Tractable Clones of Polynomials over Semigroups

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Setting

- Finite domain D
- Finite set of relations Γ over D
- $\mathrm{CSP}(\Gamma)$: Constraint Satisfaction Problems with relations from Γ

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Question:

- For which Γ is $\mathrm{CSP}(\Gamma)$ solvable in polynomial time?
- Or: Find conditions on Γ which imply (in)tractability

The Algebraic Approach

A relation R is closed under function f if

$$T_1, \ldots, T_k \in R \quad \longrightarrow \quad f(T_1, \ldots, T_k) \in R$$

Fact. [Jeavons98]

The complexity of $\mathrm{CSP}(\Gamma)$ is completely determined by the set of functions f that close all relations in Γ

Several large classes of tractable Γ have been described in terms of “closure properties”

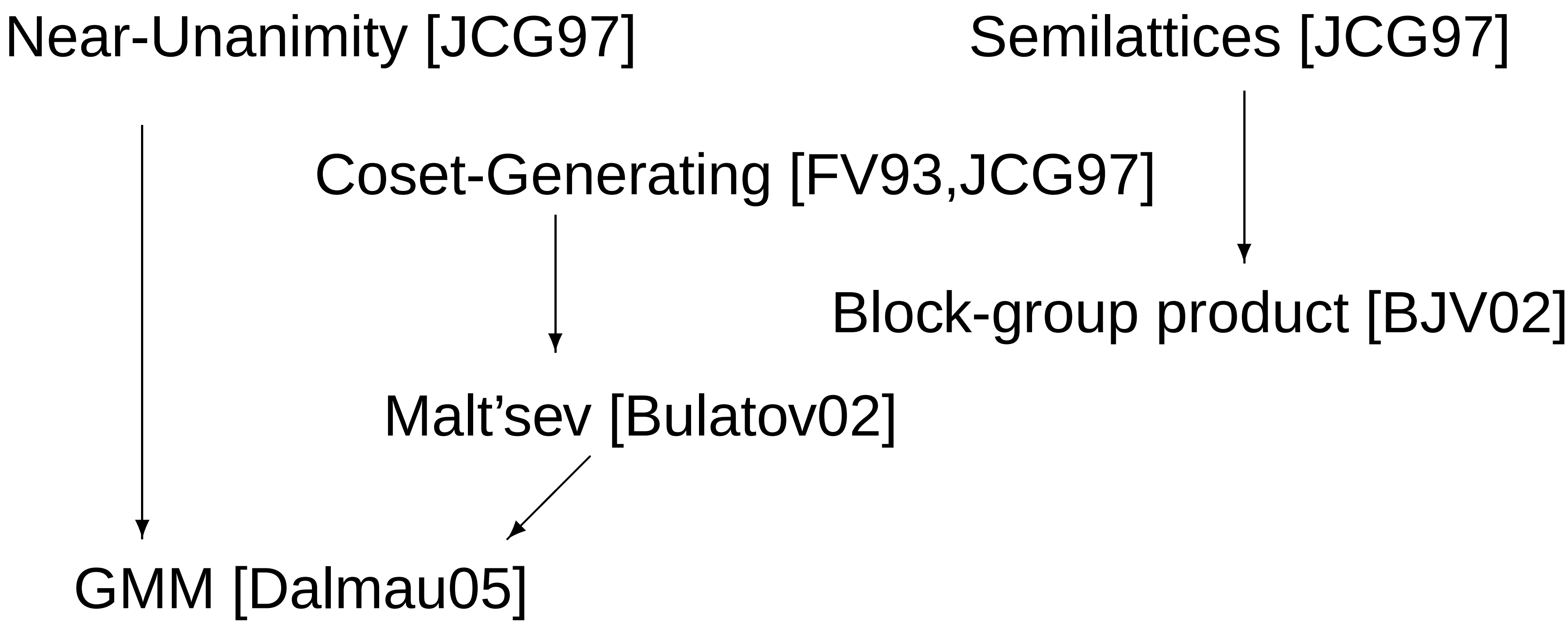
Some Tractable Operations

Near-Unanimity [JCG97]

Semilattices [JCG97]

Coset-Generating [FV93,JCG97]

Some Tractable Operations



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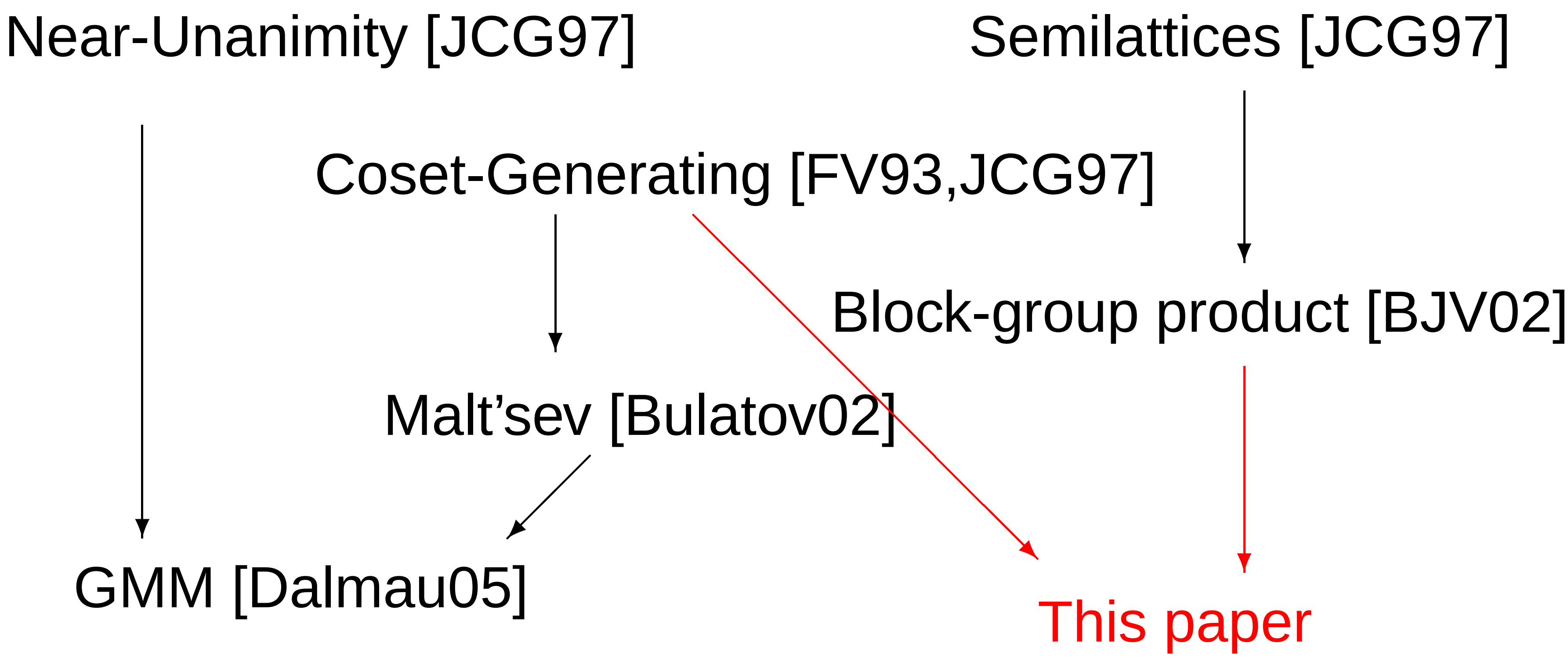

Some Tractable Operations (2)

Coset-generating op.:

$(D, \cdot)$ a group, Γ closed under $x \cdot y^{-1} \cdot z$

Semilattice op.:

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Block-group op.:

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Note: Closure under $x \cdot y$ implies closure under $x \cdot y^{-1} \cdot z$

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$(D, \cdot)$ a semilattice, Γ closed under $x \cdot y$

Block-group op.:

$(D, \cdot)$ a block-group, Γ closed under $x \cdot y$

Theorem. If $(D, \cdot)$ is a block-group and Γ is closed under $x \cdot y^{\omega-1} \cdot z$, then $\mathrm{CSP}(\Gamma)$ is tractable.

A Unifying Result

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ω: minimum such that $(x^\omega)^2 = x^\omega$

Block group: Satisfies $(x^\omega y^\omega)^\omega = (y^\omega x^\omega)^\omega$

A Unifying Result

Theorem. If $(D, \cdot)$ is a block-group and Γ is closed under $x \cdot y^{\omega-1} \cdot z$ then $\mathrm{CSP}(\Gamma)$ is tractable.

Proof idea:

- Apply first Arc-Consistency, as for semilattices.
  This places every variable in one of the groups in $(D, \cdot)$
- Then apply the coset-generating technique within each group
- Both techniques will not interfere in block-groups

Polynomials

Observation:

In the boolean domain, our algorithm solves instances containing e.g. (only) Horn clauses or (only) linear equations.

In larger domains, it solves problems that “decompose” as Horn *then* linear equations.

Polynomials

- Why only $x \cdot y$ or $x \cdot y^{\omega-1} \cdot z$?
- Polynomial: an expression of the form

$$x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdot \ldots \cdot x_{i_m}^{n_m}$$

- Interpret $\cdot$ is as product over a semigroup $S = (D, \cdot)$. Then the polynomial computes a function.
- $\mathrm{Pol}(\Gamma) =$ set of functions that close all relations in Γ
- Study $\mathrm{CSP}(\Gamma)$ when $\mathrm{Pol}(\Gamma)$ is a “clone of polynomials”

Why Study Polynomials?

- They alone explain several of the known tractable cases
- Based on semigroup operations . . .
- . . . and we have fine tools for decomposing / analyzing semigroups
- May indicate more ways to combine existing tractability paradigms

Intractable Clones of Polynomials

A condition on the semigroup:

Theorem. If S is not a block-group, then every clone of polynomials over S is NP-complete.

A condition on the clone:

Theorem. If a clone of polynomials $\mathcal{C}$ is a d-factor then $\mathcal{C}$ is NP-complete.

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- η subgroup exponent of S, $d > 1$, $d \mid \eta \mid \omega$

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- Condition preserved by composition and variable identification

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- Condition preserved by composition and variable identification
- Implies that there is a “hard” subuniverse of D preserved by $\mathcal{C}$, hence NP-completeness

Tractability within Block-groups?

$\mathcal{C}$ clone of polynomials over a block-group

"$\mathcal{C}$ not a d-factor" necessary condition for tractability

For some block-groups, it is also sufficient:

- Commutative semigroups
- Nilpotent groups

Tractability within Block-groups? (2)

Theorem. If S is a commutative semigroup and $\mathcal{C}$ a nontrivial, idempotent clone of polynomials over S, then the following are equivalent:

1) $\mathcal{C}$ is tractable

2) $\mathcal{C}$ is not a d-factor, for any d

3) $\mathcal{C}$ contains $x \cdot y^{\omega-1} \cdot z$

Tractability within Block-groups? (3)

Theorem. If S is a nilpotent group and $\mathcal{C}$ a nontrivial, idempotent clone of polynomials over S, then the following are equivalent:

1) $\mathcal{C}$ is tractable

2) $\mathcal{C}$ is not a d-factor, for any d

3) $\mathcal{C}$ contains a Malt'sev operation

Summary

- Polynomials over semigroups are a natural way of computing k-ary functions
- The polynomial $xy^{\omega-1}z$ over block-groups is sufficient for tractability; this unifies two existing results: coset-generating and block-group product
- It is also necessary for tractability of commutative clones of polynomials
- Conjecture: Its generalization, Malt'sev, is necessary over clones polynomials over groups