The Data Stream Model: Sketches and Probability Tools

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Contents

1. Streams, Approximation, Randomization

Massive data requires new kind of algorithmics

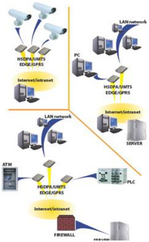
Often, approximate answers are OK. That helps!

Focus of this talk:

- (Mostly) Streaming data
- Sketches
- Counting problems

Data streams everywhere





- Telcos - phone calls
- Satellite, radar, sensor data
- Computer systems and network monitoring
- Search logs, access logs
- RSS feeds, social network activity
- Websites, clickstreams, query streams
- E-commerce
- . . .

Data streams: Concept

- Data arrives as sequence of items
- At high speed
- Forever
- Can't store them all
- Can't go back; or too slow
- Evolving, non-stationary reality

The Data stream axioms

Five Data Stream Axioms:

1. Only one pass; t-th item available at time t only
2. Small processing time per item
3. Small memory, certainly sublinear in stream length; sketches or summaries
4. Able to provide answers at any time
5. The stream evolves over time

Assumptions & Requirements

- Worst-case, adversarial, input distribution
- Difference with probabilistic assumption:
  - Items are generated probabilistically (often independently), following a probability distribution that may evolve over time
  - Implicit in Data Stream Mining and Machine Learning: Generalization!
- Randomness is in the algorithm
  - Different runs may give different answers
  - But in most runs answer is approximately correct

Counting

The Item Counting Problem

How many items have we read so far in the data stream?

To count up to t elements *exactly*, $\log t$ bits are *necessary*

Approximate solution using $\log \log t$ bits

Approximate counting: Saving 1 bit

Approximate counting, v1

Init: $c \leftarrow 0$

Update:

draw a random number $x \in [0,1]$
if $(x \leq 1/2)$ $c \leftarrow c+1$

Query: return $2c$

$\mathrm{E}[2c] = t, \qquad \sigma \simeq \sqrt{t/2}$

Space $\log(t/2) = \log t - 1 \rightarrow$ we saved 1 bit!

Approximate counting: Saving k bits

Approximate counting, v2

Init: $c \leftarrow 0$

Update:

  draw a random number $x \in [0, 1]$
  if $(x \leq 2^{-k})$ $c \leftarrow c + 1$

Query: return $2^k c$

$\mathrm{E}[c] = t/2^k, \qquad \sigma \simeq \sqrt{t/2^k}$

Memory $\log t - k \rightarrow$ we saved k bits!

$x \leq 2^{-k}$: AND of k random bits, $\log k$ memory

Approximate counting: Morris' counter

Morris' counter [Morris77]

Init: $c \leftarrow 0$

Update:

draw a random number $x \in [0,1]$

if $(x \leq 2^{-c})$ $c \leftarrow c+1$

Query: return $2^c - 2$

$E[c] \simeq \log t, \qquad \mathrm{E}[2^c - 2] = t, \qquad \sigma \simeq t/\sqrt{2}$

Memory = bits used to hold c = $\log c = \log\log t$ bits

Morris' approximate counter

- Can count up to 1 billion with $\log \log 10^9 = 5$ bits
- Problem: large variance, $\sigma \simeq 0.7\, t$

Reducing the variance, method I

Use basis $b < 2$ instead of basis 2:

- Places t in the series $1, b, b^2, \ldots, b^i, \ldots$ ("resolution" b)
- $E[b^c] \simeq t$, $\sigma \simeq \sqrt{(b-1)/2} \cdot t$
- Space $\log\log t - \log\log b$ $\quad$ ($> \log\log t$, because $b < 2$)
- For $b = 1.08$, 3 extra bits, $\sigma \simeq 0.2\, t$

Reducing the variance, method II

- Run r parallel, independent copies of the algorithm
- On Query, average their estimates
- $E[\text{Query}] \simeq t, \quad \sigma \simeq t/\sqrt{2r}$ (why?)
- Space $r \log\log t$
- Time per item multiplied by r

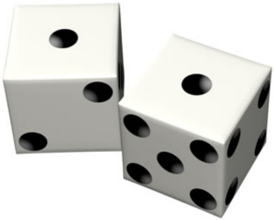
Worse performance, but more generic technique

Morris' counter: A non-streaming application

In [VanDurme+09]

- Counting k-grams in a large text corpus
- Number of k-grams grows exponentially with k
- Highly diverse frequencies
- Should fit in RAM
- Use Morris' counters (5 bits) instead of standard counters

2. Approximation. Large Deviation Bounds



Reducing the variance, general method

- Variance: $\mathrm{Var}(X) = \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$
- $\mathrm{Var}(\alpha \cdot X + \beta) = \alpha^2 \cdot \mathrm{Var}(X)$
- If X and Y independent, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
- In general, if X_i are all independent and $\mathrm{Var}(X_i) = \sigma^2$,

$$\mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

Equivalently,

$$\sigma(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{\sigma}{\sqrt{n}}.$$

Deviation Bounds

Random variables often described by expectation + variance

Suppose

$$\mathrm{E}[\text{algorithm output}] = \text{desired result}, \quad \mathrm{Var}(\text{algorithm output}) = \sigma^2$$

We usually want instead

$$|\text{algorithm output} - \text{desired result}| \leq \text{something}$$

(ε, δ)-approximation

A randomized algorithm A (ε, δ)-approximates a function $f : X \to R$ iff for every $x \in X$, with probability $\geq 1 - \delta$

- (absolute approximation) $\quad |A(x) - f(x)| < \varepsilon$
- (relative approximation) $\quad |A(x) - f(x)| < \varepsilon f(x)$

ε = accuracy; δ = confidence
Often ε, δ given as extra inputs to A

Deviation Bounds

Markov's inequality

For a non-negative random variable X and every k

$$\Pr[X \geq k\,\mathrm{E}[X]] \leq 1/k$$

Proof:

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{x} \Pr[X = x] \cdot x \geq \sum_{x \geq k} \Pr[X = x] \cdot x \\
&\geq \sum_{x \geq k} \Pr[X = x] \cdot k = k \Pr[X \geq k]
\end{aligned}$$

Deviation Bounds

Chebyshev's inequality

For every X and every k

$$\Pr[\,|X - \mathrm{E}[X]| \geq k\,] \leq \mathrm{Var}(X)/k^2$$

Equivalently,

$$\Pr[\,|X - \mathrm{E}[X]| \geq k\,\sigma(X)\,] \leq 1/k^2$$

Proof:

$$\begin{aligned} \Pr[\,|X - \mathrm{E}[X]| > k\,] &= \Pr[\,(X - \mathrm{E}[X])^2 > k^2\,] \leq \text{ (Markov)} \\ &\leq \mathrm{E}[(X - \mathrm{E}[X])^2]/k^2 = \mathrm{Var}(X)/k^2 \end{aligned}$$

Chebyshev gives (ε, δ)-approximations

Let algorithm A be such that $\mathrm{E}[A(x)] = f(x)$, $\mathrm{Var}(A(x)) \leq \sigma^2$

Algorithm $B(x)$ averages b independent copies of $A(x)$

We have $\mathrm{E}[B(x)] = f(x)$, $\mathrm{Var}(B(x)) \leq \sigma^2/b$

$$\Pr[\,|B(x) - f(x)| > \varepsilon\,] \leq \frac{\mathrm{Var}(B(x))}{\varepsilon^2} \leq \frac{\sigma^2}{b\varepsilon^2} \leq \delta$$

if we choose $b = \sigma^2 \frac{1}{\varepsilon^2} \frac{1}{\delta}$

Chebyshev gives (ε, δ)-approximations

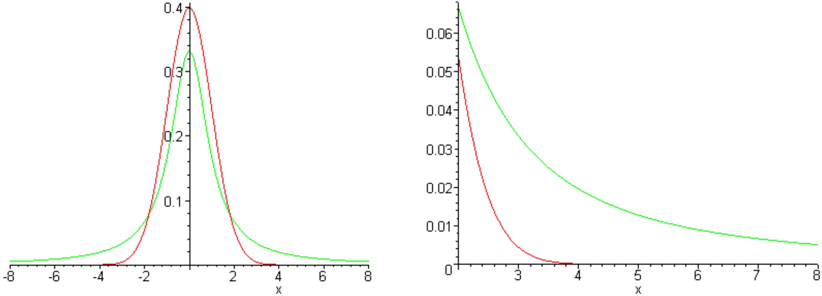
$\Pr[|X - E[X]| > k\sigma]$

$k = 1$	$k = 2$	$k = 3$	$k = 4$
≤ 1	≤ 0.25	≤ 0.11	≤ 0.07

But if X is normally distributed,

$k = 1$	$k = 2$	$k = 3$	$k = 4$
≤ 0.32	≤ 0.05	≤ 0.003	$\leq 3 \cdot 10^{-5}$

$\exp(-x^2)$ vs. $1/x^2$:



Sums of Independent Variables

- Suppose $X = \sum_{i=1}^{n} X_i$, $\mathrm{E}[X_i] = p$, $\mathrm{Var}(X_i) = \sigma^2$, all X_i independent and bounded
- By the Central Limit Theorem, $Z_n = (X - np)/\sqrt{n\sigma^2}$ tends to normal $N(0,1)$ as $n \to \infty$,
- And approximating by the normal gives

$$\Pr[\, Z_n \geq \alpha \,] \approx \exp(-\alpha^2/2)$$

- Chebyshev only gives

$$\Pr[\, Z_n \geq \alpha \,] \leq \frac{1}{\alpha^2}$$

Chernoff-Hoeffding bounds

- $X_1, X_2, \ldots X_n$ be independent random variables,
- $X_i \in [0,1]$, $\mathrm{E}[X_i] = p$,
- $X = \sum_{i=1}^{n} X_i$, so $\mathrm{E}[X] = pn$

Hoeffding bound (absolute deviation)

$$\Pr[X - pn > \varepsilon n] < \exp(-2\varepsilon^2 n)$$
$$\Pr[X - pn < -\varepsilon n] < \exp(-2\varepsilon^2 n)$$

Chernoff bound (relative deviation)

For $\varepsilon \in [0,1]$,

$$\Pr[X - pn > \varepsilon pn] < \exp(-\varepsilon^2 pn/3)$$
$$\Pr[X - pn < -\varepsilon pn] < \exp(-\varepsilon^2 pn/2)$$

Note: *Bernstein's inequality* is more general and (in essence) subsumes both

Example: Approximating the Mean

Input: ε, δ, random variable $X \in [0,1]$ (Important: bounded)

Output: (ε,δ)-approximation of $\mathrm{E}[X]$

Algorithm $A(\varepsilon,\delta)$

- Draw $n = \frac{1}{2\varepsilon^2} \ln \frac{2}{\delta}$ copies of X
- Output their average Y

Example: Approximating the Mean

- Let X_i be ith copy of X
- Then $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$, and $\mathrm{E}[Y] = \mathrm{E}[X]$
- By Hoeffding,

$$\begin{aligned} \Pr[\,|Y - \mathrm{E}[X]| > \varepsilon\,] &= \Pr[\sum_{i=1}^{n} X_i - \mathrm{E}[\sum_{i=1}^{n} X_i] > \varepsilon n] \\ &< 2\exp(-2\varepsilon^2 n) = 2\exp(-\ln(2/\delta)) = \delta \end{aligned}$$

- A different, sequential, algorithm gets (ε, δ) relative approximation using

$$O\left(\frac{1}{\varepsilon^2 \mathrm{E}[X]} \ln \frac{1}{\delta}\right)$$

samples of X
[Dagum-Karp-Luby-Ross 95, Lipton-Naughton 95]

Example: Approximating the Median

Input: ε, δ, set S of real numbers (Note: no bound assumed)

Output: some $s \in S$ whose rank in S is $(1/2 \pm \varepsilon)|S|$

Algorithm $A(\varepsilon, \delta)$

- Draw $n = \frac{1}{2\varepsilon^2} \ln \frac{2}{\delta}$ random elements from S
- Output the median of these n elements

Example: Approximating the Median

- Let X_i be 1 if ith sample has rank $\leq (1/2-\varepsilon)|S|$, 0 otherwise
- $\mathrm{E}[X_i] = 1/2-\varepsilon$
- By Hoeffding,

$$
\begin{aligned}
& \Pr[\geq n/2 \text{ draws give elements with rank} \leq (1/2-\varepsilon)|S|] \\
\leq \ & \Pr[\sum_{i=1}^{n} X_i \geq n/2] = \Pr[\sum_{i=1}^{n} X_i \geq \mathrm{E}[\sum_{i=1}^{n} X_i] + \varepsilon n] \\
\leq \ & \exp(-2\varepsilon^2 n) = \delta/2
\end{aligned}
$$

- Therefore, with probability $< \delta/2$ we draw $\geq n/2$ elements of rank $\leq (1/2-\varepsilon)|S|$. Implies median of sample $> (1/2-\varepsilon)|S|$
- Similarly the other side

Example use in Data Streams: Sampling rate

- Suppose items arrive at so high speed that we have to skip some
- Sample randomly:
  - Choose to process each element with probability α
  - Ignore each element with prob. $1 - \alpha$
- At any time t, if queried for the median, return the median of the elements chosen so far

Exercise.

Given α, δ, determine the probability ε_t such that at time t the output of the algorithm above is an (ε_t, δ)-approximation of the median on the first t elements of the stream

Improved (ε, δ)-approximation: $1/\delta$ to $\ln(1/\delta)$

Let algorithm A be such that $\mathrm{E}[A(x)] = f(x)$, $\mathrm{Var}(A(x)) \leq \sigma^2$
(Note: no bound assumed)

B: Run b independent copies of A and average results
With $b = 6\sigma^2/\varepsilon^2$ we have

$$\Pr[|B(x) - f(x)| \geq \varepsilon] < 1/6$$

C: Run c independent copies of B and take median
With $c = \dfrac{1}{2(1/2 - 1/6)^2} \ln \dfrac{2}{\delta}$ we have (Exercise: check!)

$$\Pr[|C(x) - f(x)| > \varepsilon] \leq \delta$$

Memory and runtime blowup is $b \cdot c = 27\sigma^2 \dfrac{1}{\varepsilon^2} \ln \dfrac{2}{\delta}$

A better analysis reduces constant 27 to about 4

The Distinct Element Counting Problem

How many *distinct* elements have we seen so far in the data stream?

Motivation

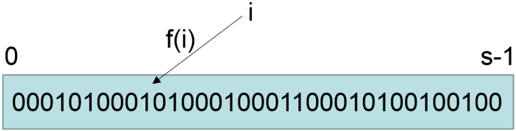
Item spaces and # distinct elements can be large

- I'm a web searcher. How many different queries did I get?
- I'm a router. How many pairs (sourceIP,destinationIP) have I seen?
  - itemspace: potentially 2^{128} in IPv6
- I'm a text message service. How many distinct messages have I seen?
  - itemspace: essentially infinite
- I'm an streaming classifier builder. How many distinct values have I seen for this attribute x?

Counting distinct elements

- Item space I, cardinality n, identified with range $[n]$
- $f_{i,t}$ = # occurrences of $i \in I$ among first t stream elements
- d_t = number of i's for which $f_{i,t} > 0$
- Often omit subindex t
- Solving *exactly* requires $O(d)$ memory
- Approximate solutions using $O(d)$, $O(\log d)$ and $O(\log \log d)$ bits

Linear counting [Whang+90] $\simeq$ Bloom filters



Init($d_{\max}$,ρ):

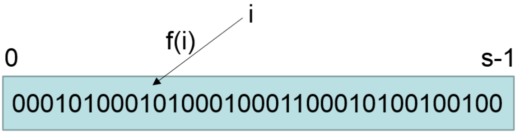
- upper bound $d_{\max} \geq d$
- $\rho < 1$, load factor
- build a bit vector B of size $s = \rho\, d_{\max}$
- choose a hash function $f : [n] \to s$

Update(x): $B[f(x)] \leftarrow 1$

Query:

- w = the fraction of 0's in B
- return $s \cdot \ln(1/w)$

Linear counting [Whang+90] $\simeq$ Bloom filters



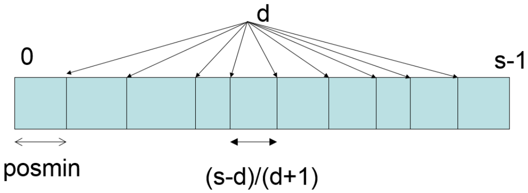
$$w = \Pr[\text{bucket } i \text{ after } d \text{ distinct elements}] = (1 - 1/s)^d \simeq \exp(-d/s)$$

$$E[\text{Query}] \simeq d, \qquad \sigma(\text{Query}) = \text{small!}$$

Issue: What is a “good” hash function?

- $f(i)$ uniformly distributed, even given all other values of f
- “Reproducibly random”
- How to get one: Later!

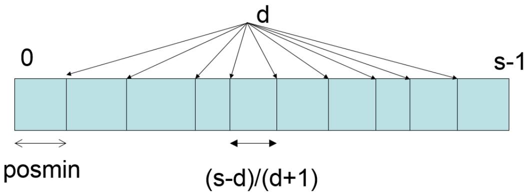
Cohen's algorithm [Cohen97]



E[gap between two 1's in B] = $(s - d)/(d + 1) \simeq s/d$

Query: return s / (size of first gap in B)

Cohen's algorithm [Cohen97]



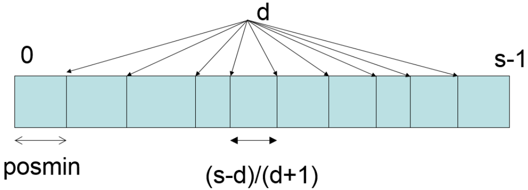
Trick: Don't store B, remember smallest key inserted in B

Init: posmin = s; choose hash function $f : [n] \to s$

Update(x): if ($f(x) <$ posmin) posmin $\leftarrow f(x)$

Query: return $s/$posmin

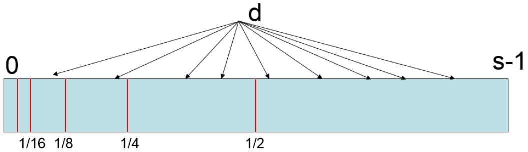
Cohen's algorithm [Cohen97]



$E[posmin] \simeq s/d \qquad \sigma(posmin) \simeq s/d$

Memory = (bits to store *posmin*) =
$\log(posmin) \leq \log s = O(\log d_{\max})$

Probabilistic Counting [Flajolet-Martin 85]



Bloom filter. But: Observe values of hash function $f(i)$, in binary

Idea: To see $f(i) = 0^{k-1}1\ldots$, about 2^k distinct values inserted

And we don't need to store B, just the smallest k

Flajolet-Martin probabilistic counter

Init: $p \leftarrow 0$

Update(x):

- let b be the position of the leftmost 1 bit of $f(x)$
- if ($b > p$) $p \leftarrow b$

Query: return 2^p

$E[2^p] = d/\varphi$, for a constant $\varphi = 0.77\ldots$
Memory = (bits to store p) = $\log p = \log\log d_{\max}$ bits

Flajolet-Martin: reducing the variance

Solution 1: Use c independent copies, average

- Problem 1: runtime multiplied by c
- Problem 2: independent runs = generate *independent* hash functions
- And we don't know how to generate several independent hash functions

Flajolet-Martin: reducing the variance

Solution 2:

- Divide stream into $c = O(\varepsilon^{-2})$ substreams
- Use first bits of $f(x)$ to decide substream for x
- Track p separately for each substream
- Same f can be used for all copies
- One sketch update per item

Memory = $O(c \log\log d_{\max}) = O(\varepsilon^{-2} \log\log d_{\max})$

Improving the leading constants

- Original [Flajolet-Martin 85]: Geometric average of estimations
- SuperLogLog [Durand+03]: Remove top 30%, then geometric average
- HyperLogLog [Flajolet+07]: Harmonic average

Standard deviation is $\simeq 1.03/\sqrt{c}$ for HyperLogLog

HyperLogLog: "cardinalities up to 10^9 can be approximated within say 2% with 1.5 Kbytes of memory"

Implementation aspects: [Heule+13]

Linear or logarithmic?

[Metwaly+08]

- "Why go logarithmic when we can go linear"
- Describe an application where extreme accuracy needed
- e.g., 10^{-4}
- For this range, linear counting uses less memory
- My take: I have ML/DM in mind; low accuracy is ok, *and* we will need to maintain *many* counts

4. Finding Frequent Elements

Heavy Hitters, Elephants, Hotlist analysis, Iceberg queries

Finding Frequent Elements

The Heavy Hitter Problem

Given a sequence S of t elements, threshold θ, find all elements with frequency $> \theta t$ - the heavy hitters

Interesting for skewed distributions

There are at most $\lfloor 1/\theta \rfloor$ heavy hitters

Good sources: [Berinde+09], [Cormode+08]

Approaches

1. Sampling: Output the heavy hitters computed in a sample
   - Uniform sample can be kept with reservoir sampling technique
   - Doable with sample size $O(1/\theta^2)$ (Hoeffding)

Solutions with memory $O(1/\theta)$:

2. Count based. We cover SpaceSaving Sketch
3. Hash based: Count-Min Sketch

The SpaceSaving sketch [Metwally+05]

- One of many counter-based methods:
  Karp-Shenker-Papadimitriou, Lossy Counter, Frequent, Sticky Sampling, GroupTest, $\ldots$
- Memory $O(1/\theta)$. Best possible
- Good update time
- Guarantee on count error
- No false negatives; but has false positives

The SpaceSaving sketch

Init(θ): Create

$k \leftarrow \lceil 1/\theta \rceil$

set of keys $K \leftarrow \emptyset$

vector *count*, indexed by K

Update(x):

if x is in K then $count[x]{+}{+}$

else, if $|K| < k$, add x to K and set $count[x] = 1$

else, replace an item with lowest count with x

and increase its count by 1

Query:

return the set K

Why Does This Work?

Claims:

Let min_t be the minimum value of a counter at time $t > 0$. Then

1. $min_t \leq t/k$
2. If $f_{x,t} > min_t$, then $x \in K$ at time t
3. For every $x \in K$, $f_{x,t} \leq count_t[x] \leq f_{x,t} + min_t$

In particular, all items with frequency over t/k are in K

Proof: By joint induction on t. Exercise: prove it!

More on SpaceSaving

Efficient implementation: StreamSummary data structure

Exercise

Without looking into the paper, propose an efficient data structure for SpaceSaving. Aim for $O(1)$ update time and $O(k) = O(1/\theta)$ items, counts, pointers, etc.

The Count-Min Sketch

[Cormode-Muthukrishnan 04]
Like SpaceSaving:

- Provides an approximation f'_x to f_x, for every x
- Can be used (less directly) to find θ-heavy hitters
- Uses memory $O(1/\theta)$

Unlike SpaceSaving:

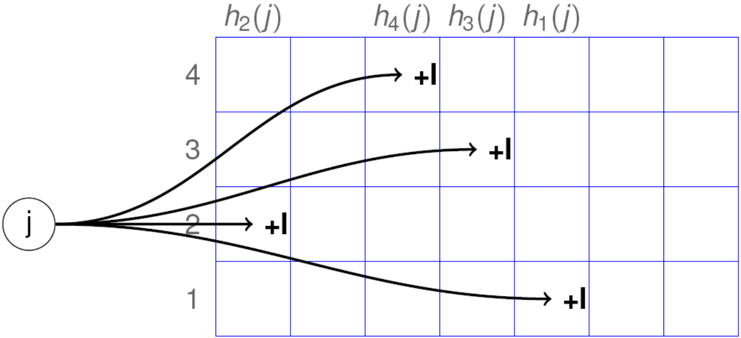
- It is randomized - hash functions instead of counters
- Supports additions and deletions
- Supports (not trivially) Heavy Hitters
- Can be used as basis for several other queries

The Count-Min Sketch

- Vector $F[n]$. Assumes $F[i] \geq 0$ for all i, at all times
- Provides estimations F' of F such that
  1. $F[i] \leq F'[i]$ for all i
  2. For every $i \in I$, $F'[i] \leq F[i] + \varepsilon|F|_1$ with probability $\geq 1 - \delta$

  where $|F|_1 = \sum_i F[i]$
- Note: $|F|_1$ may be $\ll$ stream length, if subtractions allowed
- Uses $O(\frac{1}{\varepsilon} \ln \frac{1}{\delta})$ memory words, $O(\ln \frac{1}{\delta})$ update time

The Count-Min Sketch



source: A. Bifet,

`http://albertbifet.com/comp423523a-2012-stream-data-mining/`

The Count-Min Sketch

- d independent hash functions $h_1 \ldots h_d$: $[1..n] \to [1..w]$
- one "memory cell" for each $h_j(i)$
- On instruction "$F[i]$ += v", do $h_j(i)$ += v for all $j \in 1 \ldots d$
- Estimation:

$$F'[i] = \min\{\, h_j(i) \mid j = 1..d \,\}$$

The Count-Min Sketch

$$F'[i] = \min\{\, h_j(i) \mid j = 1..d \,\}$$

- $F'[i] \geq F[i]$
  For each instruction involving i, we update all counts $h_j(i)$
  $F[i] \geq 0$ at all times for all i

- $F'[i] = F[i]$?
  No: cell $h_j(i)$ is also incremented by $k \neq i$ if $h_j(k) = h_j(i)$
- But it is unlikely that this occurs very often
- min instead of average $\rightarrow$ Markov instead of Chebyshev or Hoeffding

The Count-Min Sketch: Proof of main bound

- Fix j. Def random var $I_{ijk} = 1$ if $h_j(i) = h_j(k)$, 0 otherwise
- If h good hash function

$$E[I_{ijk}] \leq 1/\text{range}(h_j) = 1/w$$

- Def $X_{ij} = \sum_k I_{ijk} F[k]$. Then

$$E[X_{ij}] = \sum_k E[I_{ijk}] F[k] \leq |F|_1 / w$$

The Count-Min Sketch: Proof of main bound (2)

Then by Markov's inequality and pairwise independence:

$$\Pr[X_{ij} \geq \varepsilon |F|_1] \leq E[X_{ij}]/(\varepsilon |F|_1) \leq (|F|_1/w)/(\varepsilon |F|_1) \leq 1/2$$

if $w = 2/\varepsilon$. Then:

$$\begin{aligned} & \Pr[F'[i] \geq F[i] + \varepsilon |F|_1] \\ = \; & \Pr[\forall j \; : \; F[i] + X_{ij} \geq F[i] + \varepsilon |F|_1] \\ = \; & \Pr[\forall j \; : \; X_{ij} \geq \varepsilon |F|_1] \\ \leq \; & (1/2)^d = \delta \qquad \text{if } d = \log(1/\delta) \end{aligned}$$

for one fixed i. To have good estimates for all i simultaneously, use $d = \log(n/\delta)$ and use union bound

The Count-Min Sketch: Summary

- Memory is $\frac{2}{\varepsilon} \log \frac{1}{\delta}$ words
- Update time $O(\log \frac{1}{\delta})$
- Replace $\log(1/\delta)$ with $\log(n/\delta)$ if the bound needs to hold for all i simultaneously

  "$\Pr[\text{for all } i, \ldots] \leq \delta$" instead of "for all i, $\Pr[\ldots] \leq \delta$"
- Error for $F[i]$ is ε relative to $|F|_1$, not to $F[i]$

Back to Heavy Hitters

- i is a θ-heavy hitter if $F[i] \geq \theta t$
- The CM-sketch with width θ guarantees

$$F[i] \leq F'[i] \leq F[i] + \theta t$$

- So: If we output all i s.t. $F'[i] \geq \theta t$, we output all heavy hitters; no false negatives

But we can't cycle through all n candidates one by one!

Range-Sum queries

Range-sum query

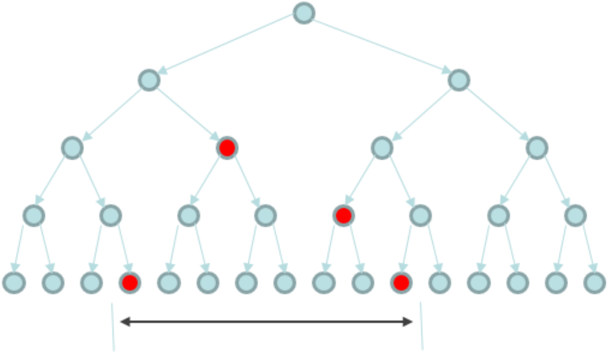
Given a, b, return $\sum_{i=a}^{b} F[i]$

Example: how many packets received came from the IP range 172.16.xxx.xxx?

We show:

- A variant of CM-sketch supports range-sum queries efficiently
- Answering range-sum queries efficiently $\longrightarrow$ finding heavy hitters efficiently

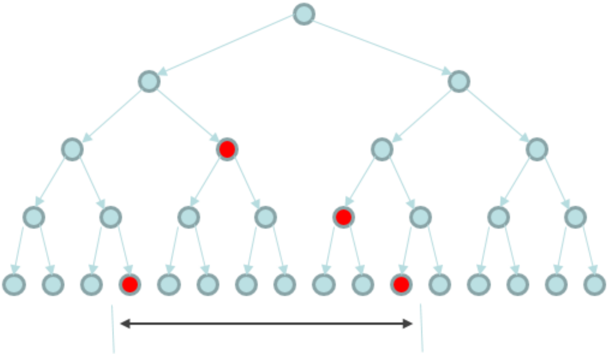
Fom CM-sketch to range-sum queries



For $p = 0 \ldots \log n$, for each $j = ...$, keep the value of $sum(j2^p \ldots (j+1)2^p - 1)$

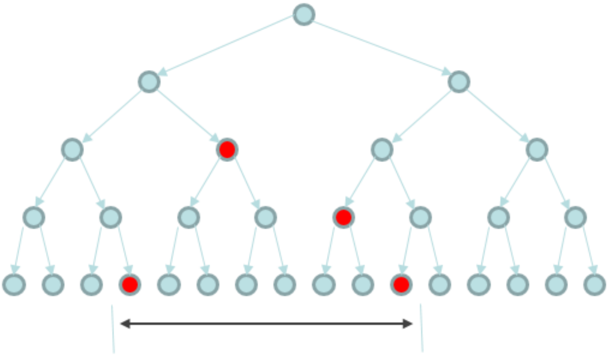
Any interval $[a, b]$ is the sum of $O(\log n)$ such values. Check it

From CM-sketch to range-sum queries



Keep one CM-sketch for each 2^p to store $sum(j2^p \ldots (j+1)2^p - 1)$ for each j

From CM-sketch to range-sum queries



When receiving *i*, update the counts for ranges where *i* lies = ancestors of *i* in the tree

When queried *sum*(*a*..*b*), decompose [*a*..*b*] as sum of such intervals, retrieve and add their sums

From Range-sum queries to heavy hitters

- Adaptively search for heavy hitters in the tree
- if a node has count $< \theta t$, do not explore its children: no heavy hitters below
- if a node has count $\geq \theta t$, explore both children
- when reaching a leaf, we know whether it's a heavy hitter
- the sum of counts at any one level of the tree is t
- no more than $1/\theta$ of them may have frequency $\geq \theta t$
- Efficiency: no more than $1/\theta$ nodes of each level are expanded

From Range-sum queries to heavy hitters

Exercise

Formalize the algorithms above:

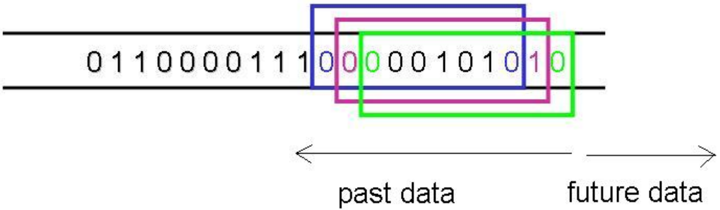
- For computing range-sum queries given CM-sketch
- Form finding all heavy hitters using range-sum queries

and tell their memory usage and update time

Other uses of CM-Sketch - Range-Sum queries

- Quantile computation: Given i, θ, find for all k the $q(k)$ such that
$$\sum_{i=1}^{q(k)} F[i] = k\theta \sum_{i=1}^{n} F[i]$$
- Reverse, histogram computation: Given f, how many i's have frequence f?"
- Inner product of two streams
- ...

5. Counting in Sliding Windows

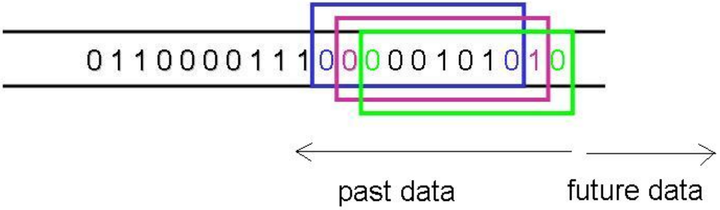


- Only last *n* items matter
- Clear way to bound memory
- Natural in applications: emphasizes most recent data
- Data that is too old does not affect our decisions

Examples:

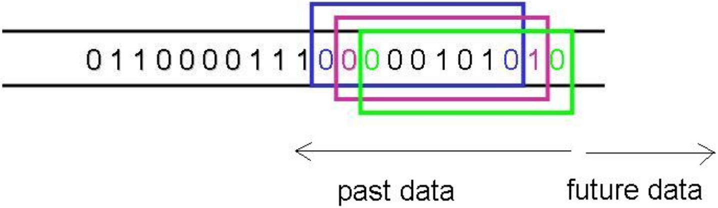
- Study network packets in the last day
- Detect top-10 queries in search engine in last month
- Analyze phone calls in last hours

Statistics on Sliding Windows



- Want to maintain mean, variance, histograms, frequency moments, hash tables, ...
- SQL on streams. Extension of relational algebra
- Want quick answers to queries at all times

Basic Problem: Counting 1's



Obvious algorithm, memory n:

- Keep window explicitly
- At each time t, add new bit b to head, remove oldest bit b' from tail,
- Add b and subtract b' from count

Fact:

$\Omega(n)$ memory bits are necessary to solve this problem exactly

Counting 1's

[Datar, Gionis, Indyk, Motwani, 2002]

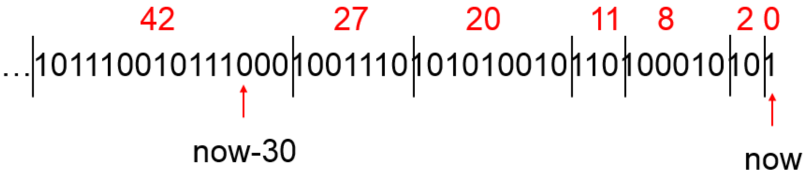
Theorem:

Estimating number of 1's in a window of length n with multiplicative error ε is possible with $O(\frac{1}{\varepsilon}\log n)$ counters $= O(\frac{1}{\varepsilon}(\log n)^2)$ bits of memory

Example:

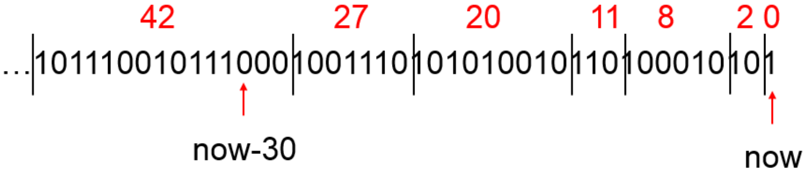
- $n = 10^6$; $\varepsilon = 0.1 \rightarrow$ 200 counters, 4000 bits

Idea: Exponential Histograms



- Each bit has a timestamp - time at which it arrived
- At time t, bits with timestamp $\leq t - n$ are *expired*
- We have up to k buckets of capacity 1, 2, 4, 8 …
- Each bucket contains the number of 1s in a subwindow, up to its capacity
- Errors: expired bits in the last bucket
- 1's in last bucket $\leq$ (1's in previous buckets) / k

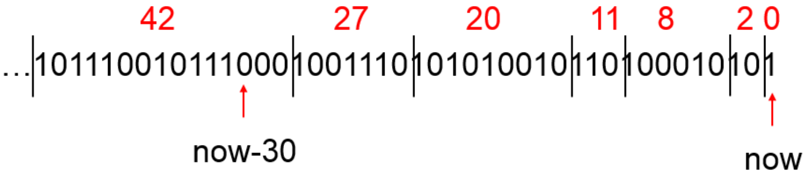
Exponential Histograms



Init: Create empty set of buckets

Query: Return total number of bits in buckets - last bucket / 2

Exponential Histograms



Insert rule(bit *b*):

- If *b* is a 0, ignore it. Otherwise, if it's a 1:
- Add a bucket with 1 bit and current timestamp *t* to the front
- for $i = 0, 1, \ldots$

  If more than *k* buckets of capacity 2^i,

    merge two oldest as newest bucket of capacity 2^{i+1},

    with timestamp of the older one
- if oldest bucket timestamp $< t - n$, drop it (all expired)

Memory Estimate

- Largest bucket needed: $k\sum_{i=0}^{C} 2^i \simeq n \to C \simeq \log(n/k)$
- Total number of buckets: $k \cdot (C+1) \simeq k \log(n/k)$
- Each bucket contains a timestamp only (perhaps its capacity, dep. on implementation)
- timestamps are in $t-n \dots t$: recycle timestamps mod n
- Memory is $O(k \log(n/k) \log n)$ bits; take $k = 1/2\varepsilon$

Generalizations

Applies also to other natural aggregates:

- Variance
- Distinct elements (using Flajolet-Martin)
- Max, min
- Histograms
- Hash tables
- Frequency moments

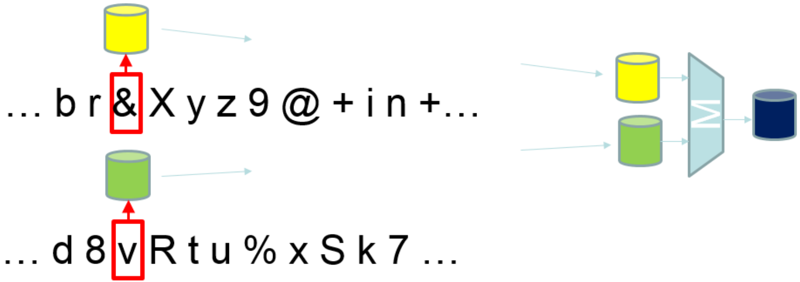
and can be combined with CM-sketch

6. Distributed Sketching

Setting:

- Many sources generating streams concurrently
- No synchrony assumption
- Want to compute global statistics
- Streams can send short summaries to central

Merging sketches



Send the sketches, not the whole stream

Merging sketches

Mergeability

A sketch algorithm is mergeable if

- given two sketches $S1$ and $S2$ generated by the algorithm on two data streams $D1$ and $D2$,
- one can compute a sketch S that answers queries correctly with respect to the concatenation of $D1$ and $D2$

Note: For frequency problems,
"for the concatenation" = "for all interleavings"

Merging sketches

All sketches we've seen are mergeable efficiently

- Bloom filters, Cohen, Flajolet-Martin, HyperLogLog
- SpaceSaving
- CM-sketch
- Exponential Histograms (though order dependent problem)

May require sites to use common random bits or hash functions

7. Wrapping up. Hash functions

- Perfect hash function: $f(i)$ cannot be guessed at all even from all other values of f
- Storing $f : A \to B$ unfeasible for large A

Wrapping up. Hash functions (2)

- Cryptographic hash functions (MD5, SHA1, SHA256, or MurmurHash) should work well, but are costly.
- Even simpler functions like linear congruential may work well in practice if not in theory — but don't use 32 bit integers if you plan to count billions!
- $O(\log n)$ bits to store such a function for $|A| = |B| = n$
- But we can't "generate many of them", e.g., to reduce variance

- Sometimes, analysis reveals that weaker notions of "good hash function" suffices
- E.g., pairwise independence suffices for CM-sketch: $f(i)$ independent of any other single $f(j)$
- (In general, will work if you use only Chebyshev or Markov)
- We can generate mutually independent, pairwise independent functions
- One can be stored with $O(\log n)$ bits

Wrapping up. Some stuff I left out

- Detecting duplicate documents
- Detecting near duplicates (LSH), minwise hashing, ...
- Sketches for geometric problems. Clustering
- Graph sketches. Counting subgraphs
- Using HyperLogLog to estimate neighborhood functions of graphs
- Sketches that are linear projections. Metric embeddings. Dimensionality reduction
- Linear algebra. PCA. Singular Value Decomposition

Wrapping up. Last words

Approximation helps

Randomness helps

Some more tools in your toolbox

`http://www.cs.upc.edu/~gavalda`

8. References and resources

With apologies to all missing papers

General Surveys on Stream Algorithmics:

- Survey by Liberty and Nelson: `http://www.cs.yale.edu/homes/el327/papers/streaming_data_mining.pdf`
- J. Ullman and A. Rajaraman, Mining of Massive Datasets, Chapter 3 - available at `http://infolab.stanford.edu/~ullman/mmds/ch4.pdf`
- A very general bibliography by K. Tufte: `http://web.cecs.pdx.edu/~tufte/410-510DS/readings.htm`
- Lecture notes by A. Chakrabarti: `http://www.cs.dartmouth.edu/~ac/Teach/CS85-Fall09/Notes/lecnotes.pdf`
- Survey by Lin and Zhang: `http://www.cse.unsw.edu.au/~yingz/papers/apweb_2008.pdf`
- Book by G. Cormode, M. Garofalakis, P. Haas, and C. Jermain: `http://dimacs.rutgers.edu/~graham/pubs/html/CormodeGarofalakisHaasJermaine12.html`
- Survey by G. Cormode: `http://dimacs.rutgers.edu/~graham/pubs/papers/sk.pdf`

8. References and resources

Approximate counting

- The original Morris77 paper: `http://dl.acm.org/citation.cfm?id=359627` also available here: `http://www.inf.ed.ac.uk/teaching/courses/exc/reading/morris.pdf`
- An analysis of Morris' counter (math intensive): `http://algo.inria.fr/flajolet/Publications/Flajolet85c.pdf`
- The application of Morris' counters to counting n-grams, by Van Durme and Lall: `http://www.cs.jhu.edu/~vandurme/papers/VanDurmeLallIJCAI09.pdf`

Large deviation bounds

- G. Lugosi: http://www.econ.upf.edu/~lugosi/anu.pdf
- A. Sinclair: http://www.cs.berkeley.edu/~sinclair/cs271/n13.pdf
- C. Shalizi list of references (much beyond the scope of this course): http://bactra.org/notebooks/large-deviations.html

Counting distinct elements

- Good general survey of distinct element counting up to 2008: Ahmed Metwally, Divyakant Agrawal, Amr El Abbadi: Why go logarithmic if we can go linear?: Towards effective distinct counting of search traffic. EDBT 2008: 618-629.
- Also general discussion on distinct element counting: `http://highscalability.com/blog/2012/4/5/big-data-counting-how-to-count-a-billion-distinct-objects-html`
- Presentation including some sketches I didn't mention: `http://www.cs.upc.edu/~conrado/research/talks/aofa2012.pdf`
- Bloom filter. K.Y. Whang, B. Vander-Zanden, H.M. Taylor, A Linear-time Probabilistic Counting Algorithm for Database Applications. ACM Trans. Database Syst., 15:2, 1990.
- Cohen's log(n) solution: Edith Cohen, Size-Estimation Framework with Applications to Transitive Closure and Reachability . FOCS 1994 and JCSS 1997.

HyperLogLog and related for distinct element counting

- The Flajolet-Martin probabilistic counter. Philippe Flajolet, G. Nigel Martin: Probabilistic Counting Algorithms for Data Base Applications. J. Comput. Syst. Sci. 31(2): 182-209 (1985). See also http://en.wikipedia.org/wiki/Flajolet-Martin_algorithm
- SuperLogLog counter (and insight on FM probabilistic counter) Durand, M.; Flajolet, P. (2003). "Loglog Counting of Large Cardinalities". Algorithms - ESA 2003. Lecture Notes in Computer Science 2832. p. 605.
- The HyperLogLog paper: Flajolet, P.; Fusy, E.; Gandouet, O.; Meunier, F. (2007). "HyperLogLog: the analysis of a near-optimal cardinality estimation algorithm". AOFA 07: Proceedings of the 2007 International Conference on the Analysis of Algorithms.
- Flajolet's contributions explained beautifully by J. Lumbroso: http://www.stat.purdue.edu/~mdw/ChapterIntroductions/ApproxCountingLumbroso.pdf

8. References and resources

HyperLogLog and related for distinct element counting (2)

- http://en.wikipedia.org/wiki/HyperLogLog
- http://research.neustar.biz/2012/10/25/ sketch-of-the-day-hyperloglog-cornerstone-of-a-big-data-i
- A live demo of hyperloglog at the web above: http://content.research.neustar.biz/blog/hll.html
- http://www.slideshare.net/sunnyujjawal/ hyperloglog-in-practice-algorithmic-engineering-of-a-state
- http://stackoverflow.com/questions/12327004/ how-does-the-hyperloglog-algorithm-work
- Important optimizations that I'd like to try: http://druid.io/blog/2014/02/18/ hyperloglog-optimizations-for-real-world-systems. html. Also here: http://research.google.com/pubs/pub40671.html

Heavy hitters - count-based approaches

- J. Vitter. Random Sampling with a reservoir. ACM Trans. on Mathematical Software, 1985.
- Good survey of heavy hitter algorithms. Radu Berinde, Graham Cormode, Piotr Indyk, Martin J. Strauss. Space-optimal Heavy Hitters with Strong Error Bounds
- Also very good survey: Graham Cormode, Marios Hadjieleftheriou. Finding Frequent Items in Data Streams. Proc. VLDB Endowment, 2008
- Richard M. Karp, Scott Shenker, Christos H. Papadimitriou. A Simple Algorithm for Finding Frequent Elements in Streams and Bags. ACM Transactions on Database Systems (TODS), Volume 28, 2003.
- The Space-Saving sketch paper. Ahmed Metwally, Divyakant Agrawal, Amr El Abbadi. Efficient Computation of Frequent and Top-k Elements in Data Streams. Intl. Conf. on Database Technology (ICDT) 2005.
- M. Charikar, K. Chen and M. Farach-Colton. ”Finding Frequent Items in Data Streams.” ICALP 2002 (conf. version) and Theoretical Computer Science 2004 (journal version)

Count-Min sketch and related

- The CM-Sketch paper. Graham Cormode and S. Muthukrishnan: An improved data stream summary: The Count-min sketch and its applications. J. Algorithms 55: 2938
- On Frugal Streaming, a neat sketch for estimating quantiles which I did not cover in the course: `http://research.neustar.biz/2013/09/16/sketch-of-the-day-frugal-streaming/`
- `http://en.wikipedia.org/wiki/Count-min_sketch`
- `https://sites.google.com/site/countminsketch/`
- `https://tech.shareaholic.com/2012/12/03/the-count-min-sketch-how-to-count-over-large-keyspac`

Counting in Sliding Windows

- Mayur Datar, Aristides Gionis, Piotr Indyk, Rajeev Motwani: Maintaining Stream Statistics over Sliding Windows. SIAM J. Comput. 31(6): 1794-1813 (2002). Conf. version in SODA 2002.
- Mayur Datar, Rajeev Motwani: The Sliding-Window Computation Model and Results. Data Streams - Models and Algorithms 2007: 149-167. `http://link.springer.com/chapter/10.1007%2F978-0-387-47534-9_8`

Mergeability

- Discussions on mergeability are a bit all over. This is sort of an overview: `http://research.microsoft.com/en-us/events/bda2013/mergeable-long.pptx`

Others (personal 1-slide selection)

- Noga Alon, Yossi Matias, Mario Szegedy: The space complexity of approximating frequency moments. J. Computer and System Sciences 58(1): 137-147 (1999). Conference version (STOC) 1996
- Paolo Boldi, Marco Rosa, and Sebastiano Vigna. HyperANF: Approximating the neighbourhood function of very large graphs on a budget. WWW, 2011.
- An application of the above to computing diameter of the Facebook graph: Lars Backstrom, Paolo Boldi, Marco Rosa, Johan Ugander, Sebastiano Vigna. Four Degrees of Separation. ACM Web Science 2012, 2012.
- A survey on streaming graph algorithms: `http://people.cs.umass.edu/~mcgregor/papers/13-graphsurvey.pdf`
- Computing SVD on streams, this will be important in streaming ML: Mina Ghashami, Edo Liberty, Jeff M. Phillips, David P. Woodruff, Frequent Directions : Simple and Deterministic Matrix Sketching. `http://arxiv.org/abs/1501.01711`
- This will also be important in streaming ML: Christos Boutsidis, Dan Garber, Zohar Karnin, Edo Liberty: Online Principal Component Analysis, SODA 2015. `http://www.cs.yale.edu/homes/el327/papers/opca.pdf`

Resources

- The MassDAL Code Bank. http://www.cs.rutgers.edu/~muthu/massdal-code-index.html
- StreamLib: https://github.com/addthis/stream-lib. Check this too: http://www.addthis.com/blog/2011/03/29/new-open-source-stream-summarizing-java-library/#.VTzMcJPl_VI
- Hokusai: https://github.com/dgryski/hokusai. I have not used it, but it looks very interesting from http://arxiv.org/ftp/arxiv/papers/1210/1210.4891.pdf and http://blog.aggregateknowledge.com/2013/09/16/sketch-of-the-day-frugal-streaming/
- Webgraph. Analysis of large graphs, contains the HyperANF and related code used for the Four-degrees-of-separation paper: http://webgraph.di.unimi.it/

8. References and resources

Resources

I have not used the following, so no guarantees of any kind (including that they still exist)

- C++: `https://github.com/hideo55/cpp-HyperLogLog/blob/master/src/hyperloglog.hpp`
- Java: `https://github.com/addthis/stream-lib/tree/master/src/main/java/com/clearspring/analytics/stream/cardinality`
- Python: `https://pypi.python.org/pypi/hyperloglog/0.0.8`
- Ruby: `https://rubygems.org/gems/hyperloglog`
- Perl: `http://search.cpan.org/~hideakio/Algorithm-HyperLogLog-0.20/lib/Algorithm/HyperLogLog.pm`
- JavaScript: `http://cnpmjs.org/package/hyperloglog`
- node.js: `https://www.npmjs.org/package/streamcount`
- `https://github.com/eclesh/hyperloglog/blob/master/hyperloglog.go`