Probabilistic analysis of algorithms: What's it good for?

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The goal

- Given some algorithm A taking inputs from some set $\mathcal{I}$, we would like to analyze the performance of the algorithm as a function of the input size (and possibly other parameters).

Why should we analyze algorithms?

- To predict the resources (time, space. ...) that the algorithm will consume
- To compare algorithm A with competing alternatives
- To improve the algorithm, by spotting the performance bottlenecks
- To explain observed behavior

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Concepts & definitions

- The performance $\mu : \mathcal{I} \to \mathbb{N}$ depends on each particular instance of the input
- We have to introduce some notion of size: $|\cdot| : \mathcal{I} \to \mathbb{N}$; we may safely assume that each $\mathcal{I}_n = \{x \in \mathcal{I} \,|\, |x| = n\}$ is finite
- Worst-case:

$$\mu^{[\text{worst}]}(n) = \max\{\mu(x) \,|\, x \in \mathcal{I}_n\}$$

- To analyze "typical behavior" or the performance of randomized algorithms, we have to assume some probabilistic distribution on the input and/or the algorithm's choices; hence, we consider the performance as a family of random variables $\{\mu_n\}_{n\geq 0}$; $\mu_n : \mathcal{I}_n \to \mathbb{N}$
- Average-case:

$$\mu^{[\text{avg}]}(n) = \mathbb{E}[\mu_n] = \sum_{k\geq 0} k\, \mathbb{P}[\mu_n = k]$$

When we assume uniformly distributed inputs

$$\mathbb{P}[x] = \frac{1}{\#\mathcal{I}_n}, \text{for all } x \in \mathcal{I}_n$$

our problem is one of counting, e.g.,

$$\mathbb{E}[\mu_n] = \frac{\sum_{x\in\mathcal{I}_n} \mu(x)}{\#\mathcal{I}_n}$$

One of the most important tools in the analysis of algorithms are generating functions:

$$A(z,u) = \sum_{n\geq 0} \sum_{k\geq 0} \mathbb{P}[\mu_n = k]\, z^n u^k$$

For the uniform distribution

$$[z^n u^k] A(z,u) = [z^n u^k] \frac{\sum_{n\geq 0} \sum_{k\geq 0} a_{n,k} z^n u^k}{[z^n] \sum_{n\geq 0} a_n z^n} = \frac{[z^n u^k] B(z,u)}{[z^n] B(z,1)}$$

with $a_{n,k} = \#\{x \in \mathcal{I} \mid |x| = n \wedge \mu(x) = k\}$ and $a_n = \#\mathcal{I}_n$

The equations before can be expressed symbolically

$$B(z,u) = \sum_{x \in \mathcal{I}} z^{|x|} u^{\mu(x)}$$

The ratio of the n-th coefficients of $B(z,u)$ and $B(z,1)$ is the probability generating function of μ_n

$$p_n(u) = \sum_{k \geq 0} \mathbb{P}[\mu_n = k]\, u^k = \frac{[z^n]B(z,u)}{[z^n]B(z,1)}$$

Taking derivatives w.r.t. u and setting $u = 1$ we get the expected value, second factorial moment, ...

$$A^{(r)}(z) = \left.\frac{\partial^r A(z,u)}{\partial u^r}\right|_{u=1}$$
$$= \sum_{n \geq 0} \mathbb{E}[\mu_n{}^{\underline{r}}]\, z^n$$

For example,

$$\mathbb{V}[\mu_n] = \mathbb{E}\left[\mu_n{}^{\underline{2}}\right] + \mathbb{E}[\mu_n] - \mathbb{E}[\mu_n]^2 = [z^n]A^{(2)}(z) + [z^n]A(z) - ([z^n]A(z))^2$$

The symbolic method

The symbolic method translates combinatorial constructions to functional equations over generating functions.

Example: Consider the counting generating function of a combinatorial class $\mathcal{A}$:

$$A(z) = \sum_{n \geq 0} a_n z^n = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}$$

If $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ then

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|} = \sum_{(\beta,\gamma) \in \mathcal{B} \times \mathcal{C}} z^{|\beta|+|\gamma|} = \left(\sum_{\beta \in \mathcal{B}} z^{|\beta|} \right) \left(\sum_{\gamma \in \mathcal{C}} z^{|\gamma|} \right)$$

$$= B(z) \cdot C(z)$$

A dictionary of (labelled) combinatorial constructions and G.F.'s

$\{\epsilon\}$	1
$\{Z\}$	z
$\mathcal{A}+\mathcal{B}$	$A+B$
$\mathcal{A}\times\mathcal{B}$	$A\cdot B$
$\mathsf{Seq}(\mathcal{A})$	$\frac{1}{1-A}$
$\mathsf{Set}(\mathcal{A})$	$\exp(A)$
$\mathsf{Cycle}(\mathcal{A})$	$\log\frac{1}{1-A}$

Complex analysis

Another important set of techniques comes from complex variable analysis.
Under suitable technical conditions, if $F(z)$ is analytic in a disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and has a single dominant singularity at $z = 1$ then

$$F(z) \sim G(z) \implies [z^n]F(z) \sim [z^n]G(z)$$

This is one of the useful transfer lemmas of Flajolet and Odlyzko (1990). Many other similar results are extremely useful when computing asymptotic estimates for the n-th coefficient of a generating function.

For example, if

$$F(z) \sim G(z) \cdot \left(1 - \frac{z}{\rho}\right)^{-\alpha} + H(z)$$

as $z \to \rho$, the dominant singularity of $F(z)$, for some analytic $G(z)$ and $H(z)$ and $\alpha \notin \{-1, -2, \ldots\}$ then

$$[z^n]F(z) \sim G(\rho)\rho^{-n}\frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O(n^{-1})\right)$$

Complex analysis

In recent years, complex analysis techniques and perturbation theory have been used to prove powerful results such as Hwang's quasi-power theorem, which allows one to prove the convergence in law to a Gaussian distribution of many combinatorial parameters in strings, permutations, trees, etc., as well as local limits and the speed of convergence.

A trivial example: Counting binary trees

A binary tree is either an empty tree (leaf) or a root together with two binary (sub)trees:

$$\mathcal{B} = \{\epsilon\} + \{Z\} \times \mathcal{B} \times \mathcal{B}$$

Hence the counting GF of binary trees is

$$B(z) = 1 + zB^2(z)$$

Solving the equation before for $B(z)$ and since $B(0) = b_0 = 1$,

$$B(z) = \begin{cases} \frac{1-\sqrt{1-4z}}{2z} & z \neq 0, \\ 1 & z = 0. \end{cases}$$

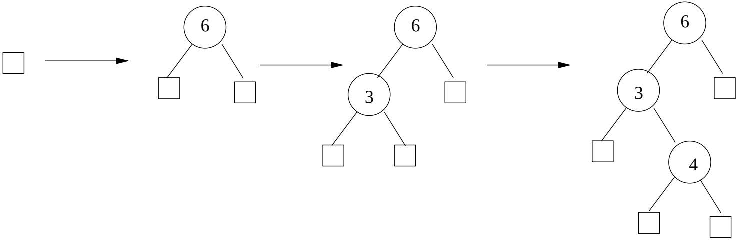
Extracting the n-th coefficient of $B(z)$ we find

$$[z^n]B(z) = \frac{\binom{2n}{n}}{n+1} \sim 4^n \frac{n^{-3/2}}{\sqrt{\pi}}$$

2^{nd} example: The average cost of building BSTs

A binary search tree T for a set of elements X contains some $y \in X$ at its root; its subtrees L and R are binary search trees recursively constructed for the sets $X^- = \{x \in X \mid x < y\}$ and $X^+ = \{z \in X \mid y < z\}$. Binary search trees (BSTs) are useful for fast lookup, and support both dynamic insertions and deletions.

To insert some new item w into a BST, we compare w to the element y at the root of T. If $w < y$ then we insert w recursively in the left subtree of T. If $y < w$ we insert recursively in the right subtree. If at any point we have to insert the element into an empty tree, we simply make w the root of the new tree.



How much does it cost to build a BST of size n?
Let $d(x,T)$ denote the depth (= edges from the root) of element x in T. It is the number of elements with which x was compared when we inserted it at some previous step.
Hence, the cost $I(T)$ to build T is

$$I(T) = \sum_{x \in T} d(x,T)$$

$I(T)$ is also called the internal path length of T.

Let $n = |T|$.

- If T is a linked list (at most one non-empty subtree per node) then $I(T) = n(n-1)/2$.
- If T is perfectly balanced then $I(T) = \Theta(n \log n)$.

In a random BST any element has the same probability of being its root; hence the probability that $|L| = k$ is $1/n$ for all k, $0 \leq k < n$.

$$\mathbb{P}[T] = \begin{cases} 1 & \text{if } T \text{ is empty,} \\ \frac{\mathbb{P}[L] \cdot \mathbb{P}[R]}{|T|} & \text{otherwise.} \end{cases}$$

$$I(T) = \begin{cases} 0 & \text{if } T \text{ is empty,} \\ I(L) + I(R) + |T| - 1 & \text{otherwise.} \end{cases}$$

$$
\begin{aligned}
\mathbb{E}[I_n] &= [z^n] I(z) \\
I(z) &= \sum_{T \in \mathcal{B}} \mathbb{P}[T]\, I(T) z^{|T|} \\
&= \sum_{(L,R) \in \mathcal{B} \times \mathcal{B}} \frac{\mathbb{P}[L]\, \mathbb{P}[R]}{|L| + |R| + 1} (I(L) + I(R) + |L| + |R|) z^{|L|+|R|+1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dI}{dz} &= \sum_{(L,R)\in\mathcal{B}\times\mathcal{B}} \mathbb{P}[L]\,\mathbb{P}[R]\,(I(L)+I(R)+|L|+|R|)z^{|L|+|R|} \\
&= \sum_{(L,R)\in\mathcal{B}\times\mathcal{B}} \mathbb{P}[L]\,\mathbb{P}[R]\,I(L)z^{|L|+|R|} \\
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&+ \sum_{(L,R)\in\mathcal{B}\times\mathcal{B}} \mathbb{P}[L]\,\mathbb{P}[R]\,|L|z^{|L|+|R|} \\
&+ \sum_{(L,R)\in\mathcal{B}\times\mathcal{B}} \mathbb{P}[L]\,\mathbb{P}[R]\,|R|z^{|L|+|R|}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dI}{dz} &= 2 \sum_{(T_1,T_2)\in\mathcal{B}\times\mathcal{B}} \mathbb{P}[T_1]\, I(T_1) z^{|T_1|}\, \mathbb{P}[T_2]\, z^{|T_2|} \\
&\quad + 2 \sum_{(T_1,T_2)\in\mathcal{B}\times\mathcal{B}} \mathbb{P}[T_1]\, |T_1| z^{|T_1|}\, \mathbb{P}[T_2]\, z^{|T_2|} \\
&= 2 \left(\sum_{T_1\in\mathcal{B}} \mathbb{P}[T_1]\, I(T_1) z^{|T_1|} \right) \cdot \left(\sum_{T_2\in\mathcal{B}} \mathbb{P}[T_2]\, z^{|T_2|} \right) \\
&\quad + 2 \left(\sum_{T_1\in\mathcal{B}} \mathbb{P}[T_1]\, |T_1| z^{|T_1|} \right) \cdot \left(\sum_{T_2\in\mathcal{B}} \mathbb{P}[T_2]\, z^{|T_2|} \right)
\end{aligned}
$$

$$\sum_{T\in\mathcal{B}} \mathbb{P}[T]\, z^{|T|} = \sum_{n\geq 0} z^n = \frac{1}{1-z}$$

$$\sum_{T\in\mathcal{B}} \mathbb{P}[T]\, |T| z^{|T|} = \sum_{n\geq 0} n z^n = z\frac{d}{dz}\frac{1}{1-z} = \frac{z}{(1-z)^2}$$

$$\frac{dI}{dz} = 2\frac{I(z)}{1-z} + \frac{2z}{(1-z)^3}$$

$$I(0) = 0$$

$$\Downarrow$$

$$I(z) = 2\frac{\ln\left(\frac{1}{1-z}\right)}{(1-z)^2} - \frac{2z}{(1-z)^2}$$

$$\mathbb{E}[I_n] = [z^n]I(z) = 2(n-1)H_n$$
$$\sim 2n\ln n + 2n\gamma + O(\log n)$$

$$H_n = \sum_{1\le k\le n} \frac{1}{k} \sim \ln n + \gamma + O(n^{-1})$$

3rd example: Insertion in K-d trees

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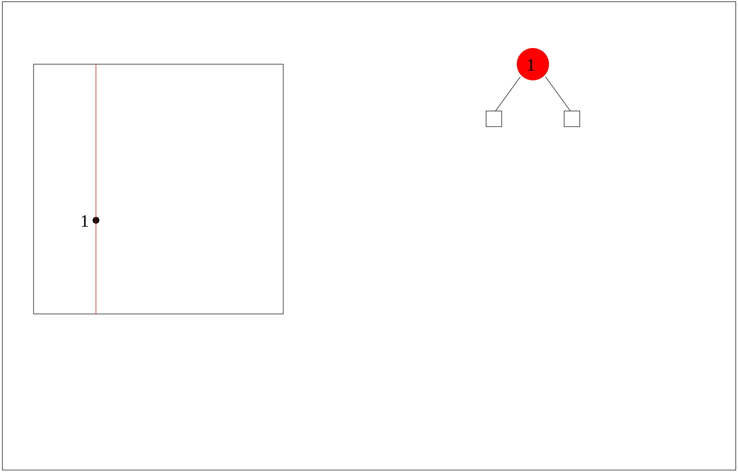

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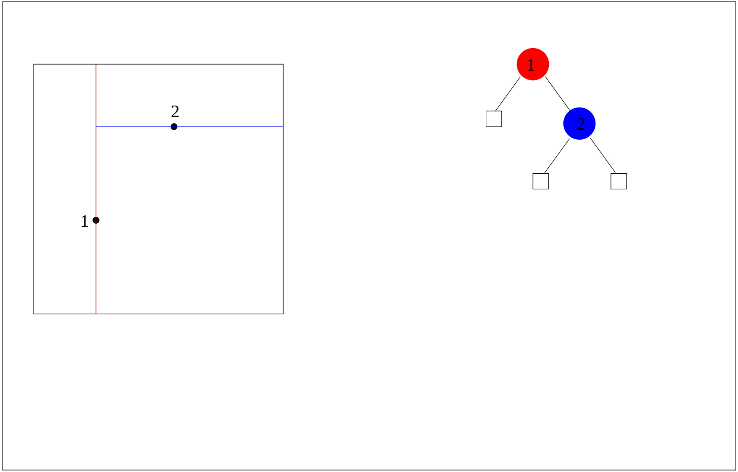

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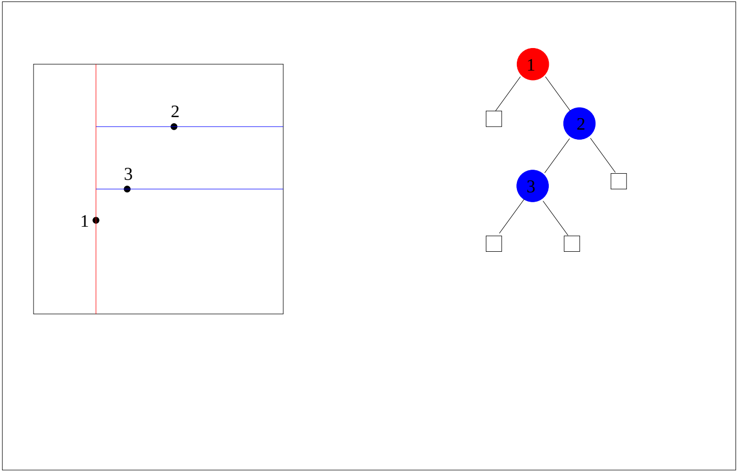

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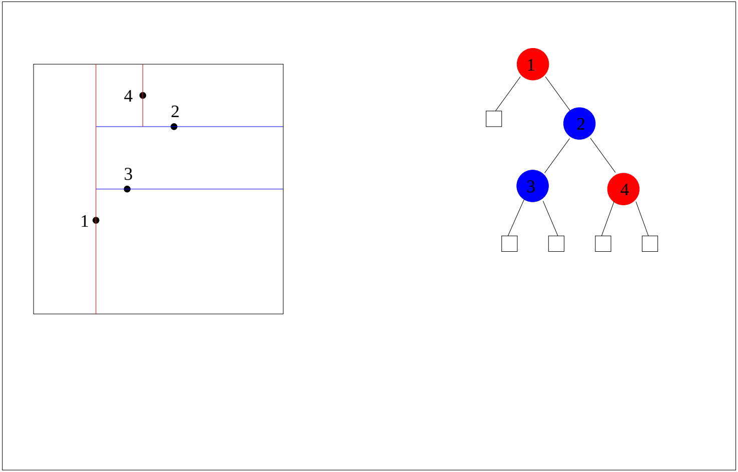

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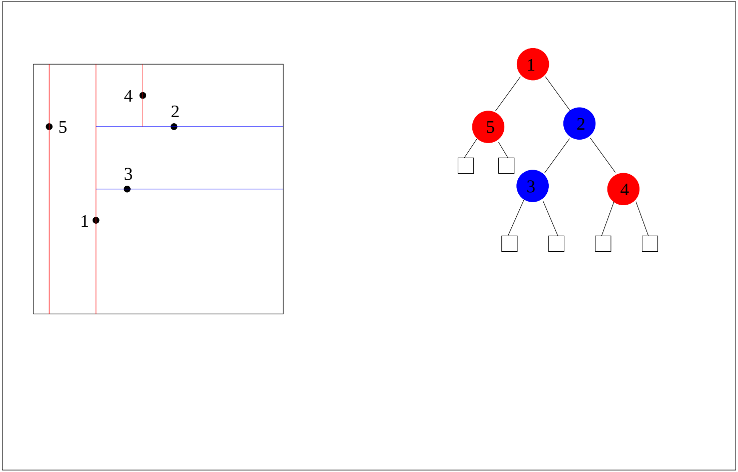


Insertion in relaxed K-d trees

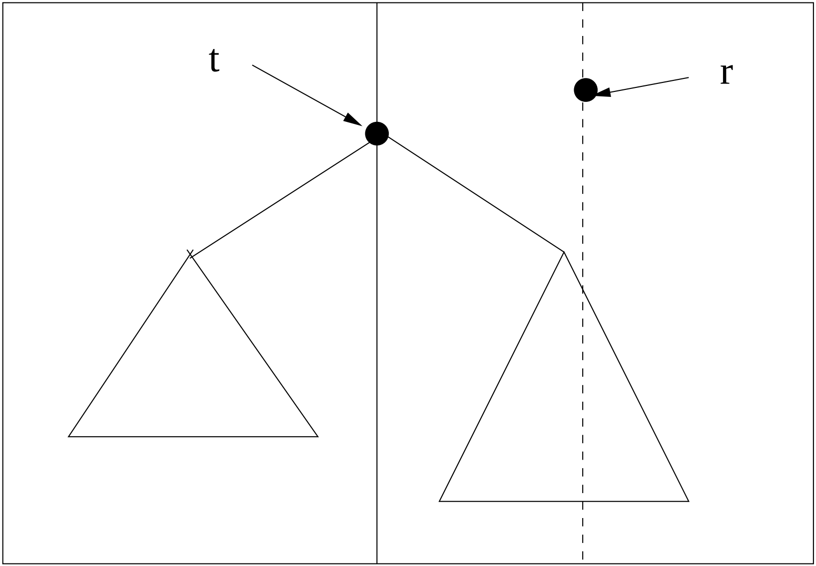
```
rkdt insert(rkdt t, const Elem& x) {
      int n = size(t);
      int u = random(0,n);
      if (u == n)
         return insert_at_root(t, x);
      else { // t cannot be empty
         int i = t -> discr;
         if (x[i] < t -> key[i])
            t -> left = insert(t -> left, x);
         else
            t -> right = insert(t -> right, x);
         return t;
      }
}
```

Deletion in relaxed K-d trees

```
rkdt delete(rkdt t, const Elem& x) {
     if (t == NULL) return NULL;
     if (t -> key == x)
        return join(t -> left, t -> right);

     int i = t -> discr;
     if (x -> key[i] < t -> key[i])
        t -> left = delete(t -> left, x);
     else
        t -> right = delete(t -> right, x);
     return t;
}
```

Split: Case #1



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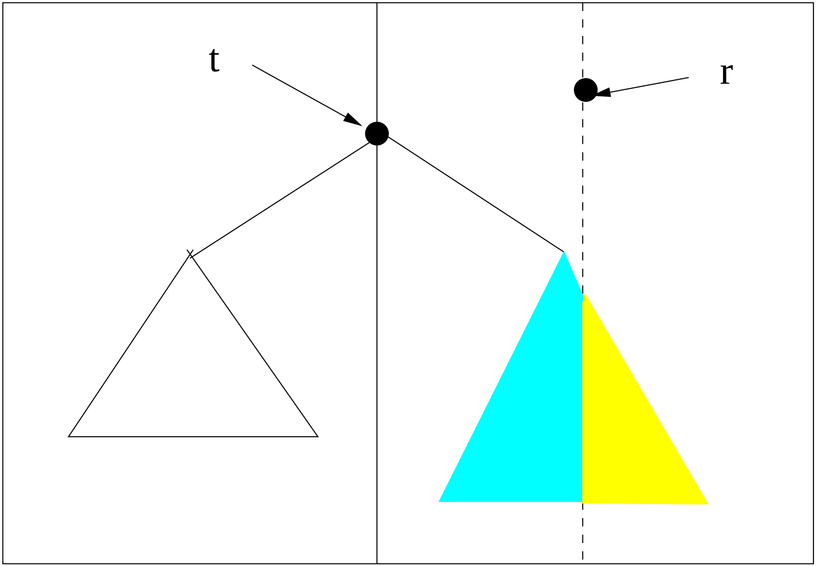

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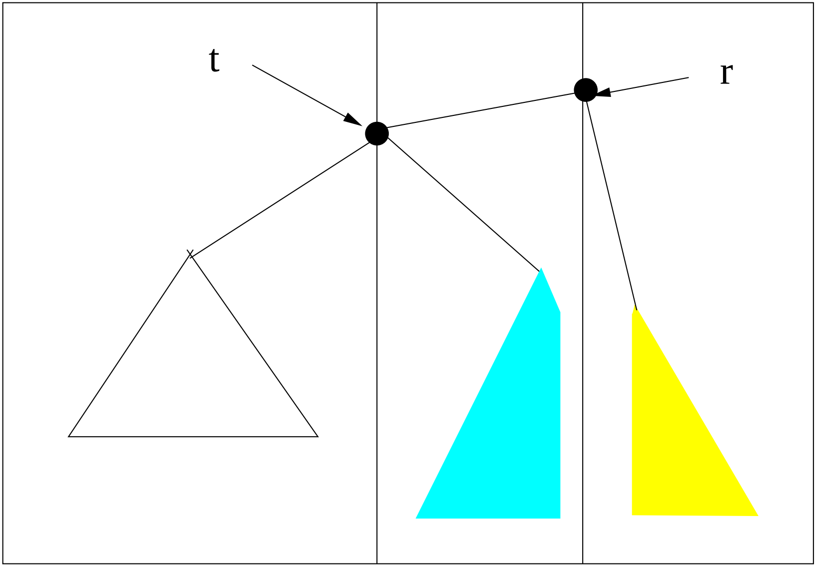
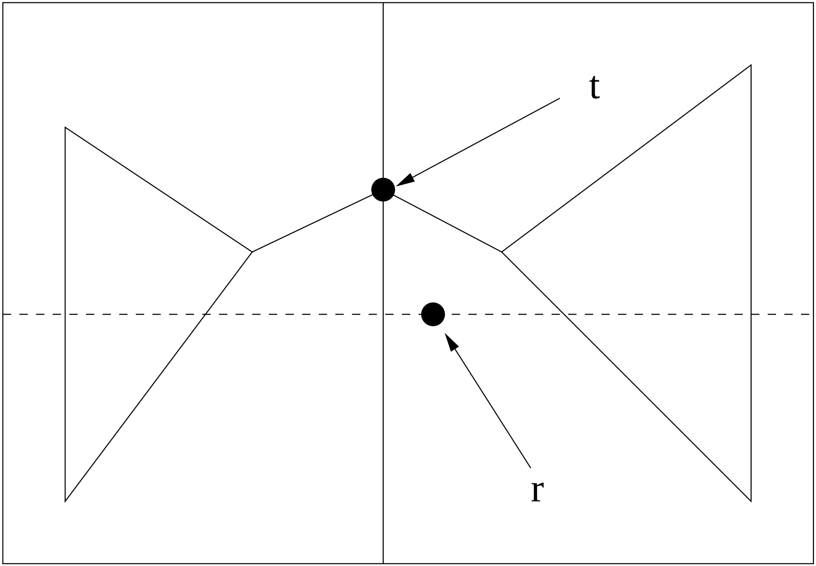


Split: Case #2



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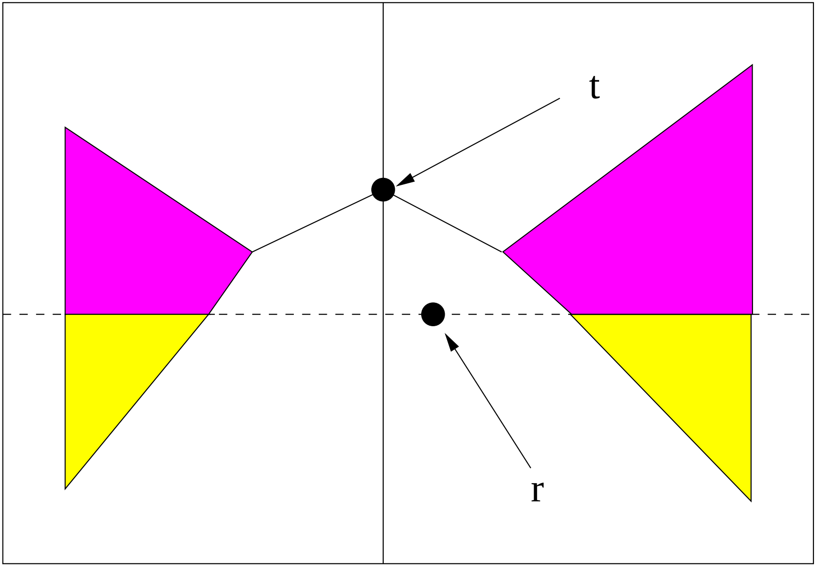

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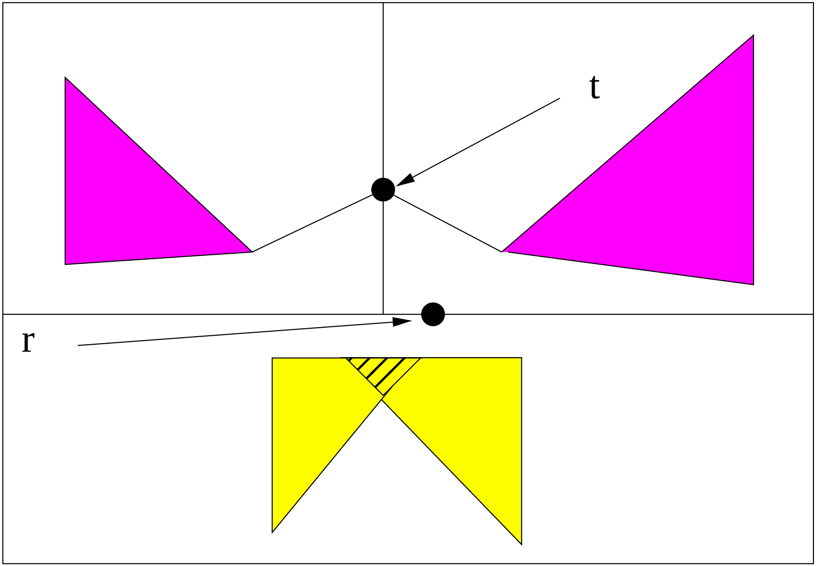

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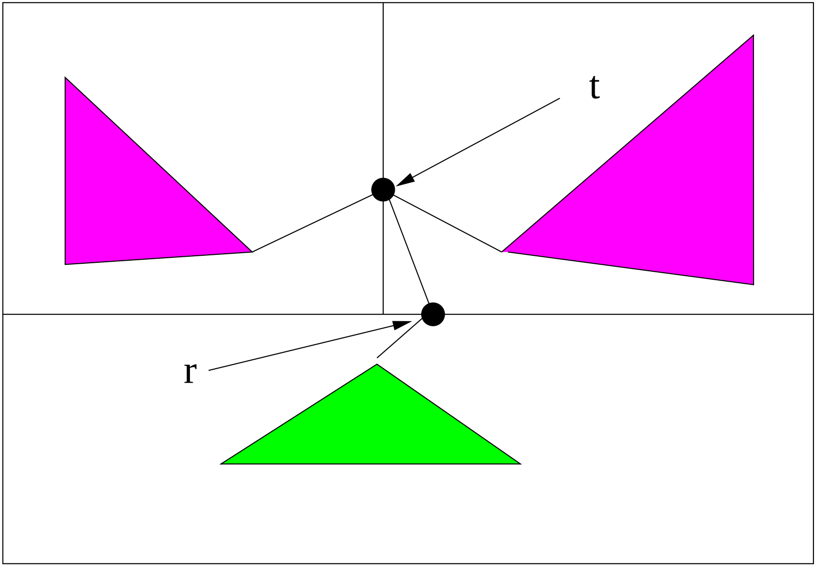

Analysis of split/join

- s_n = avg. number of visited nodes in a split
- m_n = avg. number of visited nodes in a join
- $$s_n = 1 + \frac{2}{nK} \sum_{0 \le j < n} \frac{j+1}{n+1} s_j + \frac{2(K-1)}{nK} \sum_{0 \le j < n} s_j + \frac{K-1}{K} \sum_{0 \le j < n} \pi_{n,j} m_j,$$

  where $\pi_{n,j}$ is probability of joining two trees with total size j.

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- The recurrence for s_n is

$$s_n = 1 + \frac{2}{nK} \sum_{0 \le j < n} \frac{j+1}{n+1} s_j + \frac{2(K-1)}{nK} \sum_{0 \le j < n} s_j + \frac{2(K-1)}{nK} \sum_{0 \le j < n} \frac{n-j}{n+1} m_j,$$

  with $s_0 = 0$.
- The recurrence for m_n has exactly the same shape with the rôles of s_n and m_n interchanged; it easily follows that $s_n = m_n$.

- Define
$$S(z) = \sum_{n\geq 0} s_n z^n$$
- The recurrence for s_n translates to
$$z\frac{d^2S}{dz^2} + 2\frac{1-2z}{1-z}\frac{dS}{dz} - 2\left(\frac{3K-2}{K} - z\right)\frac{S(z)}{(1-z)^2} = \frac{2}{(1-z)^3},$$
with initial conditions $S(0) = 0$ and $S'(0) = 1$.

- The homogeneous second order linear ODE is of hypergeometric type.
- An easy particular solution of the ODE is

$$-\frac{1}{2}\left(\frac{K}{K-1}\right)\frac{1}{1-z}$$

Theorem

The generating function $S(z)$ of the expected cost of split is, for any $K \geq 2$,

$$S(z) = \frac{1}{2}\frac{1}{1-\frac{1}{K}}\left[(1-z)^{-\alpha} \cdot {}_2F_1\left(\left.\begin{matrix} 1-\alpha, 2-\alpha \\ 2 \end{matrix}\,\right|\, z\right) - \frac{1}{1-z}\right],$$

where $\alpha = \alpha(K) = \frac{1}{2}\left(1 + \sqrt{17 - \frac{16}{K}}\right)$.

Theorem

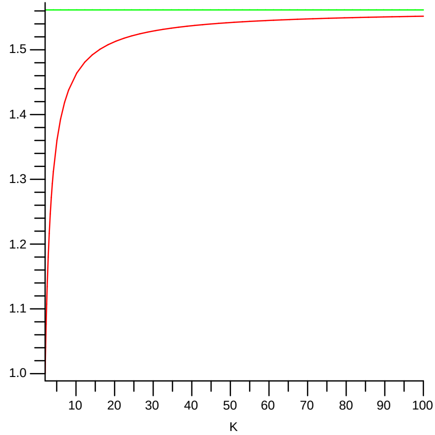
The expected cost s_n of splitting a relaxed K-d tree of size n is

$$s_n = \eta(K)\, n^{\phi(K)} + o(n),$$

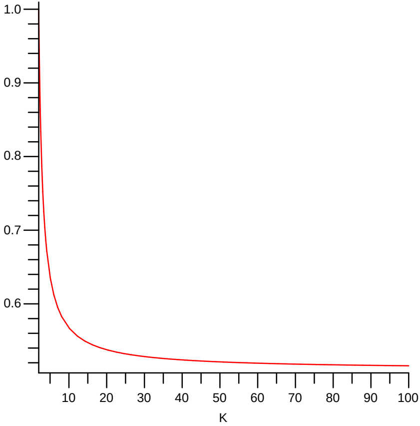
with

$$\eta = \frac{1}{2}\frac{1}{1-\frac{1}{K}}\frac{\Gamma(2\alpha-1)}{\alpha\Gamma^3(\alpha)},$$

$$\phi = \alpha - 1 = \frac{1}{2}\left(\sqrt{17-\frac{16}{K}}-1\right).$$



Plot of $\phi(K)$



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