On the Average Performance of Fixed Partial Match Queries in Random Relaxed K-d Trees

Conrado Martínez
Univ. Politècnica Catalunya, Barcelona

AofA 2014, Paris, France

–Dedicated to Ph. Flajolet 

Joint work with:

Amalia Duch

Gustavo Lau

The problem

- Input:
  - a collection of n multidimensional records, each with a K-dimensional key

$$\mathbf{x} = (x_0, \ldots, x_{K-1}) \in \mathcal{D}_0 \times \cdots \times \mathcal{D}_{K-1}$$

    stored in a suitable multidimensional data structure, e.g., a K-d tree, a K-dimensional quadtree, a K-d trie, ...
  - a query

$$\mathbf{q} = (q_0, \ldots, q_{K-1}), \quad q_i \in \mathcal{D}_i \cup \{*\}$$

    with s specified coordinates ($q_i \neq *$), $0 < s < K$, for example

$$\mathbf{q} = (0.1, *, *, 0.3, 0.07), \qquad\qquad s = 3, K = 5$$

- Output: the set of records such that $\mathbf{x}$ satisfies the query $\mathbf{q}$ (i.e., $x_i = q_i$ whenever $q_i \neq *$)

The problem

- Input:
  - a collection of n multidimensional records, each with a K-dimensional key

    $$\mathbf{x} = (x_0, \ldots, x_{K-1}) \in \mathcal{D}_0 \times \cdots \times \mathcal{D}_{K-1}$$

    stored in a suitable multidimensional data structure, e.g., a K-d tree, a K-dimensional quadtree, a K-d trie, ...
  - a query

    $$\mathbf{q} = (q_0, \ldots, q_{K-1}), \quad q_i \in \mathcal{D}_i \cup \{*\}$$

    with s specified coordinates ($q_i \neq *$), $0 < s < K$, for example

    $$\mathbf{q} = (0.1, *, *, 0.3, 0.07), \qquad s = 3, K = 5$$

- Output: the set of records such that $\mathbf{x}$ satisfies the query $\mathbf{q}$ (i.e., $x_i = q_i$ whenever $q_i \neq *$)

The problem

- Input:
  - a collection of n multidimensional records, each with a K-dimensional key

$$\mathbf{x} = (x_0, \ldots, x_{K-1}) \in \mathcal{D}_0 \times \cdots \times \mathcal{D}_{K-1}$$

    stored in a suitable multidimensional data structure, e.g., a K-d tree, a K-dimensional quadtree, a K-d trie, ...
  - a query

$$\mathbf{q} = (q_0, \ldots, q_{K-1}), \quad q_i \in \mathcal{D}_i \cup \{*\}$$

    with s specified coordinates ($q_i \neq *$), $0 < s < K$, for example

$$\mathbf{q} = (0.1, *, *, 0.3, 0.07), \qquad s = 3, K = 5$$

- Output: the set of records such that $\mathbf{x}$ satisfies the query $\mathbf{q}$ (i.e., $x_i = q_i$ whenever $q_i \neq *$)

The problem

- Input:
  - a collection of n multidimensional records, each with a K-dimensional key

    $$\mathbf{x} = (x_0, \ldots, x_{K-1}) \in \mathcal{D}_0 \times \cdots \times \mathcal{D}_{K-1}$$

    stored in a suitable multidimensional data structure, e.g., a K-d tree, a K-dimensional quadtree, a K-d trie, ...
  - a query

    $$\mathbf{q} = (q_0, \ldots, q_{K-1}), \quad q_i \in \mathcal{D}_i \cup \{*\}$$

    with s specified coordinates ($q_i \neq *$), $0 < s < K$, for example

    $$\mathbf{q} = (0.1, *, *, 0.3, 0.07), \qquad s = 3, K = 5$$

- Output: the set of records such that $\mathbf{x}$ satisfies the query $\mathbf{q}$ (i.e., $x_i = q_i$ whenever $q_i \neq *$)

K-d trees

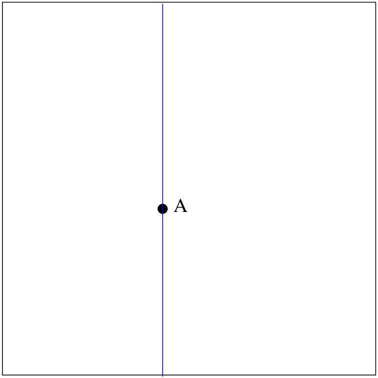
J.L. Bentley

Definition

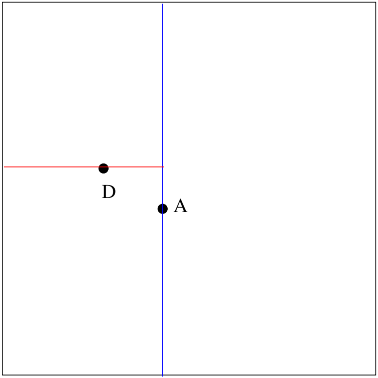
A K-d tree for a set X is either the empty tree if $X = \emptyset$ or a binary tree where:

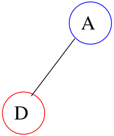
- the root contains $\mathbf{y} \in X$ and some value i (the discriminant), $0 \leqslant i < K$
- the left subtree is a K-d tree for $X^- = \{\mathbf{x} \in X \,|\, x_i < y_i\}$
- the right subtree is a K-d tree for $X^+ = \{\mathbf{x} \in X \,|\, y_i < x_i\}$

K-d trees

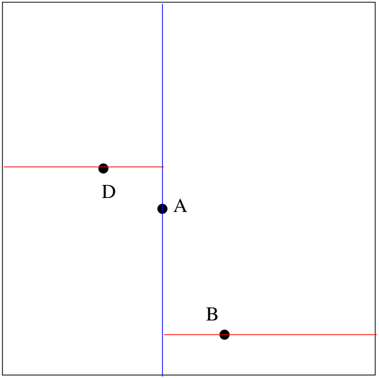


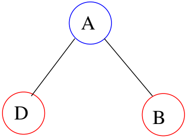
K-d trees



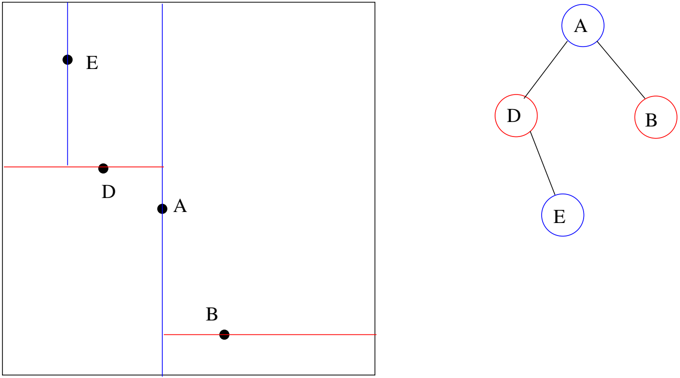


K-d trees

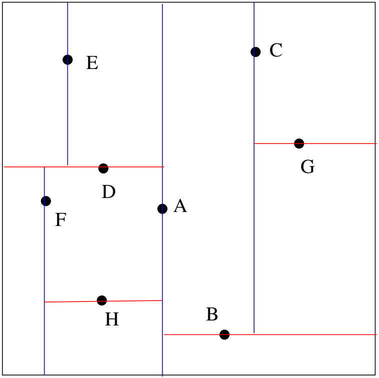


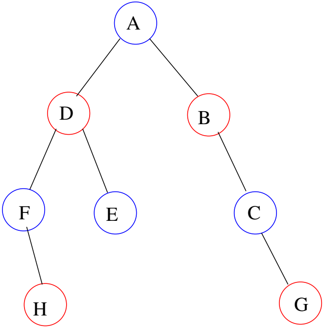


K-d trees



K-d trees





K-d trees

- All discriminants at level ℓ equal to $\ell \mod K \Rightarrow$ standard K-d trees
- Discriminants independent and uniformly drawn from $\{0, \ldots, K-1\} \Rightarrow$ relaxed K-d trees
- Discriminants chosen to cut along the longest side of the region where a new key is inserted $\Rightarrow$ squarish K-d trees
- ...

The algorithm

procedure PARTIAL_MATCH(T,$\mathbf{q}$)
  if $T = \square$ **then return**
  $i \leftarrow T.discr$; $\mathbf{x} \leftarrow T.key$
  if $\mathbf{x}$ satisfies $\mathbf{q}$ **then**
    Add $\mathbf{x}$ to the output
  if $q_i = *$ **then**
    PARTIAL_MATCH(T.left, $\mathbf{q}$)
    PARTIAL_MATCH(T.right, $\mathbf{q}$)
  else
    if $q_i < x_i$ **then**
      PARTIAL_MATCH(T.left, $\mathbf{q}$)
    else
      PARTIAL_MATCH(T.right, $\mathbf{q}$)

The probability model

- Without loss of generality: we assume $\mathcal{D}_i = [0, 1]$
- The standard probability model in this area: the n keys are drawn i.i.d from some continuous distribution in $[0, 1]^K$ and inserted into an initially empty K-d tree $\Rightarrow$ random (standard/relaxed/...) K-d tree T_n
- Equivalently: for any i, $0 \leqslant i < K$, and any $\mathbf{y}$ in the random K-d tree T_n, $\mathbf{y}$ is the rth smallest along the ith coordinate with

$$\mathbb{P}\{\#\{\mathbf{x} \in T_n \,|\, x_i \leqslant y_i\} = r\} = \frac{1}{n}$$

for $r = 1, 2, \ldots, n \Rightarrow$ shapes of K-d trees behave as binary search trees

The probability model

- Without loss of generality: we assume $\mathcal{D}_i = [0, 1]$
- The standard probability model in this area: the n keys are drawn i.i.d from some continuous distribution in $[0, 1]^K$ and inserted into an initially empty K-d tree $\Rightarrow$ random (standard/relaxed/...) K-d tree T_n
- Equivalently: for any i, $0 \leqslant i < K$, and any $\mathbf{y}$ in the random K-d tree T_n, $\mathbf{y}$ is the rth smallest along the ith coordinate with

$$\mathbb{P}\{\#\{\mathbf{x} \in T_n \,|\, x_i \leqslant y_i\} = r\} = \frac{1}{n}$$

for $r = 1, 2, \ldots, n \Rightarrow$ shapes of K-d trees behave as binary search trees

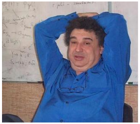
The probability model

- Without loss of generality: we assume $\mathcal{D}_i = [0, 1]$
- The standard probability model in this area: the n keys are drawn i.i.d from some continuous distribution in $[0, 1]^K$ and inserted into an initially empty K-d tree $\Rightarrow$ random (standard/relaxed/...) K-d tree T_n
- Equivalently: for any i, $0 \leqslant i < K$, and any $\mathbf{y}$ in the random K-d tree T_n, $\mathbf{y}$ is the rth smallest along the ith coordinate with

$$\mathbb{P}\left\{\#\{\mathbf{x} \in T_n \,|\, x_i \leqslant y_i\} = r\right\} = \frac{1}{n}$$

  for $r = 1, 2, \ldots, n \Rightarrow$ shapes of K-d trees behave as binary search trees

Previous work: Random queries



Ph. Flajolet

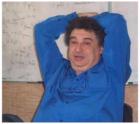
C. Puech

- For a query $\mathbf{q}$ its pattern $\mathbf{u}(\mathbf{q}) = (u_0, \ldots, u_{K-1})$ is a bitvector with $u_i = S$ if $q_i \neq *$ and $u_i = *$ if $q_i = *$. For example, $\mathbf{q} = (0.1, *, *, 0.3, 0.07) \Rightarrow \mathbf{u} = S * *SS$
- Let $\mathcal{P}_{n,\mathbf{u}}$ denote the cost (number of visited nodes) of a random partial match query with pattern $\mathbf{u}$, and let $\rho = s/K$. Then

$$P_{n,\mathbf{u}} := \mathbb{E}\left(\mathcal{P}_{n,\mathbf{u}}\right) = \beta_{\mathbf{u}} n^{\alpha} + o(n^{\alpha})$$

where $\alpha = \alpha(\rho)$ is $0 < \alpha < 1$ for any $\rho \in (0, 1)$ and $\beta_{\mathbf{u}}$ is a constant depending on the query pattern

Previous work: Random queries



Ph. Flajolet

C. Puech

- For a query $\mathbf{q}$ its pattern $\mathbf{u}(\mathbf{q}) = (u_0, \ldots, u_{K-1})$ is a bitvector with $u_i = S$ if $q_i \neq *$ and $u_i = *$ if $q_i = *$. For example, $\mathbf{q} = (0.1, *, *, 0.3, 0.07) \Rightarrow \mathbf{u} = S * * SS$
- Let $\mathcal{P}_{n,\mathbf{u}}$ denote the cost (number of visited nodes) of a random partial match query with pattern $\mathbf{u}$, and let $\rho = s/K$. Then

$$P_{n,\mathbf{u}} := \mathbb{E}\left(\mathcal{P}_{n,\mathbf{u}}\right) = \beta_{\mathbf{u}} n^{\alpha} + o(n^{\alpha})$$

where $\alpha = \alpha(\rho)$ is $0 < \alpha < 1$ for any $\rho \in (0, 1)$ and $\beta_{\mathbf{u}}$ is a constant depending on the query pattern

Previous work: Random queries

- Different data structures have different "characteristic" exponents α

Example ($s = 1, K = 2$)

- Standard K-d trees: $\alpha = 0.56\ldots$
- Relaxed K-d trees: $\alpha = 0.61\ldots$
- Squarish K-d trees: $\alpha = 0.5$
- K-d tries: $\alpha = 0.5$

- In general, $\alpha(\rho) = 1 - \rho + \vartheta(\rho)$, with $\vartheta(\rho) \geqslant 0$ "small" in $(0, 1)$
- Obs: the constant β for relaxed K-d trees only depends on s and K, not on the pattern, because of the randomly chosen discriminants

Previous work: Random queries

- Different data structures have different “characteristic” exponents α

Example ($s = 1, K = 2$)

- Standard K-d trees: $\alpha = 0.56\ldots$
- Relaxed K-d trees: $\alpha = 0.61\ldots$
- Squarish K-d trees: $\alpha = 0.5$
- K-d tries: $\alpha = 0.5$

- In general, $\alpha(\rho) = 1 - \rho + \vartheta(\rho)$, with $\vartheta(\rho) \geqslant 0$ “small” in $(0, 1)$
- Obs: the constant β for relaxed K-d trees only depends on s and K, not on the pattern, because of the randomly chosen discriminants

Previous work: Random queries

- Different data structures have different "characteristic" exponents α

Example ($s = 1, K = 2$)

- Standard K-d trees: $\alpha = 0.56\ldots$
- Relaxed K-d trees: $\alpha = 0.61\ldots$
- Squarish K-d trees: $\alpha = 0.5$
- K-d tries: $\alpha = 0.5$

- In general, $\alpha(\rho) = 1 - \rho + \vartheta(\rho)$, with $\vartheta(\rho) \geqslant 0$ "small" in $(0, 1)$
- Obs: the constant β for relaxed K-d trees only depends on s and K, not on the pattern, because of the randomly chosen discriminants

Previous work: Random queries

- Different data structures have different "characteristic" exponents α

Example ($s = 1, K = 2$)

- Standard K-d trees: $\alpha = 0.56\ldots$
- Relaxed K-d trees: $\alpha = 0.61\ldots$
- Squarish K-d trees: $\alpha = 0.5$
- K-d tries: $\alpha = 0.5$

- In general, $\alpha(\rho) = 1 - \rho + \vartheta(\rho)$, with $\vartheta(\rho) \geqslant 0$ "small" in $(0, 1)$
- Obs: the constant β for relaxed K-d trees only depends on s and K, not on the pattern, because of the randomly chosen discriminants

Previous work: Random queries

- Different data structures have different "characteristic" exponents α

Example ($s = 1, K = 2$)

- Standard K-d trees: $\alpha = 0.56\ldots$
- Relaxed K-d trees: $\alpha = 0.61\ldots$
- Squarish K-d trees: $\alpha = 0.5$
- K-d tries: $\alpha = 0.5$

- In general, $\alpha(\rho) = 1 - \rho + \vartheta(\rho)$, with $\vartheta(\rho) \geqslant 0$ "small" in $(0, 1)$
- Obs: the constant β for relaxed K-d trees only depends on s and K, not on the pattern, because of the randomly chosen discriminants

Previous work: Random queries

- Different data structures have different "characteristic" exponents α

Example ($s = 1, K = 2$)

- Standard K-d trees: $\alpha = 0.56\ldots$
- Relaxed K-d trees: $\alpha = 0.61\ldots$
- Squarish K-d trees: $\alpha = 0.5$
- K-d tries: $\alpha = 0.5$

- In general, $\alpha(\rho) = 1 - \rho + \vartheta(\rho)$, with $\vartheta(\rho) \geqslant 0$ "small" in $(0, 1)$
- Obs: the constant β for relaxed K-d trees only depends on s and K, not on the pattern, because of the randomly chosen discriminants

Previous work: Random queries

- Different data structures have different “characteristic” exponents α

Example ($s = 1, K = 2$)

- Standard K-d trees: $\alpha = 0.56\ldots$
- Relaxed K-d trees: $\alpha = 0.61\ldots$
- Squarish K-d trees: $\alpha = 0.5$
- K-d tries: $\alpha = 0.5$

- In general, $\alpha(\rho) = 1 - \rho + \vartheta(\rho)$, with $\vartheta(\rho) \geqslant 0$ “small” in $(0, 1)$
- Obs: the constant β for relaxed K-d trees only depends on s and K, not on the pattern, because of the randomly chosen discriminants

Previous work: Random queries

- Different data structures have different “characteristic” exponents α

Example ($s = 1, K = 2$)

- Standard K-d trees: $\alpha = 0.56\ldots$
- Relaxed K-d trees: $\alpha = 0.61\ldots$
- Squarish K-d trees: $\alpha = 0.5$
- K-d tries: $\alpha = 0.5$

- In general, $\alpha(\rho) = 1 - \rho + \vartheta(\rho)$, with $\vartheta(\rho) \geqslant 0$ “small” in $(0, 1)$
- Obs: the constant β for relaxed K-d trees only depends on s and K, not on the pattern, because of the randomly chosen discriminants

Previous work: Random queries

- Lots of results for random partial match queries in the literature:
  Flajolet & Puech (1986), Cunto, Lau & Flajolet (1989), Flajolet, Gonnet, Puech & Robson (1993), Kirschenhofer, Prodinger & Szpankowski (1993), Kirschenhofer & Prodinger (1994), Schachinger (1995, 2004), Duch, Estivill-Castro & Martínez (1998), Devroye, Jabbour & Zamora-Cura (2000), Neininger (2000), Martínez, Panholzer & Prodinger (2001), Chanzy, Devroye & Zamora-Cura (2001), Chern & Hwang (2006), ...

Fixed queries

- Our goal: get answers for the following question

What is the (expected) cost $P_{n,\mathbf{q}}$ of a partial match query with a given fixed query $\mathbf{q}$?

- Remarks (valid for relaxed K-d trees!):
  - We can assume that the specified coordinates are the first s coordinates: $q = (q_0, \ldots, q_{s-1}, *, *, \ldots, *)$
    A query q' which contains a permutation of the specified coordinates of q will have the same cost as $P_{n,\mathbf{q}}$
    If $q' = (q_0, \ldots, 1 - q_i, \ldots, q_{s-1}, *, *, \ldots, *)$ then $P_{n,\mathbf{q}} = P_{n,\mathbf{q}'}$

Fixed queries

- Our goal: get answers for the following question

> What is the (expected) cost $P_{n,\mathbf{q}}$ of a partial match query with a given fixed query $\mathbf{q}$?

- Remarks (valid for relaxed K-d trees!):
  - We can assume that the specified coordinates are the first s coordinates: $q = (q_0, \ldots, q_{s-1}, *, *, \ldots)$
  - A query $\mathbf{q}'$ which contains a permutation of the specified coordinates of $\mathbf{q}$ will have the same cost as $P_{n,\mathbf{q}}$

Fixed queries

- Our goal: get answers for the following question

What is the (expected) cost $P_{n,\mathbf{q}}$ of a partial match query with a given fixed query $\mathbf{q}$?

- Remarks (valid for relaxed K-d trees!):
    - We can assume that the specified coordinates are the first s coordinates: $q = (q_0, \ldots, q_{s-1}, *, *, \ldots)$
    - A query $\mathbf{q}'$ which contains a permutation of the specified coordinates of $\mathbf{q}$ will have the same cost as $P_{n,\mathbf{q}}$
    - If $\mathbf{q}' = (q_0, \ldots, 1 - q_i, \ldots, q_{s-1}, *, *, \ldots)$ then $P_{n,\mathbf{q}} = P_{n,\mathbf{q}'}$

Fixed queries

- Our goal: get answers for the following question

What is the (expected) cost $P_{n,\mathbf{q}}$ of a partial match query with a given fixed query $\mathbf{q}$?

- Remarks (valid for relaxed K-d trees!):
  - We can assume that the specified coordinates are the first s coordinates: $q = (q_0, \ldots, q_{s-1}, *, *, \ldots)$
  - A query $\mathbf{q}'$ which contains a permutation of the specified coordinates of $\mathbf{q}$ will have the same cost as $P_{n,\mathbf{q}}$
  - If $\mathbf{q}' = (q_0, \ldots, 1 - q_i, \ldots, q_{s-1}, *, *, \ldots)$ then $P_{n,\mathbf{q}} = P_{n,\mathbf{q}'}$

Fixed queries

- Our goal: get answers for the following question

What is the (expected) cost $P_{n,\mathbf{q}}$ of a partial match query with a given fixed query $\mathbf{q}$?

- Remarks (valid for relaxed K-d trees!):
  - We can assume that the specified coordinates are the first s coordinates: $q = (q_0, \ldots, q_{s-1}, *, *, \ldots)$
  - A query $\mathbf{q}'$ which contains a permutation of the specified coordinates of $\mathbf{q}$ will have the same cost as $P_{n,\mathbf{q}}$
  - If $\mathbf{q}' = (q_0, \ldots, 1 - q_i, \ldots, q_{s-1}, *, *, \ldots)$ then $P_{n,\mathbf{q}} = P_{n,\mathbf{q}'}$

Fixed queries

- Our goal: get answers for the following question

What is the (expected) cost $P_{n,\mathbf{q}}$ of a partial match query with a given fixed query $\mathbf{q}$?

- Remarks (valid for relaxed K-d trees!):
  - We can assume that the specified coordinates are the first s coordinates: $q = (q_0, \ldots, q_{s-1}, *, *, \ldots)$
  - A query $\mathbf{q}'$ which contains a permutation of the specified coordinates of $\mathbf{q}$ will have the same cost as $P_{n,\mathbf{q}}$
  - If $\mathbf{q}' = (q_0, \ldots, 1 - q_i, \ldots, q_{s-1}, *, *, \ldots)$ then $P_{n,\mathbf{q}} = P_{n,\mathbf{q}'}$

Ranks

- Given a collection X and a query $\mathbf{q}$, the rank vector is $\mathbf{r}(\mathbf{q}) = (r_0, \ldots, r_{K-1})$, with $r_i = *$ if $q_i = *$ and

$$r_i = \text{the number of } \mathbf{x} \in X \text{ with } x_i \leqslant q_i$$

- We will only write the ranks of specified coordinates $\mathbf{r} = (r_0, \ldots, r_{s-1})$
- We wil concentrate in the analysis of the cost $P_{n,\mathbf{r}} := \mathbb{E}\left(\mathcal{P}_{n,\mathbf{r}}\right)$

Ranks

- Given a collection X and a query $\mathbf{q}$, the rank vector is $\mathbf{r}(\mathbf{q}) = (r_0, \ldots, r_{K-1})$, with $r_i = *$ if $q_i = *$ and

$$r_i = \text{the number of } \mathbf{x} \in X \text{ with } x_i \leqslant q_i$$

- We will only write the ranks of specified coordinates $\mathbf{r} = (r_0, \ldots, r_{s-1})$
- We wil concentrate in the analysis of the cost $P_{n,\mathbf{r}} := \mathbb{E}\left(\mathcal{P}_{n,\mathbf{r}}\right)$

Ranks

- Given a collection X and a query $\mathbf{q}$, the rank vector is $\mathbf{r}(\mathbf{q}) = (r_0, \ldots, r_{K-1})$, with $r_i = *$ if $q_i = *$ and

$$r_i = \text{the number of } \mathbf{x} \in X \text{ with } x_i \leqslant q_i$$

- We will only write the ranks of specified coordinates $\mathbf{r} = (r_0, \ldots, r_{s-1})$
- We wil concentrate in the analysis of the cost $P_{n,\mathbf{r}} := \mathbb{E}\left(\mathcal{P}_{n,\mathbf{r}}\right)$

Ranks

Relating $P_{n,\mathbf{q}}$ and $P_{n,\mathbf{r}}$

$$\begin{aligned} P_{n,\mathbf{q}} &= \sum_{\mathbf{r}} P_{n,\mathbf{r}} \cdot \mathbb{P}\left\{\mathbf{r}(\mathbf{q}) = \mathbf{r}\right\} \\ &= P_{n,\bar{\mathbf{r}}} + \text{l.o.t.} \end{aligned}$$

with $\bar{r}_i = q_i \cdot n$ (the expected value of the rank of q_i)

Previous work: Fixed queries

Theorem (Curien & Joseph, 2011)

The cost of a partial match with $q = (t, *)$ (equiv. $q = (*, t)$) and $t \neq 0, 1$ in a random 2-dimensional quadtree satisfies

$$\lim_{n \to \infty} \frac{P_{n,\mathbf{q}}}{n^{\alpha}} = \beta \cdot \frac{\Gamma(\alpha + 2)}{\Gamma^2\left(\frac{\alpha}{2} + 1\right)} \cdot (t(1-t))^{\alpha/2}$$

with $\alpha = (\sqrt{17} - 3)/2 \approx 0.561\ldots$ *and*
$\beta =$ *constant factor for random partial match queries*

Previous work: Fixed queries

Theorem (Curien & Joseph, 2011)

The cost of a partial match with $q = (t, *)$ *(equiv.* $q = (*, t)$*) and* $t \neq 0, 1$ *in a random 2-dimensional quadtree satisfies*

$$\lim_{n\to\infty} \frac{P_{n,\mathbf{q}}}{n^{\alpha}} = \beta \cdot \frac{\Gamma(\alpha+2)}{\Gamma^2\left(\frac{\alpha}{2}+1\right)} \cdot (t(1-t))^{\alpha/2}$$

with $\alpha = (\sqrt{17}-3)/2 \approx 0.561\ldots$ *and*
$\beta =$ *constant factor for random partial match queries*

Broutin, Neininger, Sulzbach (2012, 2013): Distributional results (convergence in law to a random continuous function $\mathcal{P}(t)$), variance; also for standard 2-d trees

Previous work: Fixed queries

Theorem (Duch, Jiménez & Martínez, 2012)

The cost of a partial match with rank vector $\mathbf{r} = (r, *, *, \ldots)$ *in a random relaxed* K*-d tree*

$$P_{n,r} = \beta \frac{\Gamma(\alpha+2)}{\Gamma^2\left(\frac{\alpha}{2}+1\right)} \cdot (r(n-r))^{\alpha/2} + o(n^{\alpha})$$

with $\alpha = \alpha(1/K)$, $\alpha(\rho) = (\sqrt{9-8\rho}-3)/2$ *and* $\beta = \beta(1/K)$, *provided that*

$$0 < \lim_{n\to\infty} \frac{r}{n} < 1$$

Previous work: Fixed queries

Theorem (Duch, Jiménez & Martínez, 2012)

The cost of a partial match with rank vector $\mathbf{r} = (r, *, *, \ldots)$ *in a random relaxed* K*-d tree*

$$P_{n,r} = \beta \frac{\Gamma(\alpha + 2)}{\Gamma^2\left(\frac{\alpha}{2} + 1\right)} \cdot (r(n-r))^{\alpha/2} + o(n^\alpha)$$

with $\alpha = \alpha(1/K)$, $\alpha(\rho) = (\sqrt{9 - 8\rho} - 3)/2$ *and* $\beta = \beta(1/K)$, *provided that*

$$0 < \lim_{n \to \infty} \frac{r}{n} < 1$$

A similar result holds for standard K-d trees

Results

Theorem (AofA 2014)

The expected cost of a partial match with rank vector $\mathbf{r} = (r_0, r_1, \ldots, r_{s-1}, *, *, \ldots)$ *in a random relaxed K-d tree is*

$$P_{n,\mathbf{r}} = \beta \frac{\Gamma^{s}(\alpha+2)}{\Gamma^{2s}\left(\frac{\alpha}{2}+1\right)} \cdot \prod_{i=0}^{s-1} \left(\frac{r_i}{n}\left(1-\frac{r_i}{n}\right)\right)^{\alpha/2} \cdot n^{\alpha} + o(n^{\alpha})$$

with $\alpha = \alpha(s/K)$, $\alpha(\rho) = (\sqrt{9-8\rho}-3)/2$ *and* $\beta = \beta(s/K)$, *provided that*

$$0 < \lim_{n\to\infty} \frac{r_i}{n} < 1, \qquad 0 \leqslant i < s$$

Results

Theorem (AofA 2014)

The expected cost of a partial match with rank vector $\mathbf{r} = (*, \ldots, r_0, *, \ldots, r_1, \ldots, r_{s-1}, *, \ldots)$ *in a random standard K-d tree is*

$$P_{n,\mathbf{r}} = \beta_{\mathbf{u}(\mathbf{r})} \frac{\Gamma^s(\alpha+2)}{\Gamma^{2s}\left(\frac{\alpha}{2}+1\right)} \cdot \prod_{i=0}^{s-1} \left(\frac{r_i}{n}\left(1-\frac{r_i}{n}\right)\right)^{\alpha/2} \cdot n^\alpha + o(n^\alpha)$$

with $\alpha = \alpha(s/K)$, $\alpha(\rho)$ *the unique solution in* $[0,1]$ *of*

$$(\alpha+2)^\rho(\alpha+1)^{1-\rho} = 2$$

and provided that $0 < \lim_{n\to\infty} \frac{r_i}{n} < 1$, $0 \leqslant i < s$

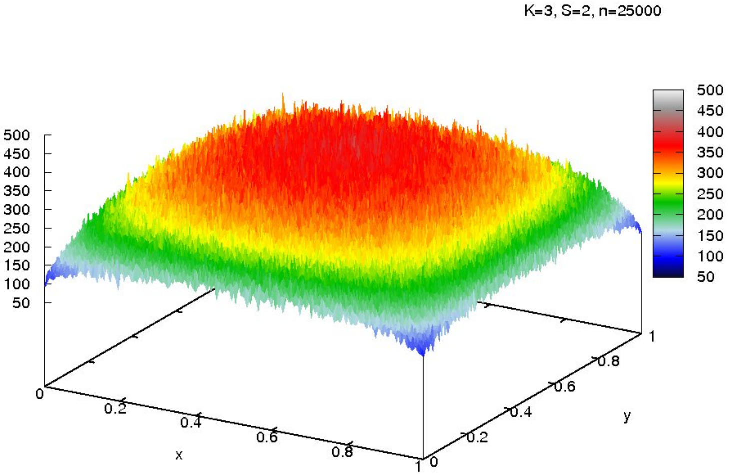
Corollary

The expected cost of a partial match with query $\mathbf{q} = (q_0, q_1, \ldots, q_{s-1}, *, *, \ldots)$ in a random relaxed K-d tree is

$$P_{n,\mathbf{q}} = \beta \frac{\Gamma^s(\alpha + 2)}{\Gamma^{2s}\left(\frac{\alpha}{2} + 1\right)} \cdot \prod_{i=0}^{s-1} \left(q_i\left(1 - q_i\right)\right)^{\alpha/2} \cdot n^{\alpha} + o(n^{\alpha})$$

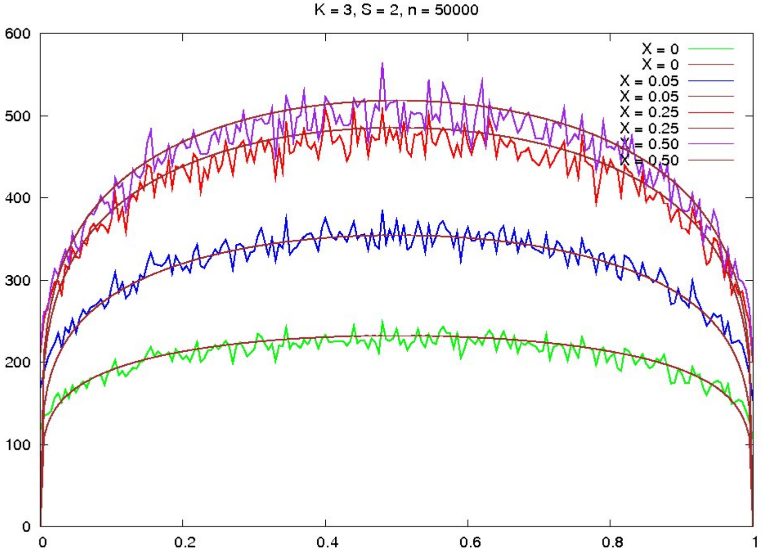
with $\alpha = \alpha(s/K)$, $\alpha(\rho) = (\sqrt{9 - 8\rho} - 3)/2$ *and* $\beta = \beta(s/K)$, *provided that* $q_i \neq 0, 1$

Results



K=3, S=2, n=25000
500
450
400
350
300
250
200
150
100
50
0
0.2
0.4
0.6
0.8
1
x
y

Results



K = 3, S = 2, n = 50000
X = 0
X = 0
X = 0.05
X = 0.05
X = 0.25
X = 0.25
X = 0.50
X = 0.50
600
500
400
300
200
100
0
0
0.2
0.4
0.6
0.8
1

Overview of the proof

Let the rank vector be $\mathbf{r} = (r_0, \ldots, r_{s-1})$. For $n > 0$

$$P_{n,\mathbf{r}} = \frac{1}{K}(A_0 + \cdots + A_{s-1} + B_0 + \cdots + B_{K-s-1})$$

with

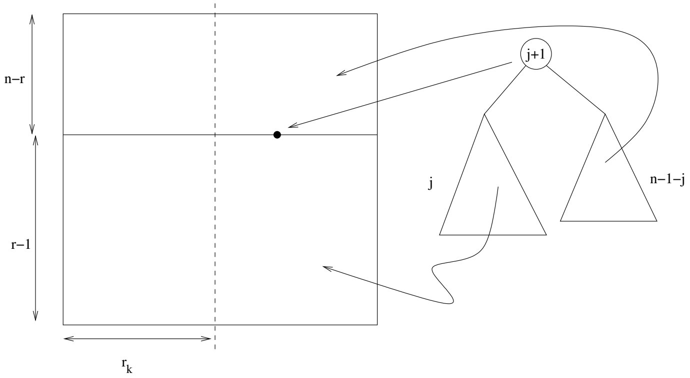
$$\begin{aligned} A_i &= \mathbb{E}\left(\mathcal{P}_{n,\mathbf{r}} \,|\, \text{root discr}{=}i\right) \\ B_i &= \mathbb{E}\left(\mathcal{P}_{n,\mathbf{r}} \,|\, \text{root discr}{=}i+s\right) \end{aligned}$$

Overview of the proof

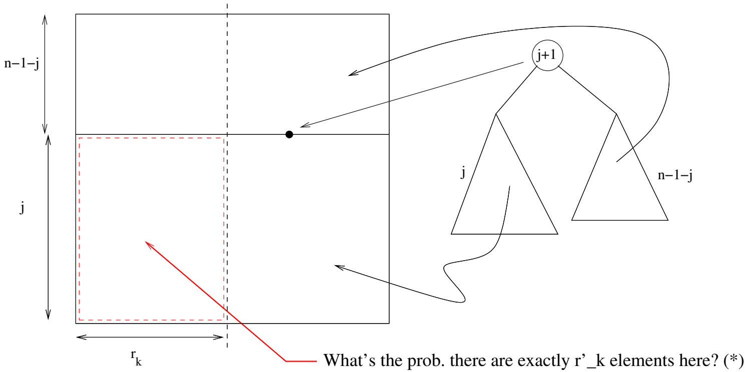
$$
\begin{aligned}
A_i &= \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}\left(\mathcal{P}_{n,\mathbf{r}}\,|\,\text{root } \mathbf{x} \text{ discr} = i \wedge \text{left subtree of size } j\right) \\
&= \frac{1}{n}\sum_{j=r_i}^{n-1} \mathbb{E}\left(\mathcal{P}_{n,\mathbf{r}}\,|\,\text{root } \mathbf{x} \text{ discr} = i \wedge \text{left subtree of size } j\right) \\
&+ \frac{1}{n}\sum_{j=0}^{r_i-1} \mathbb{E}\left(\mathcal{P}_{n,\mathbf{r}}\,|\,\text{root } \mathbf{x} \text{ discr} = i \wedge \text{left subtree of size } j\right) \\
&= 1 + \frac{1}{n}\sum_{j=r_i}^{n-1}\sum_{\mathbf{r}'} \pi_{\mathbf{r},\mathbf{r}'} P_{j,\mathbf{r}'} \frac{1}{n}\sum_{j=0}^{r_i-1}\sum_{\mathbf{r}''} \pi'_{\mathbf{r},\mathbf{r}''} P_{n-1-j,\mathbf{r}''}
\end{aligned}
$$

with $\mathbf{r}' = (r'_0, \ldots, r'_{i-1}, r_i, r'_{i+1}, \ldots, r'_{s-1})$, $\mathbf{r}'' = (r''_0, \ldots, r''_{i-1}, r_i - j - 1, r''_{i+1}, \ldots, r''_{s-1})$ and $\pi_{\mathbf{r},\mathbf{r}'}$ (resp. $\pi'_{\mathbf{r},\mathbf{r}'}$) the probability that the rank vector in the left (resp. right) subtree is $\mathbf{r}'$ (resp. $\mathbf{r}''$)

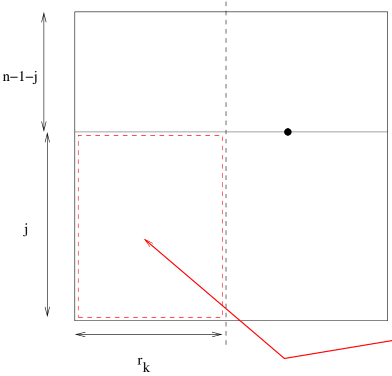
Overview of the proof



Overview of the proof

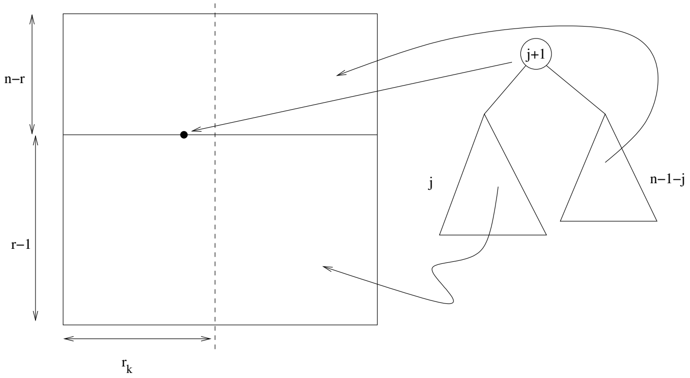


Overview of the proof

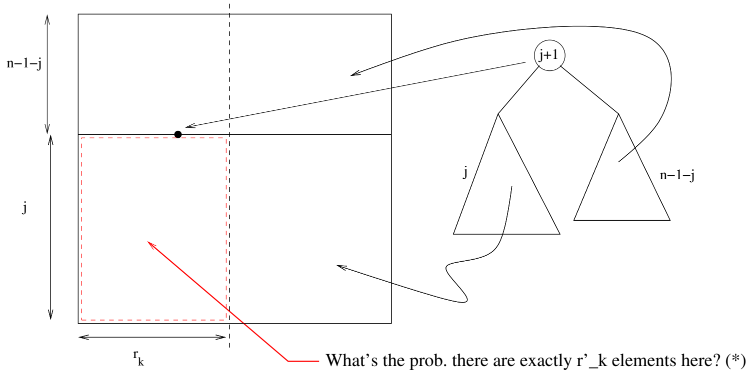


$$\frac{\binom{j}{r_k'}\binom{n-1-j}{r_k-r_k'}}{\binom{n-1}{r_k}}$$

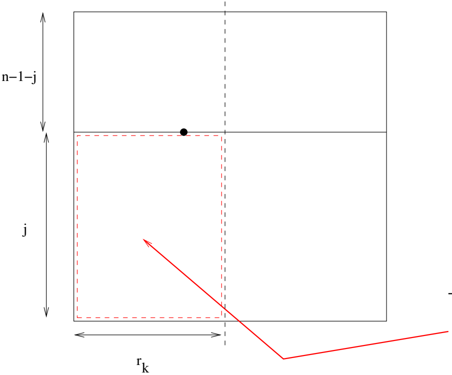
Overview of the proof



Overview of the proof



Overview of the proof



$$\frac{\binom{j}{r'_k}\binom{n-1-j}{r_k - r'_k - 1}}{\binom{n-1}{r_k - 1}}$$

Overview of the proof

$$A_i \sim 1 + \frac{1}{n}\sum_{j=r_i}^{n-1} P_{j,\bar{\mathbf{r}}^{(1)}} + \frac{1}{n}\sum_{j=0}^{r_i-1} P_{n-1-j,\bar{\mathbf{r}}^{(2)}}$$

with

$$\bar{\mathbf{r}}^{(1)} = (\bar{r}_0^{(1)}, \ldots, \bar{r}_{i-1}^{(1)}, r_i, \bar{r}_{i+1}^{(1)}, \ldots, \bar{r}_{s-1}^{(1)}),$$
$$\bar{\mathbf{r}}^{(2)} = (\bar{r}_0^{(2)}, \ldots, \bar{r}_{i-1}^{(2)}, r_i - j, \bar{r}_{i+1}^{(2)}, \ldots, \bar{r}_{s-1}^{(2)})$$

and for $k \neq i$

$$\bar{r}_k^{(1)} = \frac{j}{n} r_k, \qquad \bar{r}_k^{(2)} = \frac{n-1-j}{n} r_k$$

Terms B_i are similarly handled

Overview of the proof

Dividing by n^α ($\alpha := \alpha(s/K)$), and collecting all terms

$$\frac{P_{n,\mathbf{r}}}{n^\alpha} \sim \frac{1}{n^\alpha} + \frac{1}{K}\sum_{i=0}^{s-1}\Bigg(\frac{1}{n}\sum_{j=r_i}^{n-1}\frac{P_{j,\bar{\mathbf{r}}^{(1)}}}{j^\alpha}\frac{j^\alpha}{n^\alpha}$$

$$+\frac{1}{n}\sum_{j=0}^{r_i-1}\frac{P_{n-1-j,\bar{\mathbf{r}}^{(2)}}}{(n-1-j)^\alpha}\frac{(n-1-j)^\alpha}{n^\alpha}\Bigg)$$

$$+\frac{K-s}{K}\frac{1}{n}\sum_{j=0}^{n-1}\frac{P_{j,\bar{\mathbf{r}}^{(3)}}}{j^\alpha}\frac{j^\alpha}{n^\alpha} + \frac{P_{n-1-j,\bar{\mathbf{r}}^{(4)}}}{(n-1-j)^\alpha}\frac{(n-1-j)^\alpha}{n^\alpha}$$

Overview of the proof

We anticipate $P_{n,\mathbf{r}}/n^{\alpha} \sim f(z_0, \ldots, z_{s-1})$ with $z_i = r_i/n$ and thus, as $n \to \infty$

$$
\begin{aligned}
f(\mathbf{z}) \sim \frac{1}{K} \sum_{i=0}^{s-1} \Bigg\{ & \frac{1}{n} \sum_{j=r_i}^{n-1} \left(\frac{j}{n}\right)^{\alpha} \cdot f\left(z_0, \ldots, z_i \frac{n}{j}, \ldots\right) \\
& + \frac{1}{n} \sum_{j=0}^{r_i-1} \left(\frac{n-1-j}{n}\right)^{\alpha} \cdot f\left(z_0, \ldots, \left(z_i - \frac{j+1}{n}\right) \frac{n}{n-1-j}, \ldots\right) \Bigg\} \\
& + \frac{K-s}{K} \frac{1}{n} \sum_{j=0}^{n-1} \left(\left(\frac{j}{n}\right)^{\alpha} + \left(\frac{n-1-j}{n}\right)^{\alpha}\right) \cdot f(z_0, \ldots, z_{s-1})
\end{aligned}
$$

Overview of the proof

Passing to the limit and substituting sums by integrals

$$
\begin{aligned}
f(z_0,\ldots,z_{s-1}) \sim \frac{1}{K}\sum_{i=0}^{s-1}\Bigg\{ & \int_{z_i}^{1} x^{\alpha} f\left(z_0,\ldots,\frac{z_i}{x},\ldots,z_{s-1}\right)\,dx \\
& + \int_0^{z_i} (1-x)^{\alpha} f\left(z_0,\ldots,\frac{z_i-x}{1-x},\ldots,z_{s-1}\right)\,dx \Bigg\} \\
& + \frac{K-s}{K}\int_0^1 f(z_0,\ldots,z_{s-1})(x^{\alpha}+(1-x)^{\alpha})\,dx
\end{aligned}
$$

Overview of the proof

After some "massaging"

$$
\begin{aligned}
f(z_0,\dots,z_{s-1}) \sim{}& \\
&\frac{\alpha+2}{2s}\sum_{i=0}^{s-1}\int_{z_i}^{1} x^{\alpha} f\left(z_0,\dots,\frac{z_i}{x},\dots,z_{s-1}\right)\,dx \\
&\qquad + \int_0^{z_i}(1-x)^{\alpha} f\left(z_0,\dots,\frac{z_i-x}{1-x},\dots,z_{s-1}\right)\,dx
\end{aligned}
$$

Overview of the proof

- In order to solve the integral equation, we assume

$$f(z_0, \ldots, z_{s-1}) = \vartheta_0(z_0) \cdots \vartheta_{s-1}(z_{s-1})$$

- Because of the symmetry the problem we can safely assume $\vartheta = \vartheta_0 = \vartheta_1 = \cdots = \vartheta_{s-1}$ and $\vartheta(z) = \vartheta(1-z)$
- Furthermore, the analysis of extremal queries (when for some i, $r_i = o(n)$ or $r_i = n - o(n)$), shows that $P_{n,\mathbf{r}} = o(n^\alpha)$ in those cases and thus

$$\lim_{z \to 0} \vartheta(z) = 0$$

Overview of the proof

- The integral equation can be transformed into a second-order linear differential equation for ϑ, which can be easily solved

$$\vartheta(z) = \kappa \cdot (z(1-z))^{\alpha/2}$$

  for some constant κ.
- Last but not least, the constant factor of f can be determined using

$$\beta = \int_0^1 \int_0^1 \cdots \int_0^1 f(z_0, \ldots, z_{s-1})\, dz_0\, dz_1 \cdots dz_{s-1}$$

Concluding remarks

- We conjecture that the expected cost of partial matches follows the same pattern in many multidimensional data structures

$$P_{n,\mathbf{q}} = \beta \frac{\Gamma^s(\alpha+2)}{\Gamma^{2s}\left(\frac{\alpha}{2}+1\right)} \cdot \prod_{i=0}^{s-1} \left(q_i\,(1-q_i)\right)^{\alpha/2} \cdot n^\alpha + o(n^\alpha)$$

  with β and α derived from the analysis of random queries
- Moreover we conjecture the convergence (at least in distribution), for the sequence of r.v.'s $\{n^{-\alpha}\mathcal{P}_{n,\mathbf{q}}\}_{n\geqslant 0}$

$$\frac{\mathcal{P}_{n,\mathbf{q}}}{n^\alpha} \xrightarrow{(d)} \mathcal{P}(\mathbf{q})$$

  for a continuous random function $\mathcal{P}(\mathbf{q})$ (in s variables)

Concluding remarks

- We conjecture that the expected cost of partial matches follows the same pattern in many multidimensional data structures

$$P_{n,\mathbf{q}} = \beta \frac{\Gamma^s(\alpha+2)}{\Gamma^{2s}\left(\frac{\alpha}{2}+1\right)} \cdot \prod_{i=0}^{s-1} \left(q_i\left(1-q_i\right)\right)^{\alpha/2} \cdot n^\alpha + o(n^\alpha)$$

  with β and α derived from the analysis of random queries
- Moreover we conjecture the convergence (at least in distribution), for the sequence of r.v.'s $\{n^{-\alpha}\mathcal{P}_{n,\mathbf{q}}\}_{n \geqslant 0}$

$$\frac{\mathcal{P}_{n,\mathbf{q}}}{n^\alpha} \xrightarrow{(d)} \mathcal{P}(\mathbf{q})$$

  for a continuous random function $\mathcal{P}(\mathbf{q})$ (in s variables)

Concluding remarks

- Our on-going work is now focused in extending our analysis to other data structures and in finding a general argument which proves the "universality" of the factor

$$\beta_{\mathbf{u}} \cdot \frac{\Gamma^{s}(\alpha+2)}{\Gamma^{2s}\left(\frac{\alpha}{2}+1\right)} \cdot \prod_{i=0}^{s-1} \left(q_i\left(1-q_i\right)\right)^{\alpha/2}$$

- Partial match can be regarded as a generalization to higher dimensions of the selection of order statistics; we have heavily relied on techniques which have proven extremely useful in the analysis of classical quickselect and variants

Concluding remarks

- Our on-going work is now focused in extending our analysis to other data structures and in finding a general argument which proves the "universality" of the factor

$$\beta_{\mathbf{u}} \cdot \frac{\Gamma^s(\alpha+2)}{\Gamma^{2s}\left(\frac{\alpha}{2}+1\right)} \cdot \prod_{i=0}^{s-1} \left(q_i\left(1-q_i\right)\right)^{\alpha/2}$$

- Partial match can be regarded as a generalization to higher dimensions of the selection of order statistics; we have heavily relied on techniques which have proven extremely useful in the analysis of classical quickselect and variants

Thanks for your attention!

See you in Strobl (Austria) next June 2015