Chunksort: A Generalized Partial Sort Algorithm

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Generalized partial sorting

The problem:

- The input: An array A of n elements and p intervals $I_1 = [\ell_1, u_1]$, $I_2 = [\ell_2, u_2]$, $\ldots$, $I_p = [\ell_p, u_p]$

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Example

$p = 2,\ I_1 = [5, 8],\ I_2 = [12, 12]$

3	11	5	7	8	4	9	1	13	10	12	14	15	2	6

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⟵ gap ⟶ ⟵ block ⟶ ⟵ gap ⟶ ...

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  - Multiple selection: $I_1 = [j_1, j_1], \ldots, I_p = [j_p, j_p]$
  - Partial sorting: $p = 1$, $I_1 = [1, m]$

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- They are simple, elegant, beatiful and practical divide-and-conquer solutions to sorting and selection
- Multiple quickselect uses the divide-an-conquer principle twice to solve the multiple selection problem (Prodinger, 1995)
- Partial quicksort is a slight variation of quicksort that efficiently solves the partial sorting problem (Martínez, 2004)

Quicksort

```
void quicksort(vector<Elem>& A, int i, int j) {
    if (i < j) {
       int p = select_pivot(A, i, j);
       swap(A[p], A[l]);
       int k;
       partition(A, i, j, k);
       // A[i..k-1] ≤ A[k] ≤ A[k+1..j]
       quicksort(A, i, k - 1);
       quicksort(A, k + 1, j);
} }
```

Quickselect

```
Elem quickselect(vector<Elem>& A,
                 int i, int j, int m) {
   if (i >= j) return A[i];
   int p = select_pivot(A, i, j, m);
   swap(A[p], A[l]);
   int k;
   partition(A, i, j, k);
   if (m < k)        quickselect(A, i, k - 1, m);
   else if (m > k) quickselect(A, k + 1, j, m);
   else              return A[k];
}
```

Quicksort: The recurrence for average cost

- Probability that the selected pivot is the k-th of n elements: $\pi_{n,k}$
- Average number of comparisons Q_n to sort n elements:

$$Q_n = n - 1 + \sum_{k=1}^{n} \pi_{n,k} \cdot (Q_{k-1} + Q_{n-k})$$

Quicksort: The average cost

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- Average number of comparisons Q_n to sort n elements (Hoare, 1962):

$$Q_n = 2(n+1)H_n - 4n$$
$$= 2n \ln n + (2\gamma - 4)n + 2 \ln n + \mathcal{O}(1)$$

where $H_n = \sum_{1 \le k \le n} 1/k = \ln n + \mathcal{O}(1)$ is the n-th harmonic number.

Quickselect: The recurrence for average cost

- Average number of comparisons $C_{n,j}$ to select the j-th out of n:

$$C_{n,j} = n - 1 + \sum_{k=j+1}^{n} \pi_{n,k} \cdot C_{k-1,m} + \sum_{k=1}^{j-1} \pi_{n,k} \cdot C_{n-k,m-k}$$

Quickselect: The average cost

- Average number of comparisons $C_{n,j}$ to select the j-th out of n elements (Knuth, 1971):

$$C_{n,j} = 2\big(n + 3 + (n+1)H_n - (n+3-j)H_{n+1-j} - (j+2)H_j\big).$$

Quickselect: The average cost

- Average number of comparisons $C_{n,j}$ to select the j-th out of n elements (Knuth, 1971):

$$C_{n,j} = 2(n + 3 + (n+1)H_n \\ - (n + 3 - j)H_{n+1-j} - (j+2)H_j).$$

- This is $\Theta(n)$ for any j, $1 \leq j \leq n$.

Partial quicksort

```
void partial_quicksort(vector<Elem>& A,
                       int i, int j, int m) {
    if (i < j) {
       int p = get_pivot(A, i, j);
       swap(A[p], A[l]);
       int k;
       partition(A, i, j, k);
       partial_quicksort(A, i, k - 1, m);
       if (k < m - 1)
          partial_quicksort(A, k + 1, j, m);
}  }
```

Partial quicksort: The average cost

- Average number of comparisons $P_{n,m}$ to sort the m smallest elements out of n:

$$P_{n,m} = n - 1 + \sum_{k=m+1}^{n} \pi_{n,k} \cdot P_{k-1,m} + \sum_{k=1}^{m} \pi_{n,k} \cdot \left(P_{k-1,k-1} + P_{n-k,m-k}\right)$$

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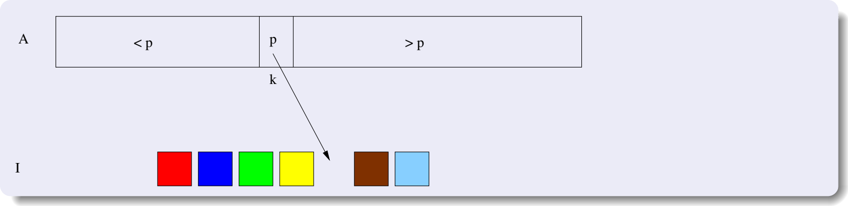
- The solution is (Martínez, 2004):

$$P_{n,m} = 2n + 2(n+1)H_n - 2(n+3-m)H_{n+1-m} - 6m + 6$$

Chunksort: An example



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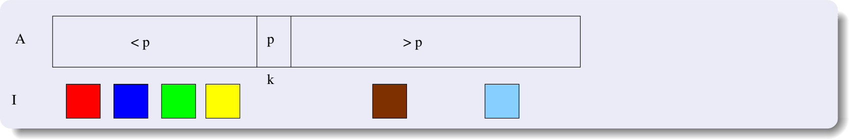

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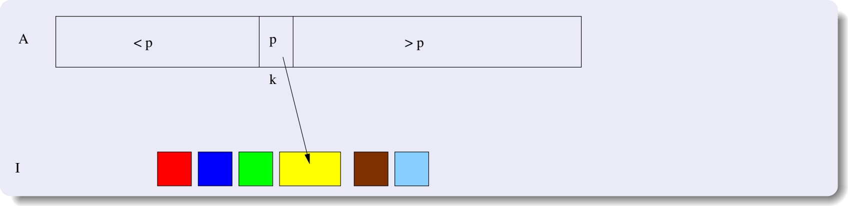

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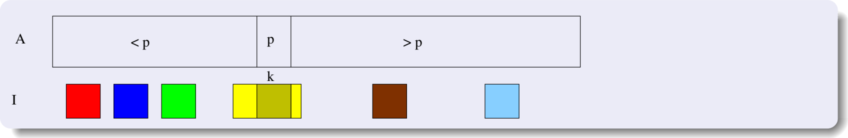

Chunksort

```
void chunksort(vector<T>& A, vector<int>& I,
               int i, int j, int l, int u) {
  if (i >= j) return;
  if (l <= u) {
    int k;  partition(A, i, j, k);
    int r = locate(I, l, u, k);
     // locate the value r such that I[r] ≤ k < I[r+1]
    if (r % 2 == 0) {
      // r = 2t  ⟹  I[r] = u_t ≤ k < ℓ_{t+1}
      chunksort(A, I, i, k - 1, l, r);
      chunksort(A, I, k + 1, j, r + 1, u);
    } else {
      // r = 2t-1  ⟹  I[r] = ℓ_t ≤ k < u_t
      chunksort(A, I, i, k - 1, l, r + 1);
      chunksort(A, I, k + 1, j, r, u);
}}}
```

Chunksort

Example (Using chunksort for partial sorting)

If $p = 1$, $I_1 = [1, m]$ then $r = 1$ whenever $k < m$; hence we make the calls

```
chunksort(A, I, i, k - 1, 1, 2);
chunksort(A, I, k + 1, j, 1, 2);
```

If $k \geq m$ then $r = 2$ and then we make the calls

```
chunksort(A, I, i, k - 1, 1, 2);
chunksort(A, I, k+1, j, 3, 2);
```

Chunksort

Example (Using chunksort for selection)

If $p = 1$, $I_1 = [j, j]$ then we will have $r = 0$ whenever $k < j$ so we call

```
chunksort(A, I, i, k - 1, 1, 0);
chunksort(A, I, k + 1, j, 1, 2);
```

If $k \geq j$ then $r = 2$ and then we make the calls

```
chunksort(A, I, i, k - 1, 1, 2);
chunksort(A, I, k + 1, j, 3, 2);
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1. Introduction
2. The algorithm
3. The analysis
4. Conclusions

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- We only count element comparisons
- Each partitioning stage needs $n-1$ comparisons of the pivot with all the other elements
- We assume that pivots are chosen at random ($\pi_{n,k} = 1/n$)
- $C_{i,j}(\{I_r, \ldots, I_s\})$ = the average number of comparisons needed to process the subarray $A[i..j]$ for the given set of intervals $\{I_r, \ldots, I_s\}$, with $i \leq \ell_r$ and $u_s \leq j$

The recurrence

$$
\begin{aligned}
C_{i,j}(\{I_r,\ldots,I_s\}) &= n-1 \\
&+ \sum_{t=r}^{s-1}\Bigg[\sum_{\ell_t\le k<u_t}\pi_{n,k}\big(C_{i,k-1}(\{I_r,\ldots,I'_t\})+C_{k+1,j}(\{I''_t,\ldots,I_s\})\big) \\
&\qquad + \sum_{u_t\le k<\ell_{t+1}}\pi_{n,k}\big(C_{i,k-1}(\{I_r,\ldots,I_t\})+C_{k+1,j}(\{I_{t+1},\ldots,I_s\})\big)\Bigg] \\
&+ \sum_{i\le k<\ell_r}\pi_{n,k}C_{k+1,j}(\{I_r,\ldots,I_s\}) \\
&+ \sum_{\ell_s\le k\le j}\pi_{n,k}C_{i,k-1}(\{I_r,\ldots,I_s\}),
\end{aligned}
$$

with $I'_t = [\ell_t, k-1]$ and $I''_t = [k+1, u_t]$.

How to solve the recurrence ...

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- First we have $p = 1$ and $I_1 = [a, b]$ and we translate the recurrence for $C_{i,j}(\{[a, b]\})$ into a functional equation for

$$C_{[a,b]}(u, v) = \sum_{1 \le i \le j} C_{i,j}(\{[a, b]\}) u^i v^j,$$

which is actually a first-order linear differential equation

How to solve the recurrence ...

- Then you can do a similar thing for $p = 2$, by introducing

$$C_{[a,b],[c,d]}(u,v) = \sum_{1 \le i \le j} C_{i,j}(\{[a,b],[c,d]\})u^i v^j,$$

which satisfies a similar ODE but the independent term now involves $C_{[a,b]}(u,v)$ and $C_{[c,d]}(u,v)$

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- A pattern emerges here, so that one can obtain a general form for the functional equation satisfied by $C_{\{I_1,\dots,I_p\}}(u,v)$
- Solve and extract $[u^i v^j] C_{\dots}(u,v)$

. . . But how I actually did solve it

I guessed the solution from the known solutions to the algorithms which chunksort generalizes and I proved it by induction. . .

Theorem

The average number of element comparisons $C_n(\{I_1,\ldots,I_p\}) \equiv C_{1,n}(\{I_1,\ldots,I_p\})$ needed by chunksort given the intervals $\{I_1,\ldots,I_p\}$ is

$$\begin{aligned}C_n &= 2n + u_p - \ell_1 + 2(n+1)H_n - 7m - 2 + 15p\\&\quad - 2(\ell_1+2)H_{\ell_1} - 2(n+3-u_p)H_{n+1-u_p}\\&\quad - 2\sum_{k=1}^{p-1}(\overline{m}_k+5)H_{\overline{m}_k+2},\end{aligned}$$

where

- $\overline{m}_k = |\overline{I}_k| = \ell_{k+1} - u_k - 1$
- $m_k = |I_k| = u_k - \ell_k + 1$
- $m = m_1 + m_2 + \cdots + m_p$

Chunksort vs. quicksort+quickselect

- For small p ($p = 1, 2$) it is perfectly reasonable to solve the problem using quickselect to find the beginning and end of each block, and then sort each block using quicksort
- The order of magnitude of the average cost of chunksort and this alternative is similar; but there are significative diferences for the second order terms
- For example, if $p = 1$ and $I_1 = [\alpha \cdot n - f(n), \alpha \cdot n + f(n)]$ for some $\alpha < 1/2$ and $f(n) = o(n)$ then chunksort makes $2(1 - \alpha)n$ comparisons less

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- The formula for the average cost of chunksort generalizes the corresponding formulas for special cases: quicksort, quickselect, partial quicksort, multiple quickselect, . . .
- Despite being simple and "efficient", chunksort should not be used as a substitute for the specialized algorithms (maybe it could be used for the less frequent tasks of multiple selection or partial sorting)

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- It is interesting to analyze the cost of the algorithm when taking into account the cost $\Theta(\log p)$ of locating the pivot's position in the array of intervals
- I would like to know about possible applications for chunksort; e.g., partial quicksort has been used to improve significantly the practical performance of Kruskal's algorithm for minimum spanning trees

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Thank you for your attention