players unhappy

Social welfare function: Kemeny's Rule

- Unhappiness: For a ranking σ on A.
  - Let $n_{a \succ b}$ be the number of voters who prefer a to b
  - Player i is unhappy when $a \succ_\sigma b$ but $b \succ_i a$.
  - For $(a \succ_\sigma b)$, σ makes $n_{b \succ a}$ players unhappy
  - Define the total unhappiness of σ as

$$K(\sigma) = \sum_{a \succ_\sigma b} n_{b \succ a}$$

Social welfare function: Kemeny's Rule

- Unhappiness: For a ranking σ on A.
  - Let $n_{a \succ b}$ be the number of voters who prefer a to b
  - Player i is unhappy when $a \succ_\sigma b$ but $b \succ_i a$.
  - For $(a \succ_\sigma b)$, σ makes $n_{b \succ a}$ players unhappy
  - Define the total unhappiness of σ as

$$K(\sigma) = \sum_{a \succ_\sigma b} n_{b \succ a}$$

- Select the ranking σ^* with minimum total unhappiness.

Social welfare function: Kemeny's Rule

- Unhappiness: For a ranking σ on A.
  - Let $n_{a \succ b}$ be the number of voters who prefer a to b
  - Player i is unhappy when $a \succ_\sigma b$ but $b \succ_i a$.
  - For $(a \succ_\sigma b)$, σ makes $n_{b \succ a}$ players unhappy
  - Define the total unhappiness of σ as

$$K(\sigma) = \sum_{a \succ_\sigma b} n_{b \succ a}$$

- Select the ranking σ^* with minimum total unhappiness.
- Social choice: The top alternative in σ^*

Voting rules: Copeland and Maximin

Voting rules: Copeland and Maximin

- x beats y in a pairwise election if a strict majority of voters prefer x to y.

Voting rules: Copeland and Maximin

- x beats y in a pairwise election if a strict majority of voters prefer x to y.
- Copeland
  - $Score(x) = \#$alternatives x beats in pairwise elections
  - elect x^* with the maximum score

Voting rules: Copeland and Maxmin

- x beats y in a pairwise election if a strict majority of voters prefer x to y.
- Copeland
  - $Score(x) = \#$alternatives x beats in pairwise elections
  - elect x^* with the maximum score
- Maximin
  - $Score(x) = min_y n_{x \succ y}$
  - elect x^* with the maximum score

Which rule to use?

- We just introduced infinitely many rules
- How do we know which is the "right" rule to use? Axioms, Characterization theorems, Impossibility Theorems
- Impossibility versus Computational hardness

Condorcet winner

- Recall: x beats y in a pairwise election if a strict majority of voters prefer x to y.

Condorcet winner

- Recall: x beats y in a pairwise election if a strict majority of voters prefer x to y.
  The majority preference prefers x to y

Condorcet winner

- Recall: x beats y in a pairwise election if a strict majority of voters prefer x to y.
  The majority preference prefers x to y
- A Condorcet winner is an alternative that beats every other alternative in pairwise election

Condorcet winner

- Recall: x beats y in a pairwise election if a strict majority of voters prefer x to y.
  The majority preference prefers x to y
- A Condorcet winner is an alternative that beats every other alternative in pairwise election
- A Condorcet paradox happens when the majority preference has a cycle.

Condorcet Paradox: Example

Condorcet Paradox: Example

N	1	2	3	Majority Pref
	a	c	b	$a \succ b$
	b	a	c	$b \succ c$
	c	b	a	$c \succ a$

Condorcet Paradox: Example

N	1	2	3	Majority Pref
	a	c	b	$a \succ b$
	b	a	c	$b \succ c$
	c	b	a	$c \succ a$

Also known as Dodgson's Paradox (Alice in Wonderland by Charles L. Dodgson alias Lewis Carroll)

Condorcet consistency

- If a Condorcet winner exists, it is unique.

Condorcet consistency

- If a Condorcet winner exists, it is unique.
- A voting rule is Condorcet consistent if it always selects the Condorcet winner if one exists.

Condorcet consistency

- If a Condorcet winner exists, it is unique.
- A voting rule is Condorcet consistent if it always selects the Condorcet winner if one exists.
- Among rules we just saw
  - All positional scoring rules (plurality, Borda, . . .), plurality with runoff, STV, are NOT Condorcet consistent.
  - Kemeny, Copeland, Maximin ARE Condorcet consistent.

Condorcet consistency

- If a Condorcet winner exists, it is unique.
- A voting rule is Condorcet consistent if it always selects the Condorcet winner if one exists.
- Among rules we just saw
  - All positional scoring rules (plurality, Borda, . . .), plurality with runoff, STV, are NOT Condorcet consistent.
  - Kemeny, Copeland, Maximin ARE Condorcet consistent.
  - What is the complexity of Existence of Condorcet winner, obtaining the Condorcet winner . . .

Strategy-proofness

- A voting rule is strategy-proof if there exists no profile where some voter can obtain a preferred outcome by changing her preferences.
- Which voting rules are strategy-proof?
- Do they have good properties?
- When they are not, can the manipulation be computed easily?

Problems

E-manipulation: Given a set C of candidates, a set V of nonmanipulative voters, a set S of manipulative voters, with $S \cap V = \emptyset$, and a candidate $c \in C$. Is there a way to set the preference lists of the voters in S such that, under election system E, c is the (a) winner?

Problems

E-manipulation: Given a set C of candidates, a set V of nonmanipulative voters, a set S of manipulative voters, with $S \cap V = \emptyset$, and a candidate $c \in C$. Is there a way to set the preference lists of the voters in S such that, under election system E, c is the (a) winner?
E-Bribery: Given a set C of candidates, a set V of voters, a candidate $c \in C$, and a nonnegative integer k. Is there a way to set the preference lists of at most k voters such that, under election system E, c is the (a) winner?

Problems

E-Control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is the (a) winner?

Problems

E-Control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is the (a) winner?

E-Destructive control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is not a winner?