Noise induced transitions in fluid neural networks

Jordi Delgado a, b, Ricard V. Sole b

a Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya, Pau Gargallo 5, 08028 Barcelona, Spain
b Complex Systems Research Group, Departament de Física i Enginyeria Nuclear, Universitat Politècnica de Catalunya, Sor Eulàlia d'Anitzu s/n. Campus Nord, Mòdul B4, 08034 Barcelona, Spain

Received 23 September 1996; revised manuscript received 18 February 1997; accepted for publication 24 February 1997
Communicated by A.P. Fordy

Abstract

We study the autosynchronization of temporal patterns of activity in Leptothorax ants, using to model this phenomenon the formalism of fluid neural networks. It is known that autosynchronization is involved in complex phenomena observed in ant colonies, such as task allocation and mutual exclusion. We have numerical evidence that this happens at the critical point of a noise induced transition. © 1997 Published by Elsevier Science B.V.

Keywords: Noise induced transitions; Fluid neural networks

Noise induced nonequilibrium phase transitions [1] have been shown to occur in a wide spectrum of physical and nonphysical systems. For real systems noise is never strictly zero and, together with intrinsic nonlinearities, can lead to qualitative changes in the stationary behaviour of the system under consideration. The influence of external noise is able to generate drastic modifications of macroscopic properties and it can even lead to noise induced critical points [1]. Examples of noise induced transitions have been explored in physical, chemical and biological systems [1‑3]. As far as we know, however, there is no example in the literature of a well defined noise induced transition phenomenon in social behaviour. In this Letter we show that a well defined noise induced transition occurs in the behavior of some species of ant colonies. Such a transition point, as we discuss below, has been shown to be related with the emergence of some computational properties.

The phenomenon under consideration has been observed in the temporal patterns of colony behavioral activity of Leptothorax ants. Experimental work with ants of different species shows that these ants exhibit cyclical bursts of activity within the nest [4]. There have been several attempts at providing an explanation of how these cycles arise [5], all of them emphasizing the self-organizing character of these oscillations, which emerge through local and individual interactions. There is experimental evidence that this autosynchronization phenomenon underlies some complex activities in ant colonies, for example mutual exclusion and task allocation [6]. This phenomenon has attracted considerable attention as a new example of generic self-organization in complex systems [7].

The aim of this Letter is to make clear enough that these self-organized activity cycles could be a consequence of the subtle interplay between the nonlinearities of the system and the external noise that causes the spontaneous activation of ants. The formalism that will be used is that of fluid neural networks (FNN...
for short, see Refs. [7–9]), a spatially distributed system of mobile formal neurons, based on the thought-provoking analogies between ant colonies and neural collectives, that has been successful at providing an explanation of the above mentioned cyclical bursts. Some predictions of FNN have been experimentally tested [10].

In FNN the standard approach of neural networks is used, but a new set of rules defining local movement and individual activation is also introduced. A set of \( N \) automata is used. The state of each automaton (say the \( i \)th one) is described through a continuous state variable \( S_i(t) \in \mathbb{R} \), at each time step \( t \in \mathbb{N} \). Each element can move on a \( L \times L \) two-dimensional lattice. A set of rules is defined:

(a) **Neural network structure**: the automata are considered as some kind of “neurons” (ants) and interactions are described as in neural networks by means of a sigmoidal function \( \Phi(x) \). If \( S_i(t) \) is a given automaton (the spatial dependence is omitted for simplicity), the new states are updated following

\[
S_i(t+\Delta t) = \Phi \left( g \left( J_{ii} S_i(t) + \sum_{i \neq j \in B(i)} J_{ij} S_j(t) - \Theta_i \right) \right)
\]

where \( B(i) \) are the nearest automata, located in the neighborhood defined by the eight nearest lattice sites, and \( J_{ii} \neq 0 \). For simplicity we use the threshold \( \Theta_i = 0 \), and we take \( \Phi(z) = \tanh(gz) \) with \( g = 0.1 \).

(b) **Spontaneous activation**: each automaton can be either active or inactive and, if active, it moves randomly to one of the eight nearest cells (if no space is available, no movement takes place). In our model a given automaton will be active if \( S_i(t) > \theta \) and inactive otherwise. Here this threshold is \( \theta = 10^{-16} \). Once an automaton becomes inactive, it can return to the active state with a spontaneous activity level \( \omega \); here we use \( S_0 = 0.1 \) with some probability \( p_s \).

(c) **Coupling matrix**: the coupling matrix \( J \) is not fixed. Connections are local and changing over time as a consequence of movement. They are also state-dependent, i.e. \( J_{ij} \) will be a simple function of the states of the actually interacting pair \( (i,j) \) of automata, i.e. \( J_{ij} = f(a_i^l, a_j^l) \), where \( a_i^l = \Theta(S_i(t) - \theta) \).

In our case, where two basic states are defined, i.e. “active” and “inactive”, the connection matrix reduces to the following 2 \( \times \) 2 table,

\[
\Lambda = \begin{pmatrix}
\lambda_{11} & \lambda_{10} \\
\lambda_{01} & \lambda_{00}
\end{pmatrix}
\]

At a given time step, the interaction \( J_{ij} \) between the \( i \)th and the \( j \)th elements is equal to \( \lambda_{a_i^l a_j^l} \in \Lambda \) by depending of the activity states of the given elements. More precisely, \( J_{ij} \) will be equal to: \( \lambda_{11} \) when both ants are active, to \( \lambda_{10}, \lambda_{01} \) when one is active and the other inactive and to \( \lambda_{00} \) if both automata are inactive. In this Letter we take for simplicity \( \lambda_{a_i^l a_j^l} = 1 \).

Our choice is based in the observation of ant colonies. Self-interaction and positive feedback (with local excitability) play an essential role in colony dynamics [11].

When a given individual is in isolation, Eq. (1) reads

\[
S_i(t+\Delta t) = \tanh(\mu S_i(t))
\]

(here \( \mu = gJ_{ii} \), and \( J_{ii} > 0 \)). In the linear approximation, i.e. \( S(t+\Delta t) = [\delta S(0)] S(t) \), we obtain

\[
S(t+\Delta t) = \mu S(t).
\]

If we define activity as previously, starting from \( S(0) = S_0 \) the state after \( \tau \) iterations will be \( S(\tau) = S_0 \mu^\tau \), so the time for a given element to inactivate (\( S(\tau) < 0 \)) will be

\[
\tau = \frac{\log(\Gamma)}{\log(gJ_{ii})},
\]

where \( \Gamma = \theta/S_0 \). If we consider an isolated individual, we can compute the probability of having \( n \) spontaneous activations in a time interval of length \( T \), provided \( T \) is long enough, and \( \Delta t \) is very small. We will exploit the analogy between our randomly active “ant” and a model neuron with absolute refractoriness \( \tau \), subject to a Poisson sequence of excitatory zero-width point inputs with arrival rate \( p_a \). In short, following Ref. [12], to compute the above mentioned probability \( P_n(T, \tau) \) we start from a well known similar result for \( \tau = 0 \), i.e. the Poisson process, where we have

\[
P_n(T) = \frac{(p_a T)^n e^{-p_a T}}{n!}
\]

for \( n = 0, 1, 2, \ldots \). Then, to calculate \( P_n(T, \tau) \) we replace each zero width pulse occurring at, say, \( t_i \) by a pulse of duration \( \tau \) beginning at \( t_i \); we consider different cases depending on the relation among \( T, n \) and \( \tau \), and, after some tedious calculations, we finally obtain
Fig. 1. Temporal behavior of $\rho^A_1$, with parameters $L = 50$, $S_u = 0.1$, $g = 0.1$, $\theta = 10^{-16}$, $p_k = 0.01$ and (A) $\rho = 0.10$; (B) $\rho = 0.15$; (C) $\rho = 0.20$; (D) $\rho = 0.25$; (E) $\rho = 0.30$; (F) $\rho = 0.35$.

$$P_n(T, \tau) = \Theta[T - (n + 1)\tau]$$

$$\times \left[ 1 - e^{-p_1 T - (n-1)\tau} \sum_{k=0}^{n-1} p_k^k [T - (n-1)\tau]^k/k! \right]$$

$$- \Theta[T - n\tau] \left( 1 - e^{-p_1 T - n\tau} \sum_{k=0}^{n} p_k^k (T - n\tau)^k/k! \right).$$  (4)
where \( \Theta(x) \) is the Heaviside step function \( (\Theta(x) = 0 \text{ if } x \leq 0, \text{ and } \Theta(x) = 1 \text{ if } x > 0) \).

Although the individual automaton is easily analysed, the collective system, i.e. \( N \) automata with interactions, is much more difficult. We will characterize the dynamical behavior of FNN by

\[
\rho_i^+ = \frac{1}{N} \sum_{j=1}^{N} \Theta[S_j(t) - \theta],
\]

i.e. the density of active elements at each time step, where \( \rho_i^+ \in [0, 1] \). We define also the total density of automata as \( \rho = N/L^2 \). See Fig. 1 for a sample of the temporal behavior of \( \rho_i^+ \) for different \( \rho \). At low \( \rho \) we have a very disordered pattern, which becomes more and more ordered as \( \rho \) is increased. Despite our focusing on a quantity that apparently does not include the spatial dependence, this dependence is partially responsible of the dynamics of \( \rho_i^+ \) (see below).

In a previous paper [9] it was shown that a first approach to a characterization of a transition in the dynamical behavior of the FNN was to measure the Shannon–Kolmogorov entropy (SKE) of the following probability distribution \( p(j, \rho) \): let \( T_j \) the number of time steps where exactly \( j \) elements were active \( (j = 0, 1, \ldots, N) \); now, if \( T \) is the total number of time steps, the relative frequency of \( j \) simultaneously active elements will be \( p(j, \rho) = T_j/T \). The SKE is then defined as

\[
S(\rho) = - \sum_{i=0}^{N} p(i, \rho) \log p(i, \rho).
\]

It is expected that SKE will provide us with a measure of the diversity of macroscopic states. \( S(\rho) \) has an upper and a lower bound: \( S(\rho) \in [0, \log(N + 1)] \). At low densities, \( S(\rho) \) will grow until the onset of collective oscillations, where it will decrease due to the highly correlated temporal patterns (Fig. 2). See Ref. [9] for details on the SKE of FNN.

The time evolution of \( \rho_i^+ \) is a stochastic process because of the spontaneous activation of inactive automata. The process \( \rho_i^+ \) is non-Markovian due to the existence of the time interval in which the automaton is active. In the absence of interactions this time interval is the above mentioned \( \tau \): the number of active automata at time \( t \) will be those active at \( t - 1 \) plus those inactive at \( t - 1 \) that spontaneously become active, minus those that were active at \( t - \tau \); however, if we consider the interactions between automata, this time interval is altered by the patterns of interaction. It is the interplay between this \( \tau(\rho, \rho_a) \) and the “velocity” of activity propagation (where we have the “hidden” spatial dependence of \( \rho_i^+ \)) that will allow us to do some analytical work, in spite of the difficulties implied by the non-Markovian character of \( \rho_i^+ \) (this work will be published elsewhere).

In this Letter we will focus on the numerical evidence of a noise induced transition at the point of maximum SKE. Noise induced transitions are a phenomenon observed in systems coupled to a fluctuating environment, where fluctuations depend on the state of the system. Roughly, the effect of this system-randomness coupling is that noise can induce macroscopic ordered behavior on the system, which becomes manifest through a qualitative change on the stationary probability density of the process. We have computed, for various parameters sets, an estimation of the stationary probability density (histogram) for \( \rho_i^+ \): \( P(\rho^+) \). The stationarity has been assured using the results of Ref. [9] for the transient length. After the transient has finished, we have used \( T = 2 \times 10^4 \) time steps to compute the above mentioned probability density. The final result has been obtained averaging over different initial conditions (see Fig. 3).

The computed \( P(\rho^+) \) for different values of \( (\rho_a, \rho) \)
makes clear the relation between the shape of $P(\rho^+)$ and the dynamical behavior of $\rho^+_t$. If we have a low density $\rho$ or a low activation probability $p_a$, $\rho^+_t$ will be zero, except some irregular bursts of activity that cannot propagate, so the histogram will have a clear maximum at $\rho^+ = 0$. At high enough values of $\rho$, the rapid propagation of activity resulting from the interaction between automata will make them remain active almost all the time (if we have not a very low $p_a$). But at intermediate values of both $p_a$ and $\rho$, the interplay between the spontaneous activation and the propagation of the activity will make $\rho^+_t \in (0, 1)$. As we can see in Fig. 3, the qualitative change of shape of $P(\rho^+)$, where the transition occurs, consists of a displacement of the maximum of $P(\rho^+)$ toward the boundary $\rho^+ = 1$, until there is not any extrema in the interval $(0, 1)$. In the examples of Fig. 3 we can observe that the above mentioned qualitative change takes place between $\rho = 0.15$ and $\rho = 0.25$ for $p_a = 0.01$ (Figs. 3A(c) and 3A(d)) and between $\rho = 0.25$ and $\rho = 0.35$ for $p_a = 0.001$ (Figs. 3B(c) and 3B(d)), which was already suggested by the SKE in Fig. 2.

These phenomena are precisely the type of phenomena for which noise induced transitions theory [1] is well suited, since it shows us how to define an order parameter to accurately determine the nature and location (in parameter space) of the transition.

The numerical evidence of a noise induced transition introduced in the above suggests us an approach to the definition of an order parameter\(^1\). An appropriate indicator of a noise induced transition is, following standard theory [1], the extrema of the stationary probability law, which correspond to the macroscopic phases of the system. Then, in FNN we can define $\rho_m^+$ such that

$$P(\rho_m^+) = \max_{\rho^+ \in (0, 1)} P(\rho^+).$$

As we have checked extensively in numerical simulations, this quantity will be equivalent to find $\rho_m^+$ equating to zero the first derivative of $P(\rho^+)$ if $\rho_m^+ \in (0, 1)$, and will give us the correct value of either $\rho_m^+ = 0$ or $\rho_m^+ = 1$ otherwise. Finally we will define the order parameter (Fig. 4)

$$\Gamma(\rho, p_a) = 1 - \rho_m^+,$$

where obviously the $(\rho, p_a)$ dependence comes from $P(\rho^+)$. In Fig. 4 we can see the order parameter for the FNN whose histograms $P(\rho^+)$ we have seen in Fig. 3. The points where takes place the transition are, for $p_a = 0.01$, i.e. Fig. 4A, $\rho_c \simeq 0.215$, and for

\(^1\)In a previous paper, Sole et al. [9] put forward an order parameter $\Gamma(\rho) = \log(N + 1) - S(\rho)$ based on the assumption that at low $\rho$ the SKE will be maximum due to the practical absence of interactions between the automata. This has been found to be not as general as was supposed, because of at certain values of $p_a$ and in spite of the low $\rho$, interactions are frequent enough to make $S(\rho)$ have values below its maximum.
Once we have an order parameter $I'(\rho, p_a)$ we can explore the parameter space $(p_a, \rho)$ to locate the transition boundaries and to construct a qualitative phase diagram, which is shown in Fig. 5. There are three clear areas in the diagram, corresponding to the three shapes that can have $P(\rho^+)$, with the associated three values of $I'(\rho, p_a)$: $I'(\rho, p_a) = 0$ in region II, where we have self-organized oscillatory behavior. This is so because once all the automata are active, they will remain in this state until each automaton, approximately $r'(\rho, p_a)$ steps after, becomes inactive again. Since the propagation is very fast, any spontaneous activation will initiate the process again (see Figs. 1D, 1E, and 1F). $I'(\rho, p_a) = 1$ in region III where we have almost total inactivity, due to either a very low $p_a$, where spontaneous activation is a very rare event, or very low $\rho$, where activity cannot propagate. $I'(\rho, p_a) \in (0, 1)$ in region I. In region I we have an intermediate activity level almost all the time, but with a very irregular behavior (see Figs. 1A, 1B and 1C).

Previous work on FNN [9] has emphasized the density dependent transition from disorder for low densities to ordered oscillating behavior for higher densities, focusing the study on certain measures, as the SKE, that are good enough to suggest the existence of a transition, but fails at linking the numerical work with any well developed theory of phase transitions. In this Letter we have shown that the phenomena under study can be characterized numerically in such a way that the transition is equally well detected, but also provides us with the possibility of a best understanding, relating the phenomenology of FNN to the well understood phenomena of noise induced transitions. This finding paved the way to the possibility of analytic work, which is being accomplished through the work on the two factors that are mainly influenced...
by $p$ and $p_a$, i.e. the mean time an automaton is active and the velocity of activity propagation, which are responsible of the dynamic behavior of $p^\uparrow$. Moreover, if the time an automaton is active is much less than the time the activation needs to reach the whole system (this will occur if we have a growing $L$), we will not have any transition at all, so we think what we have is a much more interesting phenomenon than a “simple” noise induced transition, we have what has been called a finite size (noise) induced transition [13].

Finally, we can relate the phenomenon described in this Letter with the noise induced transition found behind another mechanism for swarm behaviour, stigmergy [14], by Millonas [15] (for a recent study of (second order) phase transitions in swarm models see also Ref. [16]). This suggests a new way by which swarms could compute: noise induced computation (this is discussed more in depth in Ref. [17]).

Acknowledgement

The authors would like to thank Ton Sales, Jordi Bascompte, Susanna C. Manrubia, Bartolo Luque, Ferran Mazzanti and Brian Goodwin for several comments and suggestions. This work has been supported by grants of the Universitat Politècnica de Catalunya (PR9407), the Generalitat de Catalunya (JD, FI/93-3008) and the CICYT (PB94-1195).

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