1. (1) Briefly describe the difference between deterministic and randomised algorithm, and name two examples of algorithms that are not deterministic.

2. (2) Consider a random graph $G \in G_{n,p}$, give an expression to compute the probability that a given vertex in G has degree $= 0$.

3. (2) Let n be a power of 2 integer. Give an algorithm such that using a fair coin, it can generate u.a.r. an integer r with $0 \leq r < n$. Prove that indeed any r is generated u.a.r., i.e with probability $1/(n-1)$.

4. (2) Let X_G be a random variable counting the number of edges in a random graph $G \in G_{n,p}$. Find the values of p and δ such that with high probability $X_G \in [\frac{n^2p}{2} - \delta, \frac{n^2p}{2} + \delta]$, for large n. Give the details of the proof.

5. (2) The 10^5 students in the Barcelona public university system, are constituting a selected e-club named EPLFP. To be admitted into EPLFP a student has to set a personal web page, and prove that it has at least 250 followers. Do you you think the random graph $G_{n,p}$ is a suitable model to study the properties and evolution of the social network EPLFP?

6. (2) Assume that we have a biased coin, i.e. ($\mathbf{Pr}[H] \neq 1/2$). How would you use this coin to simulate a fair coin ($\mathbf{Pr}[H] = \mathbf{Pr}[T] = 1/2$)?

7. Consider the following protocol for broadcasting data to a set of n host on an unreliable network: The server send a packet to the first host. If it does not receive an acknowledge in one second, it sends another packet, and so on. Once it receives an acknowledge from the first host, it goes on to the second host. Assume that, for each packet sent to host i, the probability than an acknowledgement is received by the server in one second is p, and that all of these "acknowledge-received" events are independent. Let X_i be the number of packages sent to i host, so X_i has a geometric distribution with parameter p. Define X as the total number of packages sent to n host, $X = \sum_{i=1}^{n} X_i$.

   (a) (1) What is $\mathbf{E}[X]$?

   (b) (2) Give un (exponentially decreasing) upper bound on the probability that X exceeds twice its expectation.

8. (3) We throw n balls to m bins. independently and u.a.r. Let X_i be an indicator rv for bin i containing $\leq n/2m$ balls, and let X be the number of bins containing no more than $\mathbf{E}[X] \leq m\left(\frac{1}{e}\right)^{n/8m}$.

9. Suppose that we can obtain independent samples $X_1, X_2, \ldots X_n$ of a random variable X and that we want to use these samples to estimate $\mathbf{E}[X]$. Using t samples, we use $(\sum_{i=1}^{n} X_i/t)$ for our estimate of $\mathbf{E}[X]$. We want the estimate to be within $\epsilon\mathbf{E}[X]$ from the true value of $\mathbf{E}[X]$ with probability $\geq (1-\delta)$. Due to dependences, we may not be able to use Chernoff bound directly to bound how good our estimate is if X is not a $0/1$ random variable. We develop an alternative approach that requires only having a bound on the $\mathbf{Var}[X]$. Let $r = \sqrt{\mathbf{Var}[X]}/\mathbf{E}[X]$.

   (a) (3) Show using Chebyshev that $O(r^2/\epsilon^2\delta)$ samples are sufficient to solve the problem (*Hint: Compute* $\mathbf{E}[Y_t]$ *and* $\mathbf{Var}[Y_t]$ *in terms of* X. *Use Chebyshev on* Y_t *to bound* $\mathbf{Pr}\left[\left|(\sum_{i=1}^{t} X_i - \mathbf{E}[X])/t\right| < \epsilon\mathbf{E}[X]\right]$ *(be carefull it is tricky). Take* $t \geq r^2/\epsilon^2\delta$)

   (b) (2) Suppose that we need only a weak estimate that is within $\epsilon\mathbf{E}[X]$ of $\mathbf{E}[X]$, with probability at least $3/4$. Argue that. $O(r^2/\epsilon^2)$ samples are enough for this weak estimate. (*Hint: take* $t \geq 4r^2/\epsilon^2$)

Recall Chernoff's bounds

Let $X_1 \ldots X_n$ be independent Bernuilli r.v. with $\Pr X_i = 1 = p_i$, and let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbf{E}[X]$. Then:

1. For any $\delta > 0$,

$$\mathbf{Pr}[X \leq (1-\delta)\mu] < \left(\frac{e^{\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu},$$

$$\mathbf{Pr}[X \geq (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

2. For any $\delta \in (0,1]$

$$\mathbf{Pr}[X \leq (1-\delta)\mu] < e^{-\mu\delta^2/2},$$

$$\mathbf{Pr}[X \geq (1-\delta)\mu] < e^{-\mu\delta^2/4}.$$

3. If each for every $1 \leq i \leq n$, $0 \leq X_i \leq 1$, then for every $\delta > 0$

$$\mathbf{Pr}[|X-\mu| \geq \delta\mu] \leq 2e^{-\delta^2\mu/(2+\delta)}.$$