Universitat Politècnica de Catalunya
Facultat d'Informàtica de Barcelona

Name, First Name ID number

Degree: Grau en Enginyeria Informàtica **Academic year:** Q1 2021–2022 (Mid-term Exam)
Course: Randomized Algorithms (RA-MIRI) **Date:** November 4th, 2020
Time: 1h 30m

1. **(2 points)**

   You are participating in a contest. You have three closed doors in front of you. Behind one of the doors, there is a lastest model cell phone. The other two rooms are empty. You choose one of the doors, you'll win whatever is behind the chosen door. The host —who knows what do the rooms contain— opens one of the other two doors to show you an empty room. Then he makes a deal: "do you stick to your initial choice or do you want to change?"

   What shall you do? Explain your answer. Give the probability of winning the cell for both strategies: (1) stick to your initial choice; (2) change your initial choice.

   SOLUTION:

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(Answer to Question 1 here.)

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2. **(2.5 points)** Let X and Y be two independent identically distributed geometric random variables, $X, Y \sim \text{Geom}(p)$. Compute:

   (a) $\mathbb{E}[X+Y]$

   (b) $\mathbb{V}[X+Y]$

   (c) $\mathbb{P}[X \leq k]$

   (d) $\mathbb{E}[\min(X,Y)]$

   Useful formula: if $x \neq 1$

$$\sum_{k=a}^{b} x^k = \frac{x^{b+1} - x^a}{x-1}.$$

   Another formula which might be of help: If Z is a discrete positive random variable then

$$\mathbb{E}[Z] = \sum_{k \geq 0} \mathbb{P}[Z > k]\,.$$

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(Answer to Question 2 here.)

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3. **(2.5 points)** Suppose we have three polynomials $p(x)$, $q(x)$ and $r(x)$. Suppose $\deg(p) = \deg(q) = n$ and $\deg(r) = 2n$, and we want to check whether $p(x) \cdot q(x) = r(x)$ or not. We assume that all coefficents are $O(\log n)$ bits and any arithmetic operation has cost $O(1)$. Then computing the product $p(x) \cdot q(x)$ has cost $\Theta(n^2)$ (naïvely) or with cost $\Theta(n \log n)$ (using the Fast Fourier Transform); we would want to avoid such costly computation. We have a random number generator that will produce a random integer of m bits, that is, in the range $[0, 2^m - 1]$, in time $\Theta(m)$. Take $m = c \log_2 n$ for some $c > 1$.

   Design an efficient Montecarlo randomized algorithm that will always correctly answer that $p(x) \cdot q(x) = r(x)$ if that is the case, and that if $p(x) \cdot q(x) \neq r(x)$ it might wrongly answer that $p(x) \cdot q(x) = r(x)$, but that will happen with probability at most $1/n^{c-1}$. In other words, if the algorithm answers $p(x) \cdot q(x) \neq r(x)$ then it is the case for sure, whereas if the algorithm answers that $p(x) \cdot q(x) = r(x)$ then it might be wrong, albeit with very small probability ($O(1/n^{c-1})$). Give the cost of your algorithm (number of arithmetic operations used, should be $\ll n^2$).

SOLUTION:

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(Answer to Question 3 here.)

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4. **(3 points)** Median-of-$(2t+1)$ quickselect is a variant of quickselect which picks uniformly at random and without replacement a sample of $2t+1$ elements from the array and uses the median of these $2t+1$ elements as the pivot for each recursive stage (as long as $n \geq 2t+1$). So the probability $\pi_{n,j}$ that the pivot choosen is the j-th smallest element is not uniform, while

$$\pi_{n,j} = \frac{1}{n}, \qquad \text{for all } j,\ 1 \leq j \leq n,$$

in ordinary randomized quickselect, when the pivot of each stage is choosen u.a.r.

When analyzing the expected cost of ordinary quickselect we set up a recurrence for the expected number $f_n = \mathbb{E}[F_n]$ of comparisons to select an element of random rank as follows:

$$\begin{aligned} f_1 &= f_0 = 0 \\ f_n &= n - 1 + \sum_{j=1}^{n} \pi_{n,j} \times \mathbb{E}[\#\text{ of comparisons} \,|\, \text{pivot is the } j\text{th element}] \\ &= n - 1 + \sum_{j=1}^{n} \pi_{n,j} \left(\frac{j-1}{n} f_{j-1} + \frac{n-j}{n} f_{n-j} \right) = n - 1 + \frac{2}{n^2} \sum_{j=0}^{n-1} j f_j. \end{aligned}$$

Similar steps can be applied in the case of median-of-$(2t+1)$ quickselect, but the so called *splitting probabilities* $\pi_{n,j}$ are different. Also the number of comparisons of each recursive stage will be larger because we need some comparisons to find the median of the sample. However, t is a constant and hence the number of comparisons in a recursive stage of the algorith is $n + \mathcal{O}(1)$.

(a) Calculate the probability $\pi_{n,j}$ that the pivot is the j-th smallest element, when it is choosen as the median of $2t+1$ random elements selected without replacement from the array of n elements.

(b) Set up the recurrence for the expected number $f_n^{(t)}$ of comparisons to select an element of random rank for quickselect with median-of-$(2t+1)$.

(c) Identify a shape function for the recurrence. Apply the continuous master theorem to solve the recurrence. Show that $f_n^{(t)} = \frac{2t+3}{t+1} n + o(n)$ (for ordinary quickselect we have $f_n = f_n^{(0)} = 3n + o(n)$).

Useful formulas:

- If k is constant with respect to x and $x \to \infty$ then

$$\binom{x}{k} \sim \frac{x^k}{k!}$$

- (Beta integral) For any $m, n \geq 0$,

$$\int_0^1 x^m (1-x)^n \, dx = \frac{m!n!}{(m+n+1)!}$$

SOLUTION:

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(Answer to Question 4 here.)