Complexity: Problems and Classes

Maria Serna

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Algorithmics: Basic references

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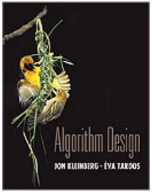

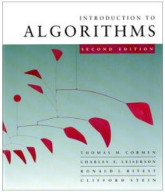

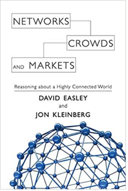

Computational Complexity: Basic references

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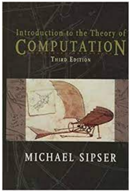

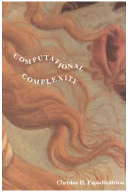

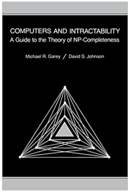

Growth of functions: Asymptotic notations

We consider only functions defined on the natural numbers.

$$f, g : \mathbb{N} \to \mathbb{N}$$

O-notation

For a given function $g(n)$

$$O(g(n)) = \{f(n) \mid \text{there exists a positive constant } c \text{ and } n_0 \geq 0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

Equivalently, the set of functions that verify

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$\begin{aligned} 5n^3 + 2n^2 &= O(2^n) \\ 5n^3 + 2n^2 &= O(n^4) \\ 2^n &= O(2^{2n}) \\ 2^n &= O(2^{n \log n}) \end{aligned}$$

It is used for asymptotic upper bound.

Although $O(g(n))$ is a set we write $f(n) = O(g(n))$ to indicate that $f(n)$ is a member of $O(g(n))$

Ω-notation

For a given function $g(n)$

$$\Omega(g(n)) = \{f(n) \mid \text{there exists positive constants } c \text{ and } n_0 \text{ such that} \\ 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}$$

Equivalently, the set of functions that verify

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

$$\begin{aligned} 5n^3 + 2n^2 &= \Theta(n^3) \\ 5n^3 + 2n^2 &= \Omega(n^3) \\ 5n^3 + 2n^2 &= \Omega(n^2) \\ 2^n &= \Omega(2^{n/2}) \end{aligned}$$

It is used for asymptotic lower bound.

Θ-notation

For a given function $g(n)$

$$\Theta(g(n)) = \{f(n) \mid \text{there are positive constants } c_1,\ c_2, \text{ and } n_0 \geq 0 \ \}$$
$$\text{such that } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}$$

Equivalently, the set of functions that verify

$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$5n^3 + 2n^2 = \Theta(n^3)$$

$$5n^3 + 2n^2 \notin \Theta(n^2)$$

It is used for asymptotic equivalence

o-notation

For a given function $g(n)$

$$o(g(n)) = \{f(n) \mid \text{for any positive constant } c \text{ there is } n_0 \geq 0 \text{ such that} \\ 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

Note that $f(n) = O(n)$ implies $f(n) \leq cg(n)$ asymptotically for some c but $f(n) = o(n)$ implies $f(n) \leq cg(n)$ asymptotically for any c and when $f(n) = o(g(n))$ it holds that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

It is used for asymptotic upper bounds that are not asymtotically tight.

ω-notation

$f(n) \in \omega(g(n))$ iff $g(n) \in o(f(n))$

Algorithm's analysis

- Time
- Space

Algorithm $\mathcal{A}$ on input x takes time $t(x)$.
$|x|$ denotes the size of input x.

Definition

The cost function of algorithm $\mathcal{A}$ is a function from $\mathbb{N}$ to $\mathbb{N}$ defined as

$$\mathcal{C}_{\mathcal{A}}(n) = \max_{|x|=n} t(x)$$

Fundamental growth functions

- Polynomial time
- Exponential time

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- Similar definitions replacing time by space
  Most used PSPACE polynomial space

Problem types

- Decision
  Input x
  Property $P(x)$

  Example: Given a graph and two vertices, is there a path joining them?

- Function
  Input x
  Compute y *such that* $Q(x, y)$

  Example: Given a graph and two vertices, compute the minimum distance between them.

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  Coding inputs/outputs on alphabet Σ a deterministic algorithm solving a problem determines a function
  $f : \Sigma^* \to \Sigma^*$ s.t., for any x, $Q(x, f(x))$ is true.

Decision problem classes

- Undecidable
  No algorithm can solve the problem.
- Decidable
  There is an algorithm solving them.
  - P:
    There is an algorithm solving it with polynomial cost.
  - EXP
    There is an algorithm solving it with exponential cost.
  - PSPACE
    There is an algorithm solving it within polynomial space.

NP: non-deterministic polynomial time

It is possible to define a certificate y and a property $P(x, y)$ such that

- If x is an input with answer yes, there is y such that $P(x, y)$ is true,
- $P(x, y)$ can be decided in polynomial time, given x and y.
- y has polynomial size with respect to $|x|$.

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Problems with a polynomial time verifier
$\{x \mid \exists y P(x, y)\}$

Some decision problems

Bipartiteness (BIP)
Given a graph determine whether it is bipartite.

Perfect matching (PMATCH)
Given a graph determine whether it has a perfect matching.

Hamiltonian Cycle (HC)
Given a graph determine whether it has a Hamiltonian circuit.

In which classes?

NP-hardness

- It is an open question whether P = NP or NP = EXP. Most believed is that P $\neq$ NP
- Π is NP-hard means that a polynomial time algorithm for Π can be reused to solve in polynomial time any problem in P.
- Decision problem A is NP-complete iff $A \in$ NP and A is NP-hard. Look at Garey and Johnson's book for a big list of NP-hard/complete problems.

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- The NP-hardness of a problem is assessed through reductions.

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This type of reduction is calle many-one polynomial reduction, sometiomes we use $\leq_m^p$ to distinguish from other reducibilities.

Completeness

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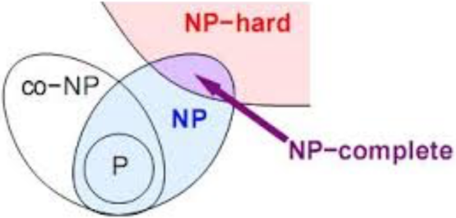
- P, NP, PSPACE, EXP are closed under $\leq$.
- A NP-complete problem is a problem complete for NP under $\leq$.
- $\leq$ is a transitive relation.

NP-completeness

A problem A is NP-complete if:

1. $A \in$ NP, and
2. for every $B \in$ NP, $B \leq A$.

If for every $B \in$ NP, $B \leq A$ but $A \notin$ NP then A is said to be NP-hard.



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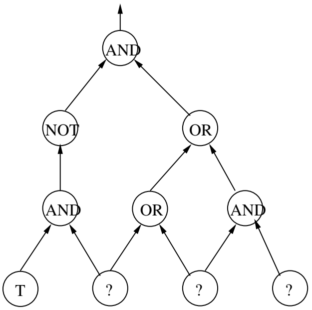
- *Majority conjecture:* P $\neq$ NP
- To prove a problem is NP-complete, we just have to find a reduction from a problem known to be NP-complete.
- We need as a seed a first NP-complete problem.

CIRCUIT SAT.

CIRCUIT SAT: Given a Boolean circuit with gates *AND*, *OR*, *NOT*, and the input gates *T*, *F* and *?*, and one output gate. Is there an an assignment to the input gates (?), such that the circuit evaluates to T?



For example if the input to ? is T,F,T, the output is F
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- There is a way to feed c and get output 1 iff there is a valid cerificate.

The seminal theorem: Cook-Levin's Theorem

Therefore, given any instance x for A, we can construct in poly-time instance C of CIRCUIT SAT whose known inputs are the bits of x, and whose unknown inputs are the bits of x, and such that the output of C is 1 iff $\mathcal{A}$ outputs YES on input (x, c).

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Theorem (Cook-Levin's theorem)

CIRCUIT-SAT is NP-complete.

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- For example, for $X = \{x_1, x_2, x_3\}$,

$$\phi = (x_1 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (x_2 \vee x_3)$$

is satisfiable, take $T(x_1) = T(x_2) = 0$, $T(x_3) = 1$ then $T(\phi) = 1$.

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- k-SAT. Given a boolean formula in CNF $\phi = \bigwedge_{i=1}^{m}(C_i)$ in where each clause is a disjunction of exactly k literals, is ϕ satisfiable?
  Ex. 3-SAT
  $\phi = (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee \bar{x_2} \vee x_4) \wedge (\bar{x_1} \vee \bar{x_2} \vee \bar{x_4}) \wedge (x_1 \vee \bar{x_3} \vee \bar{x_4}) \wedge (\bar{x_1} \vee \bar{x_2} \vee \bar{x_3}) \wedge (\bar{x_2} \vee x_3 \vee x_4) \wedge (\bar{x_1} \vee x_2 \vee \bar{x_3}) \wedge (\bar{x_2} \vee x_3 \vee \bar{x_4}) \wedge (x_2 \vee \bar{x_3} \vee \bar{x_4}) \wedge (\bar{x_2} \vee \bar{x_3} \vee \bar{x_4})$
  $T(x_1) = 1$, $T(x_2) = T(x_3) = T(x_4) = 0$, satisfies the previous instance.

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SAT problem and variations

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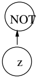
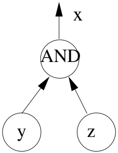
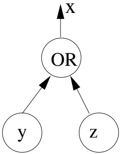


$(x \vee z) \wedge (\bar{x} \vee \bar{z})$

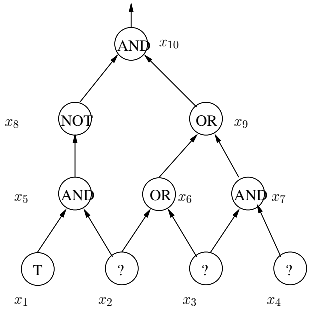


$(\bar{x} \vee y) \wedge (\bar{x} \vee z) \wedge (x \vee \bar{y} \vee \bar{z})$



$(\bar{y} \vee x) \wedge (\bar{z} \vee x) \wedge (z \vee y \vee \bar{x})$

Example.



$(x_1)\wedge$
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- Therefore, CIRCUIT SAT $\leq_m^p$ SAT.

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Theorem

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- The replacements depend on the size k of clause C_j.

SAT $\leq$ 3-SAT

- If $k = 1$, $C_j = z$, we add variables $\{y_{j1}, y_{j2}\}$ and clauses

$$C'_j = \{(z \vee y_{j1} \vee y_{j2}), (z \vee \bar{y}_{j1} \vee y_{j2}), (z \vee y_{j1} \vee \bar{y}_{j2}), (z \vee \bar{y}_{j1} \vee \bar{y}_{j2})\}.$$

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- If $k = 3$, we add $C_j = (z_1 \vee z_2 \vee z_3)$ to ϕ'.

SAT $\leq$ 3-SAT

- If $k > 3$, $C_j = (z_1 \vee z_2 \vee \cdots \vee z_k)$, add variables $\{y_{j1}, y_{j2}, \ldots, y_{jk-3}\}$ and the clauses

$$C'_j = \{(z_1 \vee z_2 \vee y_{j1}), (\bar{y}_{j1} \vee z_3 \vee y_{j2}), \ldots, (\bar{y}_{jk-3} \vee z_{k-1} \vee z_k)\}$$

- A satisfying assignment for C_j must made $T(z_i) = 1$ for at least one z_i. Then the assignment T' such that, $T(z_i) = 1$, $T'(y_{j1}) = \cdots = T'(y_{ji-2}) = 1$ and $T'(y_{ji-1}) = \cdots = T'(y_{jk-3}) = 0$ satisfies C'_j.
- On the other hand, if T' satisfies C'_j, there is z_i such that $T'(z_i) = 1$ (otherwise there would be a $y_{j\ell}$ such that $T'(y_{j\ell}) = 1 = T'(\bar{y}_{j\ell})$)

Example

Input to SAT: $\phi = (\bar{x_1}) \wedge (\bar{x_1}, \bar{x_2}) \wedge (\bar{x_1}, x_3, \bar{x_4}) \wedge (x_1, x_2, \bar{x_3}, x_4, x_5)$
$C_1' = (\bar{x_1}, y_{11}, y_{12}) \wedge (\bar{x_1}, \bar{y}_{11}, y_{12}) \wedge (\bar{x_1}, y_{11}, \bar{y}_{12}) \wedge (\bar{x_1}, \bar{y}_{11}, \bar{y}_{12})$
$C_2' = (\bar{x_1}, \bar{x_2}, y_2) \wedge (\bar{x_1}, \bar{x_2}, \bar{y}_2)$
$C_3' = (\bar{x_1}, x_3, \bar{x_4})$
$C_4' = (x_1, x_2, y_{41}) \wedge (\bar{y}_{41}, x_3, y_{42}) \wedge (\bar{y}_{42}, x_4, x_5)$

Then $f(\phi) = C_1' \wedge C_2' \wedge C_3' \wedge C_4'$
with $X' = \{x_1, x_2, x_3, x_4, x_5, y_{11}, y_{12}, y_2, y_3, y_{41}, y_{42}\}$

3-SAT is NP-complete

Theorem

3-SAT is NP-complete

Proof.

- The above construction can be done in polynomial time, therefor SAT $\leq_m^p$ 3-SAT, 3-SAT is NP-hard
- On the other hand 3-SAT is a subproblem of SAT, do 3-SAT$\in$ NP

□

The k-SAT problem

Theorem

For $k \geq 3$, k-SAT is NP-complete

Proof.

- We have just show that 3-SAT is NP-complete.
- Assume that ℓ-SAT is NP-complete.
- To reduce ℓ-SAT $\leq_m^p$ $(\ell + 1)$-SAT, for each clause C_j (with ℓ literals)
    - Add variable $\{y_j\}$
    - and add clauses
$$C_j' = \{(C_j \vee y_j), (C_j \vee \bar{y}_j)\}.$$
    - C_j is satisfiable iff C_j is satisfiable.

□

The 2-SAT problem

2-SAT: Given a Boolean formula ϕ, where each clause has exactly 2 literals, is ϕ satisfiable?

- A clause in a 2-SAT instance has the form $(y \vee z)$ for some literals y, z.

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- A 2-SAT formula can be seen as a collection of implications.
- To show that 2-SAT $\in$ P, we construct a kind of reduction to a problem related to the strongly connected components of a digraph.

Strongly connected components of a digraph

- A digraph G is strongly connected iff $\forall u, v \in V$, there is a path from u to v ($u \rightsquigarrow v$) and there is a path from v to u ($v \rightsquigarrow u$).
- We can determine if G is strongly connected in $O(n+m)$ time.
- When $\vec{G}$ not strongly connected, we can find its strongly connected components in $O(n+m)$ steps.

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- When $\vec{G}$ not strongly connected, we can find its strongly connected components in $O(n + m)$ steps.
- Recall, that by collapsing each strongly connected component of $\vec{G}$ to a node, and removing multiple edges and loops, the remaining digraph is acyclic.

2-SAT $\leq$ P

Let ϕ be a 2-SAT instance on a set X of n variables and with m clauses ($|\phi| = 2m$).

2-SAT$\leq$ P

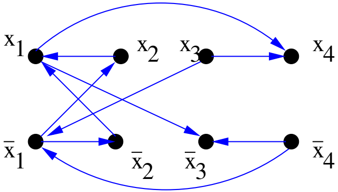
Let ϕ be a 2-SAT instance on a set X of n variables and with m clauses ($|\phi| = 2m$).
Define G_ϕ as follows:

- V: $2n$ nodes, two for each variable (x and $\bar{x}$).
- $\vec{E}$, has $2m$ edges, for each $C_i = (\alpha \vee \beta)$, add edges $\bar{\alpha} \to \beta$ and $\bar{\beta} \to \alpha$.

Notice that G_ϕ collects all implications in ϕ.

Examples

$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_1 \vee x_2)$
$\wedge(\bar{x}_3 \vee x_4) \wedge (\bar{x}_1 \vee x_4)$
which is satisfiable.



$\phi = (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2)\wedge$
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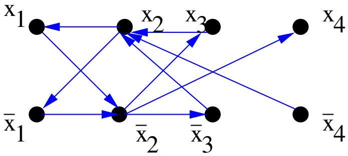
which is not satisfiable.



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ϕ is satisfiable iff no strongly connected component in G_ϕ contains nodes x and $\bar{x}$, for $x \in X$

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2-Sat $\in P$

Proof.

To build G_ϕ takes $O(m)$.

The strongly connected components of G_ϕ can be obtained in $O(m+n)$. □