Rank Selection in Multidimensional Data

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Joint work with:

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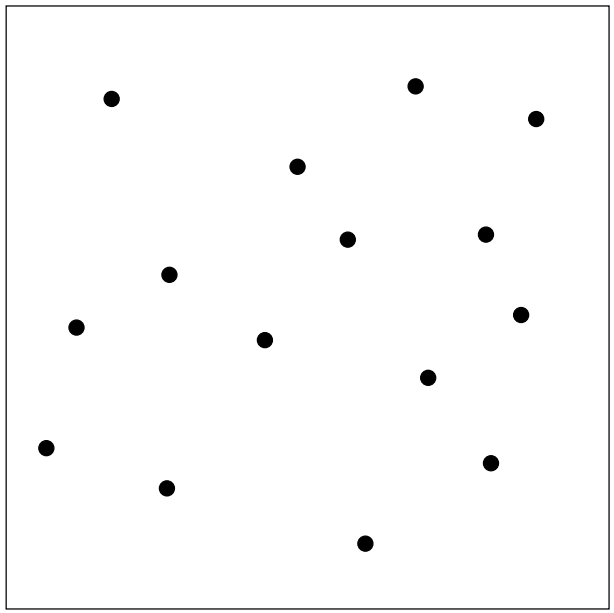
Introduction

The problem: Given a collection of n multidimensional records, each with K coordinates, and values i, $1 \leqslant i \leqslant n$, and j, $1 \leqslant j \leqslant K$, find the i-th record along the j-th coordinate



n = 15
K = 2

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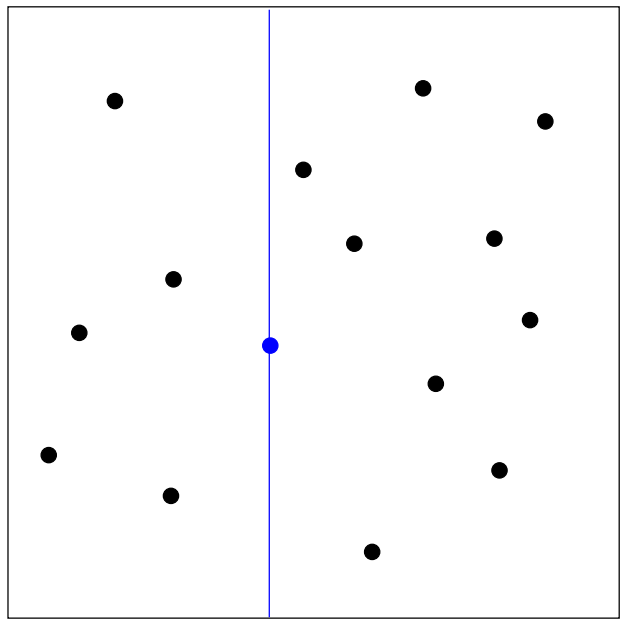

$n = 15$ $i = 6$
$K = 2$ $j = 1$

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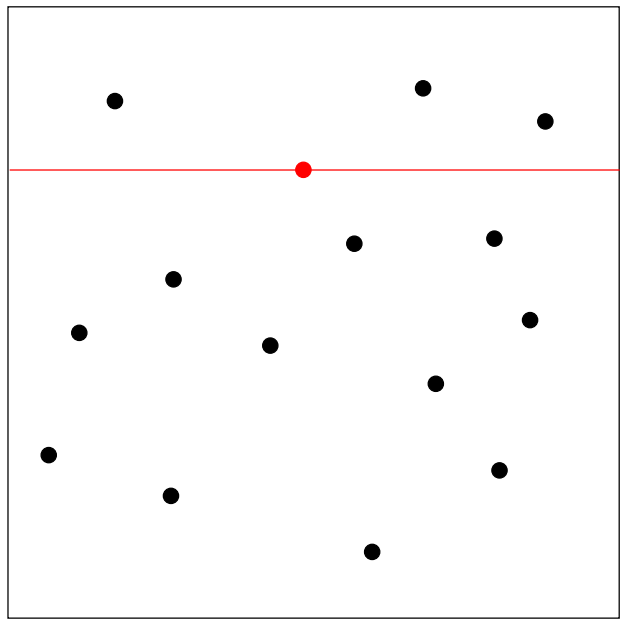


$n = 15$ $i = 12$
$K = 2$ $j = 2$

Introduction

- What if the collection is organized in some multidimensional index? (e.g., a K-d tree, a quadtree, ...)
- If $K = 1$ and the collection of n records is stored in some kind of binary search tree $\Rightarrow$ (expected) time $\Theta(\log n)$, using some little extra space
- We look for an algorithm that uses space $\Theta(n)$, independent of K
- The data structure for the n records should efficiently support usual spatial queries, e.g., orthogonal range search
- We assume w.l.o.g. the n records are points from $[0, 1]^K$

K-d trees

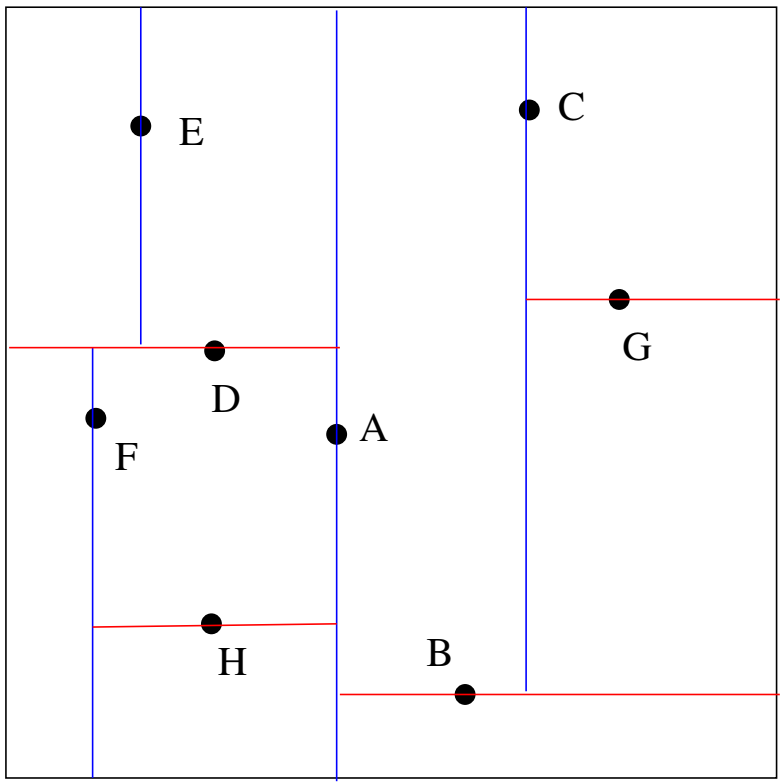
J.L. Bentley

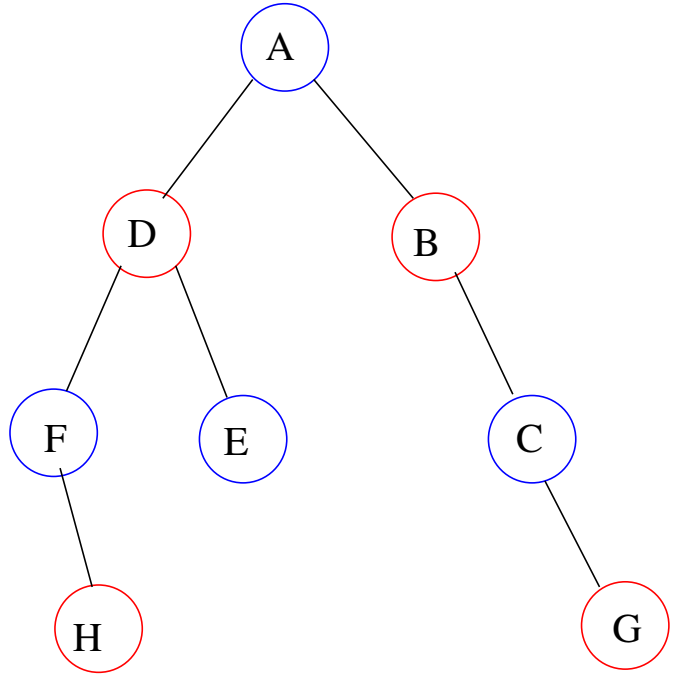
Definition

A K-d tree for a set $X \subset [0,1]^K$ is either the empty tree if $X = \emptyset$ or a binary tree where:

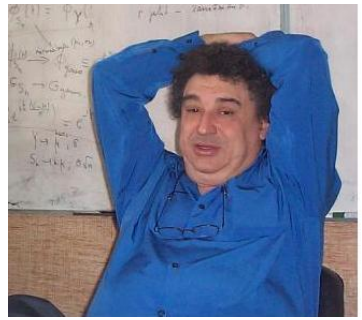
- the root contains $y \in X$ and some value j, $1 \leqslant j \leqslant K$
- the left subtree is a K-d tree for $X^- = \{x \in X \,|\, x_j < y_j\}$
- the right subtree is a K-d tree for $X^+ = \{x \in X \,|\, y_j < x_j\}$

K-d trees





K-d trees



Ph. Flajolet

C. Puech

- In a partial match query we are given a query $q = (q_1, \ldots, q_K)$ where s coordinates are specified and $K - s$ are "don't cares"
- The goal is to find all records in a collection that satisfy the query
- Flajolet and Puech (1986) showed that a partial match in a random standard K-d tree of size n has expected cost $\Theta(n^{\alpha(s/K)})$, where $\alpha(x) = 1 - x + \phi(x)$, $0 \leqslant \phi(x) < 0.07$
- Similar results have been proved for other variants of K-d trees, quadtrees, etc.

K-d trees

L. Devroye

- Orthogonal range queries ask for all records falling inside an hyperrectangle (with sides parallel to the axis); their expected cost has been analyzed by Chanzy, Devroye and Zamora-Cura (2001) and Duch and Martínez (2002):

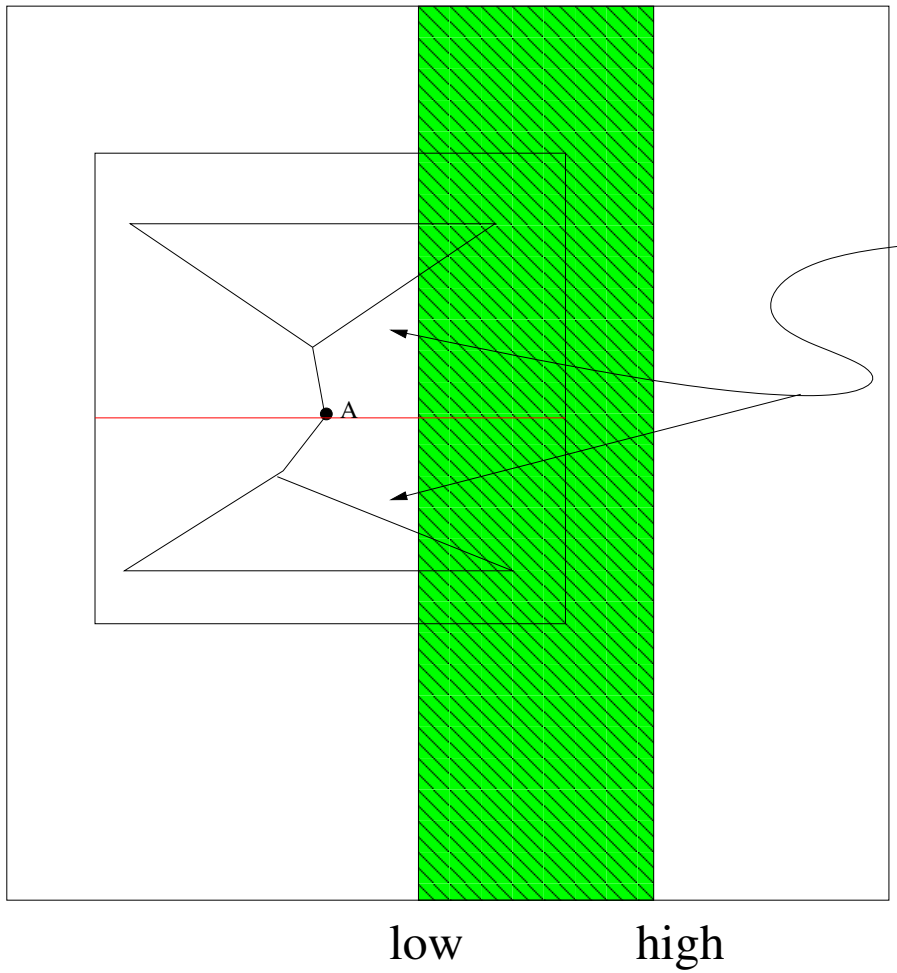
$$n \cdot \text{volume of query} + n^{\alpha(1/K)} \cdot \text{perimeter of query} + \text{l.o.t.}$$

The algorithm

Our algorithm has three main steps

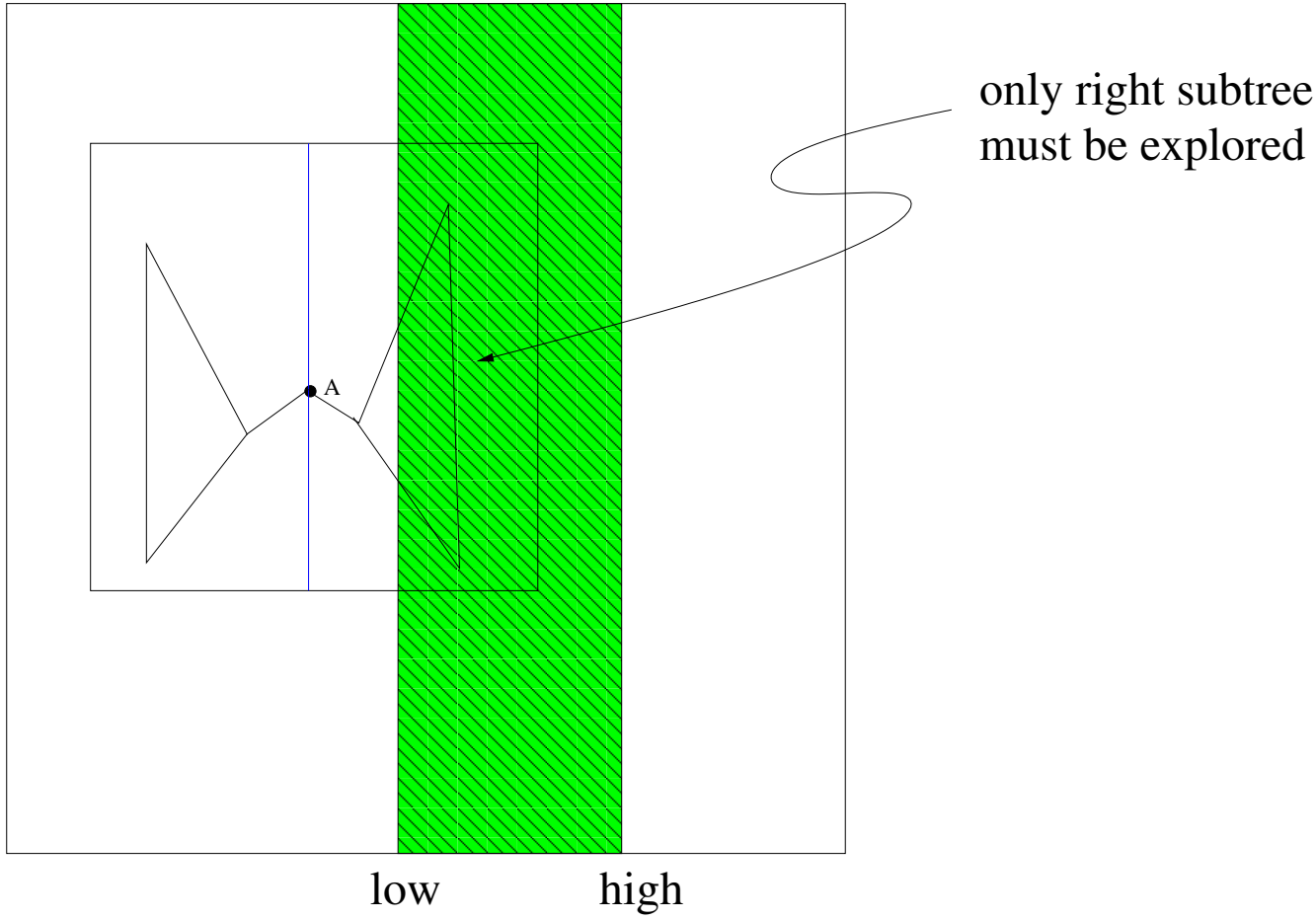
- The main loop starts with a strip $x_j \in [low, high] = [0, 1]$ and explores the K-d tree, reducing the strip in such a way that it always contains the i-th record along coordinate j
- When the main loop finishes, it has found the sought element (if it is stored in a node that discriminates w.r.t. j) or the strip does only contain nodes discriminating w.r.t. a coordinate $\neq j$; if needed, the second step performs an orthogonal range search to locate all records within the strip
- A conventional selection algorithm is used to find the sought element among the elements reported in the previous step

The algorithm: main loop

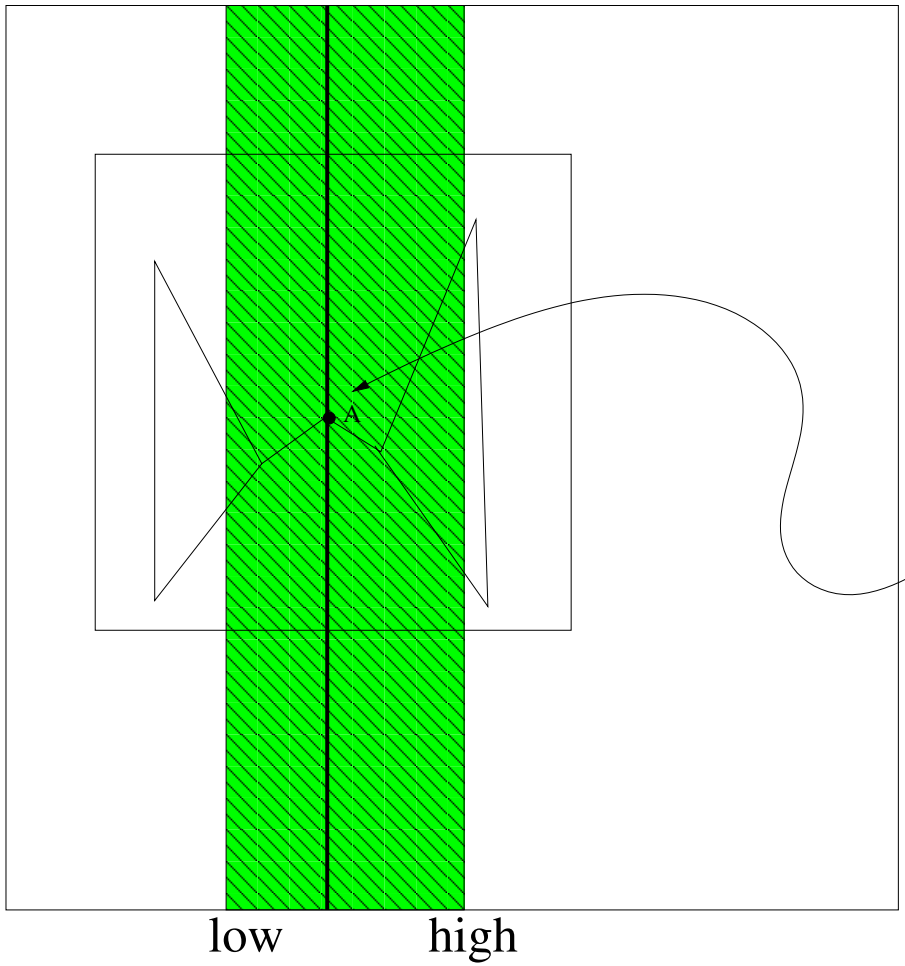


both subtrees
must be explored

The algorithm: main loop

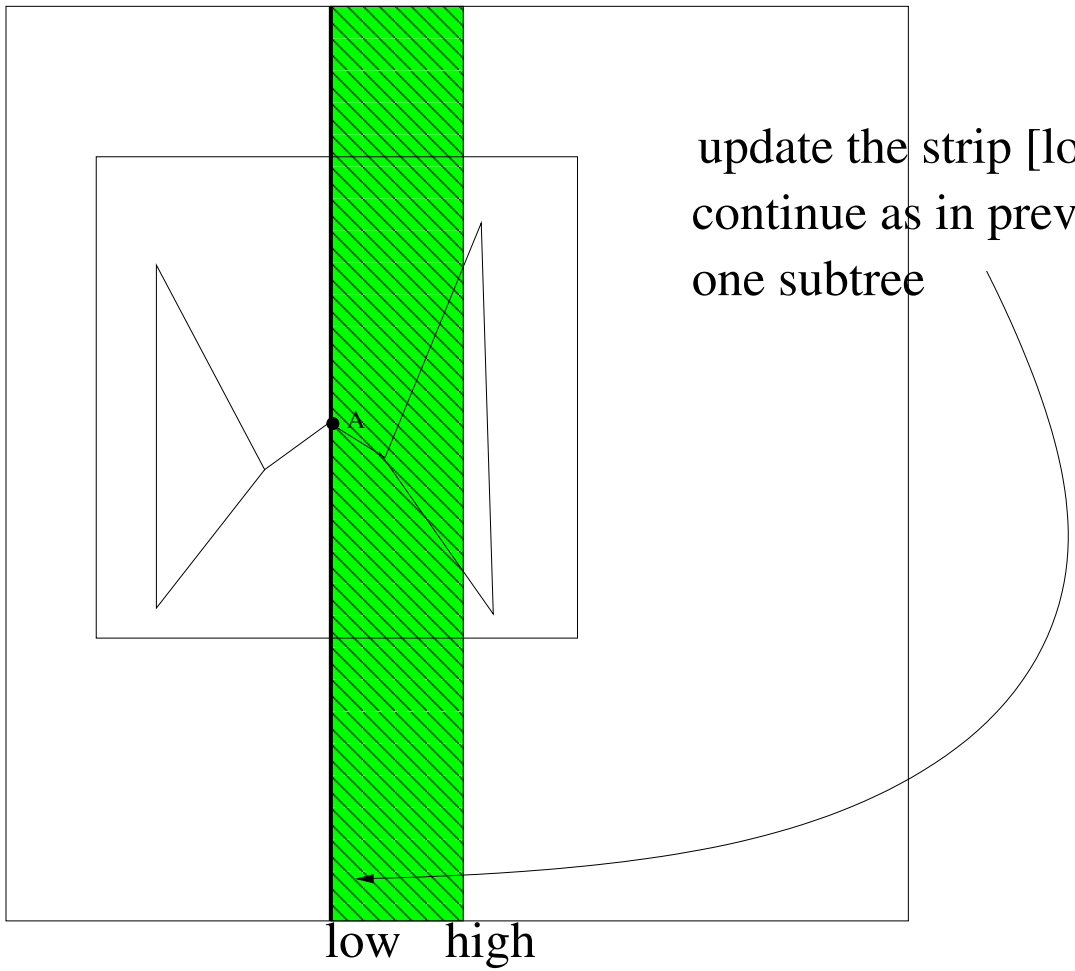


The algorithm: main loop



find the rank of the element
along coordinate j =
count how many
elements are below

The algorithm: main loop



update the strip [low, high] and
continue as in previous case exploring
one subtree

Analysis

Hypothesis for the analysis: The n records are independently drawn from a continuous distribution in $[0, 1]^K$ (standard probability model for random K-d tree)

Analysis

Five key observations

1. The number of visited nodes in the main loop is at most the number of nodes visited by an orthogonal range search with the strip $[low, high]$
2. The cost of a call to BELOW is that of a partial match with a single specified coordinate
3. The expected number of calls to BELOW is $\Theta(\log n)$
4. The main loop finds the sought point when the node discriminates along j-th coordinate or the strip $[low, high]$ contains it and no point that discriminates with respect to j
5. The strip contains $\Theta(1)$ points on average

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Theorem

The expected cost T_n *of* KDSELECT *to select the* i*-th smallest element along the* j*-th coordinate in a random relaxed* K*-d tree of size* n*, for random uniformly distributed* i *and* j*,* $i \sim$ *Unif*$(1, n)$*,* $j \sim$ *Unif*$(0, K-1)$*, is*

$$T_n = \Theta(n^\alpha \log n),$$

where $\alpha = \alpha(1/K)$ *satisfies* $1/K \leqslant \alpha(K) < 1$ *for all* $K \geqslant 2$*.*

$$\alpha(x) = \frac{1}{2}\left(\sqrt{9-8x}-1\right)$$

Analysis

To achieve a good expected performance for a call to BELOW, it is necessary that each node contains the size of the subtree rooted at that tree

procedure BELOW(T, j, z)
  if T $= \square$ **then return** 0
  if T.$\mathrm{discr} \neq$ j **then**
    $$c \leftarrow \begin{cases} 1 & \textbf{if } T.\mathrm{key}[j] \leqslant z, \\ 0 & \text{otherwise.} \end{cases}$$
    return BELOW$(z, j, T.\mathrm{left})$ + BELOW$(z, j, T.\mathrm{right})$ + c
  else
    if $z <$ T.key[j] **then return** BELOW$(z, j, T.\mathrm{left})$
    else return $T.\mathrm{left}.\mathrm{size}$ + BELOW$(z, j, T.\mathrm{right})$

Analysis: A Refinement

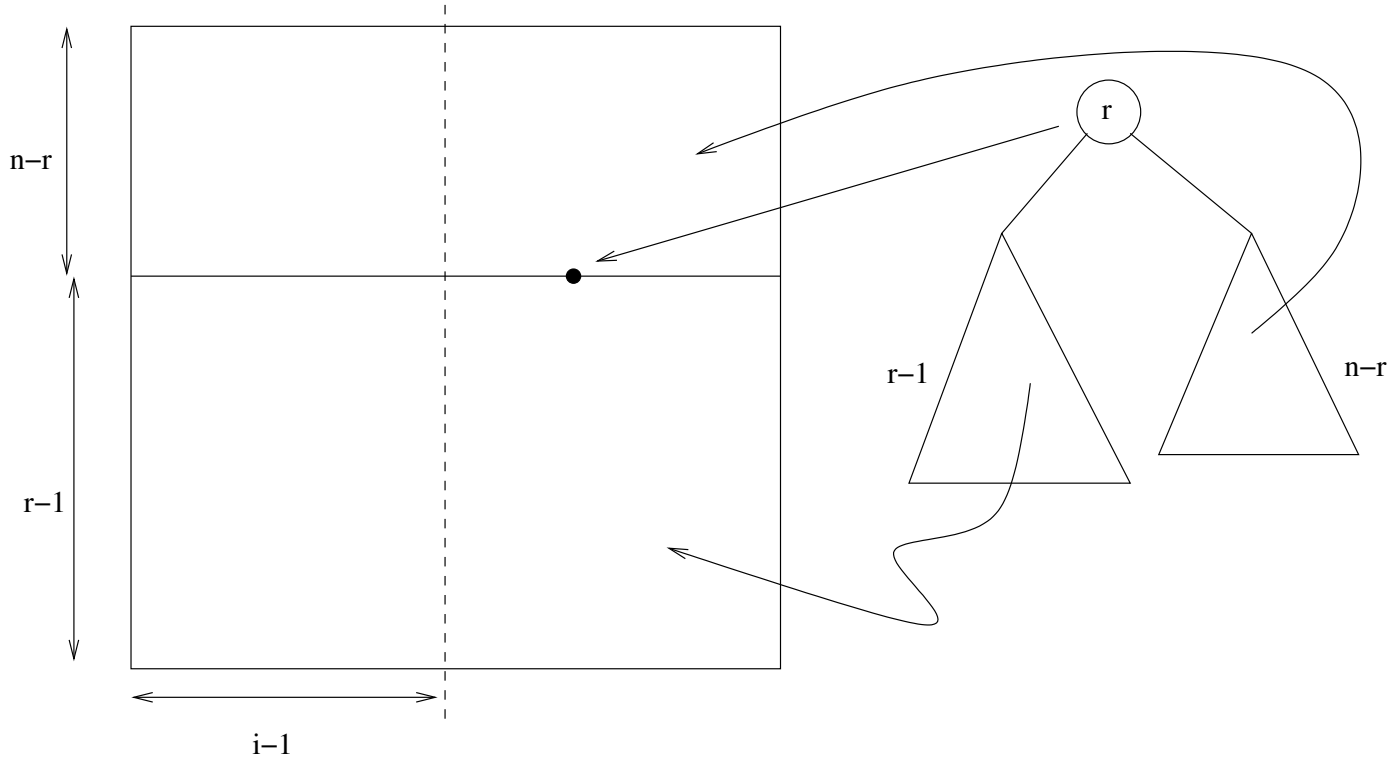
Let $C_{n,i}$ denote the expected cost of a call BELOW(T, j, z) in a random relaxed K-d tree T of size n when exactly $i-1$ of its elements satisfy $x_j \leqslant z$.
Then, for $n > 0$ and $1 \leqslant i \leqslant n+1$

$$C_{n,i} = 1 + \frac{1}{K} \cdot \frac{1}{n} \cdot \left[\sum_{r=1}^{i-1} C_{n-r,i-r} + \sum_{r=i}^{n} C_{r-1,i} \right]$$

$$+ \frac{K-1}{K} \cdot \frac{1}{n} \left[\sum_{r=1}^{n} \left\{ \frac{n+1-i}{n} \sum_{k} \frac{\binom{r-1}{k-1}\binom{n-r}{i-k}}{\binom{n-1}{i-1}} \cdot (C_{r-1,k} + C_{n-r,i+1-k}) \right.\right.$$

$$\left.\left. + \frac{i-1}{n} \sum_{k} \frac{\binom{r-1}{k-1}\binom{n-r}{i-k-1}}{\binom{n-1}{i-2}} (C_{r-1,k} + C_{n-r,i-k}) \right\} \right].$$

Analysis: A Refinement



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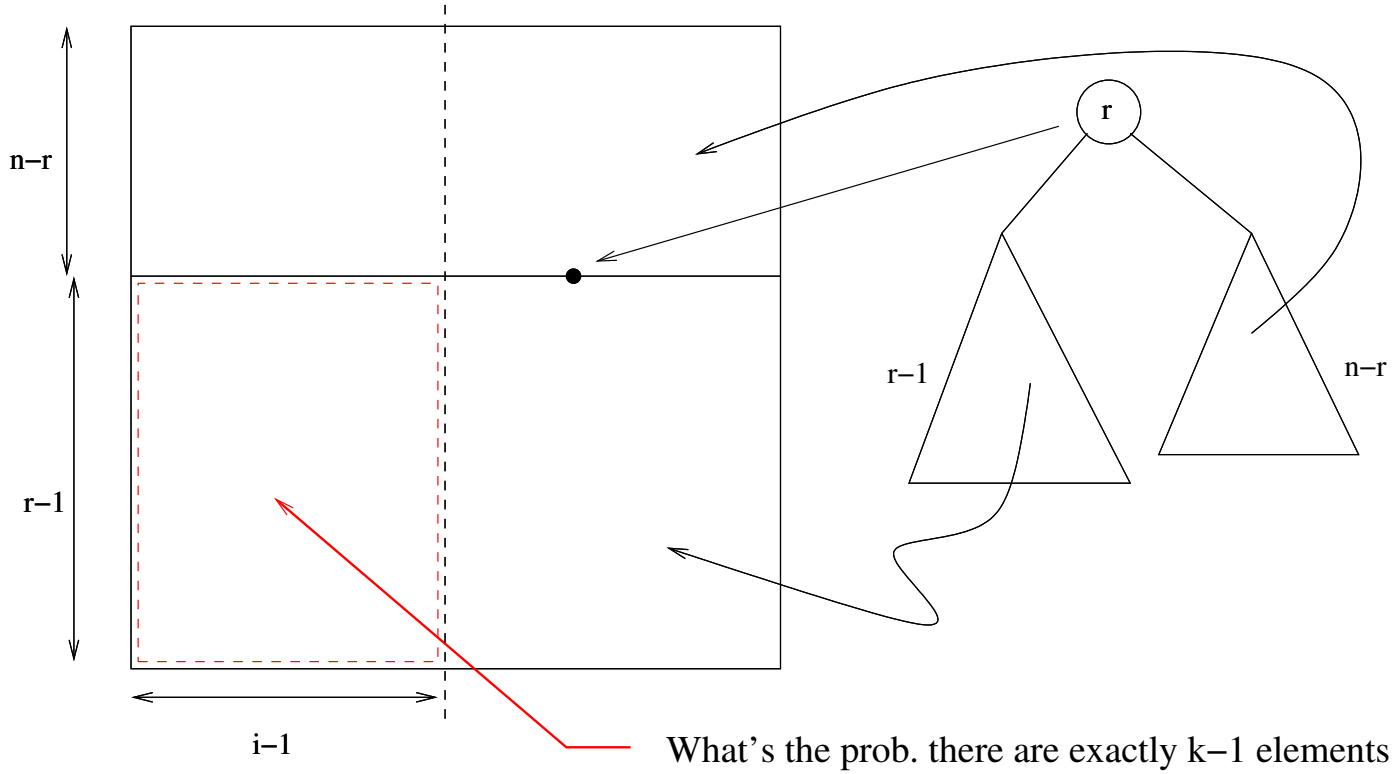

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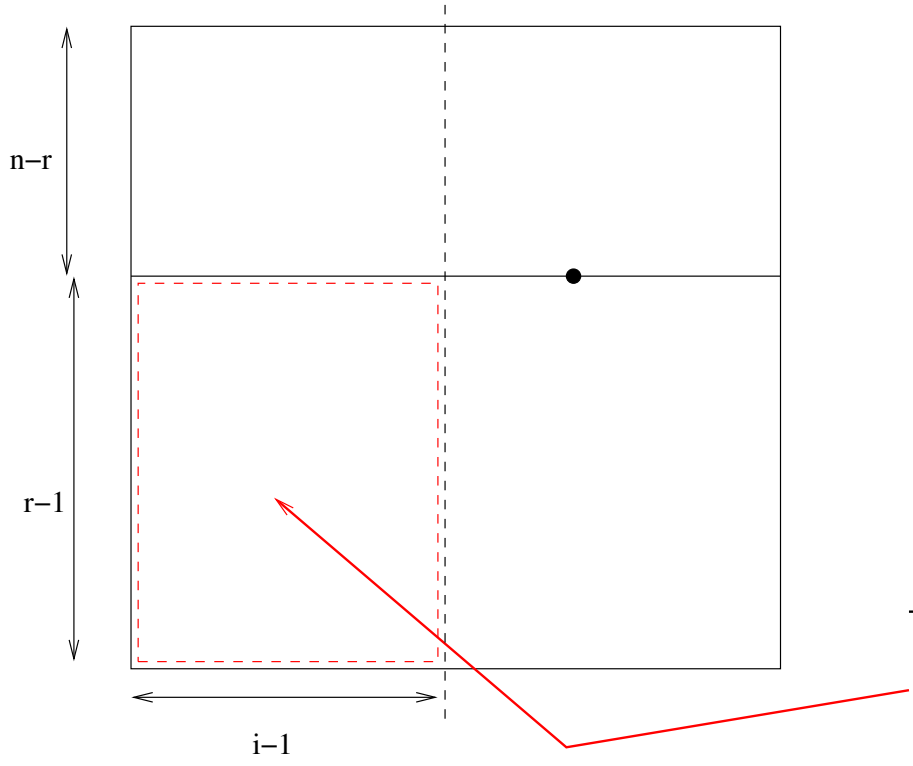


$$\frac{\binom{r-1}{k-1}\binom{n-r}{i-k}}{\binom{n-1}{i-1}}$$

Analysis: A Refinement

Hopes to solve the recurrence for $C_{n,i}$ are dim ...
However we can prove

$$f(x) = \lim_{n\to\infty, i/n\to x} C_{n,i}/n^{\alpha},$$

for $0 < x < 1$ exists and the recurrence leads to

$$f(x) = \frac{1}{K - \frac{2(K-1)}{\alpha+1}} \left[\int_0^x f\left(\frac{x-u}{1-u}\right)(1-u)^{\alpha}\,du + \int_x^1 f\left(\frac{x}{u}\right) u^{\alpha}\,du \right].$$

Analysis: A Refinement

The equation for $f(x)$ can be solved yielding

$$f(x) = \eta \cdot (x(1-x))^{\alpha/2}, \qquad 0 \leqslant x \leqslant 1,$$

for some constant η. Determination of η follows from

$$\int_0^1 f(x)\,dx = \beta$$

where β is the constant factor of the main order term in the expected cost of a random partial match:

$$\eta = \frac{\Gamma(2\phi+2)}{\Gamma^2(\phi+1)}\beta = \frac{\Gamma(2\phi+2)}{\Gamma^2(\phi+1)}\frac{\Gamma(2\alpha+1)}{(1-1/K)(1+\alpha)\alpha^3\Gamma^3(\alpha)}.$$

Analysis: A Refinement

From the analysis of $f(x)$ it follows

$$C_{n,i} = \eta \cdot \left(\sqrt{i(n+1-i)}\right)^{\alpha}.$$

For standard K-d trees a similar formula holds, now with α the characteristic exponent in the expected cost of partial matches in standard K-d trees and the constant η which depends also on j, the specified coordinate.

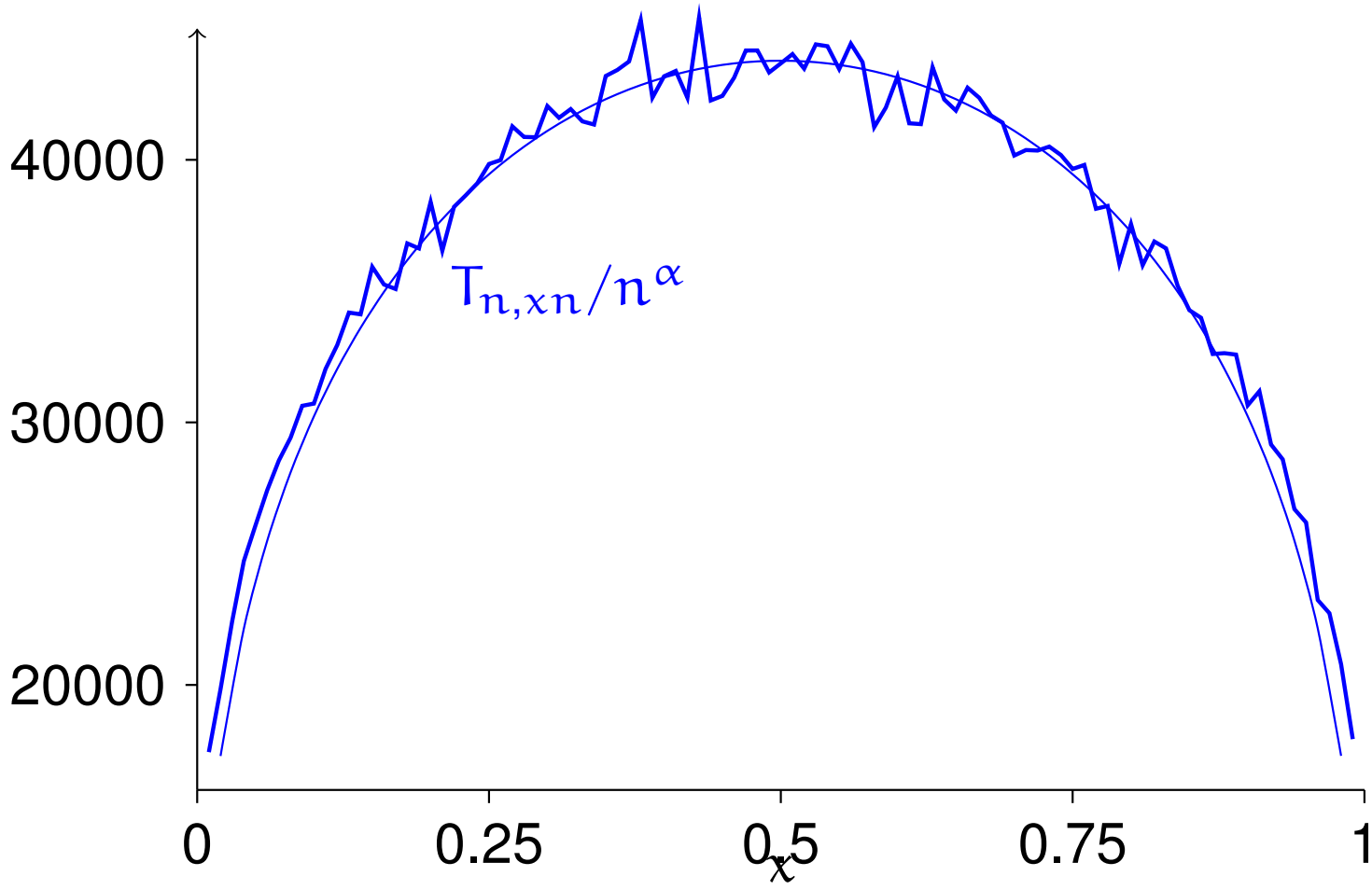
Analysis: A Refinement

The refined analysis of BELOW is the building block for a refined analysis of KDSELECT. We can show that the expected cost $T_{n,i}$ to find the i-th record along some given coordinate in a random relaxed K-d tree of size n is

$$T_{n,i} \sim n^{\alpha}(f(i/n)\ln(i) + f(1-i/n)\ln(n+1-i) - f(i/n)) + o(n^{\alpha}),$$

with $f(x) = \eta \cdot (x(1-x))^{\alpha/2}$.

Analysis: A Refinement



A plot of experimental values of $T_{n,xn}/n^{\alpha}$ (thick line) as a function of $x = i/n$, versus the theoretical predictions (thin line), for relaxed 2-d trees.

Final remarks

- A simple algorithm with sublinear expected cost
- It can easily be extended to many other multidimensional data structures
- Very little overhead: storing the size of each subtree is not very space consuming and it can also be sucessfully used for balancing (e.g., randomized relaxed K-d trees)
- Experiments show that it is competitive in practice compared to alternative solutions, for reasonably low dimensions (when K grows, $\alpha(\mathrm{K}) \rightarrow 1$)

Thanks for your attention!