Extension Orderings*

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Abstract. In this paper we study how to extend a collection of term orderings on disjoint signatures to a single one, called an *extension ordering*, which preserves (part of) their properties. Apart of its own interest, e.g. in automated deduction, extension orderings turn out to be a new method to obtain simple and constructive proofs for modularity of termination of TRS. Three different schemes to define extension orderings are given. The first one to deal with reduction orderings, the second one to extend simplification orderings and the last one for total reduction orderings. This provides simpler and more constructive proofs for some known modularity results for (simple and total) termination of rewriting as well as the first —to our knowledge— results for rewriting modulo equational theories. Finally, our technique is applied to extend an ordering on a given signature to a new one on the signature enlarged with new symbols.

1 Introduction

General methods for proving termination are crucial for decision procedures and for using rewriting-like methods in theorem proving and programming. A well-known related problem in rewrite theory is to prove the modularity of termination of *term rewrite systems* (TRS's), that is, given two terminating rewrite systems R_1 and R_2, to show that their union $R_1 \cup R_2$ is also terminating. This property was shown to be false in general, even when the two rewrite systems do not share any function symbols [17]. Therefore many efforts have been devoted to finding sufficient conditions under which termination is modular. Most existing results impose syntactic (or abstract) conditions on (one of) the two TRS's (e.g. [16, 5, 11, 8, 13, 6]), and some other ones impose restrictions on the method used to show the termination of the involved TRS's ([9, 7]).

In this context it is quite surprising that, although usually termination of TRS's is proved by actually building a *reduction ordering* in which the rewrite relation is contained (cf. [4]), in most modularity proofs for termination such an ordering is not explicitly built, that is, in this sense these proofs are not constructive. Hence, for simplicity and to improve the understanding of the problem, it is worth to look at it from a more constructive point of view. This can give information about the weakest conditions ensuring modularity and why they are

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needed. As a good example for this viewpoint, *total termination* (i.e. the rewrite relation is included in some total reduction ordering) was proved modular provided that one of the TRS's is conservative in [7]. The reason for this requirement seems to be rather technical, and the proof does provide no intuition of its need. Here we prove the same result in a simple way, actually building a total ordering containing the combined system. This gives some intuition about the restriction.

Our main idea is to generalize the combination of TRS's to the combination of orderings. Instead of proving that the ordering induced by the combined rewrite system is well-founded, we build a well-founded ordering and prove that it includes the combined TRS. This is done by means of *extension orderings*. By an extension ordering of a collection of orderings $\succ_1, \ldots, \succ_n$ on disjoint signatures, we mean another ordering $\succ$ s.t. $\succ_i \subseteq \succ$ for all i. Actually, an extension ordering is interesting when it preserves (part of) the properties satisfied by the initial orderings.

In section 3 we define an extension ordering $\succ_e$ on ground terms, which preserves well-foundedness of the $\succ_i$. Moreover, if every $\succ_i$ is monotonic, then $\succ_e$ fulfills a property called *quasi-monotonicity* which suffices for termination of rewriting. As expected, due to the non-modularity of termination of *disjoint union* of TRS's, $\succ_e$ cannot be lifted, in general, to terms with variables. However, lifting is possible for right-linear systems. This gives us a simple constructive proof for modularity of right-linear TRS's.

Section 4 is on simple termination (i.e. the rewrite relation is included in some simplification ordering). We define an extension ordering $\succ_s$ preserving monotonicity and the subterm property to prove the modularity of simple termination (this was first proved in a non-constructive way by [9]).

In section 5 an extension ordering preserving well-foundedness, monotonicity and totality on ground terms is given, allowing us to prove the results of [7].

In section 6 these extension orderings are proved to preserve also E-compatibility for certain classes of equational theories E. Therefore we can obtain the the first —as far as we know— modularity results for rewriting modulo equational theories. This is a first example of new results that can be derived from our extension orderings approach.

In section 7 we extend any ordering, to allow new function symbols. These results can sometimes be used to prove termination of *hierarchical unions* of TRS's [3, 14].

In section 8 some possible improvements to the present results are commented and the general application of the method in termination proofs is discussed.

2 Preliminaries

We adopt the standard notations and definitions for term rewriting given in [4]. Here we recall some of them.

$|t|$ is the size (number of symbols) of t. $\mathcal{V}ars(t)$ and $\mathcal{M}vars(t)$ are respectively the set and the multiset of variables of t. Operations on sets and multisets are denoted ambiguously. $t|_p$ is the subterm of t at position p; $t[s]_p$ is obtained from

t by replacing $t|_p$ by s, and $\mathcal{H}ead(t)$ is the topmost function symbol of t. $=_E$ is the congruence generated on $\mathcal{T}(\mathcal{F})$ by the equations E.

Below let s and t (possibly indexed) be terms in $\mathcal{T}(\mathcal{F})$ and f a function symbol in $\mathcal{F}$. A (strict partial) ordering on $\mathcal{T}(\mathcal{F})$ (a transitive irreflexive relation) $\succ$ fulfills the *subterm property* if $f(\ldots t \ldots) \succ t$ for all f and t. It is *monotonic* if $s \succ t$ implies $f(\ldots s \ldots) \succ f(\ldots t \ldots)$. A monotonic ordering fulfilling the subterm property is a *simplification ordering* and for finite signatures it is *well-founded*: there is no infinite sequence $t_1 \succ t_2 \succ \ldots$ An ordering $\succ$ is *E-compatible* if $s' =_E s \succ t =_E t'$ implies $s' \succ t'$. Finally we say that an ordering is total on (the E-congruence classes of) $\mathcal{T}(\mathcal{F})$ if $s \neq t$ ($s \neq_E t$) implies $s \succ t$ or $t \succ s$ for all terms $s, t \in \mathcal{T}(\mathcal{F})$. Any well-founded monotonic ordering $\succ$ total on $\mathcal{T}(\mathcal{F})$ is a simplification ordering.

An ordering on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is *stable* under substitutions if $s \succ t$ implies $s\sigma \succ t\sigma$ for all terms s and t and substitutions σ. A well-founded monotonic ordering stable under substitutions is called a *reduction ordering*. An ordering $\succ$ on $\mathcal{T}(\mathcal{F})$ can always be lifted to an ordering $\succ_v$ (or simply $\succ$) on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, which is stable under substitutions in the following way: if s and t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$, then $s \succ t$ if $s\sigma \succ t\sigma$ for all substitutions σ with range in $\mathcal{T}(\mathcal{F})$.

Let $\succ$ be a relation on terms. Then $max_{\succ}(S)$ is the set of all maximal terms in the set of terms S, i.e. the set S' s.t. $t \in S'$ iff there is no s in S such that $s \succ t$. When working modulo an equational theory E, $max_{\succ}(S)$ is any set obtained from S' as defined above by choosing one element in each E-congruence class.

Given an ordering $\succ$ on terms and a congruence relation $=$, the lexicographic extension $\succ^{lex}$ of $\succ$ wrt. $=$ for n-tuples is defined as: $\langle s_1, \ldots, s_n \rangle \succ^{lex} \langle t_1, \ldots, t_n \rangle$ iff $\exists k$ s.t. $s_1 = t_1, \ldots, s_{k-1} = t_{k-1}$ and $s_k \succ t_k$.

The extension to finite multisets of a congruence relation $=$ is defined as: $\{s_1, \ldots, s_m\} = \{t_1, \ldots, t_n\}$ iff $m = n$ and there exists some permutation π of $1 \ldots m$, s.t. $s_{\pi(i)} = t_i$ for all $i : 1 \ldots n$. The extension, wrt. a congruence $=$, of an ordering $\succ$ to finite multisets, denoted by $\succ\!\!\succ$, is defined by: $M = \{s_1, \ldots, s_m\} \succ\!\!\succ \{t_1, \ldots, t_n\} = N$ iff $M \supset N$ or $s_i \succ t_{j_1} \wedge \ldots \wedge s_i \succ t_{j_k}$ and $(M \setminus \{s_i\} \underline{\succ\!\!\succ} N \setminus \{t_{j_1}, \ldots, t_{j_k}\}$ for some $i = 1 \ldots m$ and $1 \leq j_1 < \ldots < j_k \leq n$ $(k = 0 \ldots n)$, where $M \underline{\succ\!\!\succ} N$ means $M \succ\!\!\succ N$ or $M = N$.

If $\succ$ is a well-founded ordering on terms then $\succ^{lex}$ and $\succ\!\!\succ$ are well-founded orderings on n-tuples of terms and on finite multisets of terms respectively, more precisely $\succ^{lex}$ and $\succ\!\!\succ$ preserve irreflexivity, transitivity and well-foundedness.

A term rewrite system (TRS) is a set of rules $l \rightarrow r$ where l and r are terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ with $\mathcal{V}ars(l) \supseteq \mathcal{V}ars(r)$. The union of two rewrite systems R_1 and R_2 is called a *disjoint union* (or *direct sum*), and denoted by $R_1 \uplus R_2$, if R_1 and R_2 do not share any symbol. For a given TRS R and terms s and t in $\mathcal{T}(\mathcal{F}, \mathcal{X})$, we say that s *rewrites* into t, written $s \rightarrow_R t$ if $s|_p = l\sigma$ and $t = s[r\sigma]_p$ for some rule $l \rightarrow r \in R$, position p in s and substitution σ. Moreover, s rewrites modulo E into t, denoted by $s \rightarrow_{R/E} t$ if $s =_E s'$, $s'|_p = l\sigma$ and $t = s'[r\sigma]$, for some rule $l \rightarrow r \in R$, term s', position p in s' and substitution σ. R is *terminating* (resp. *E-terminating*) over $\mathcal{T}(\mathcal{F})$ if $\rightarrow_R$ (resp. $\rightarrow_{R/E}$) is well-founded over $\mathcal{T}(\mathcal{F})$. The embedding relation $\succ_{emb}$ is the smallest simplification ordering, i.e. the relation

$\rightarrow^{+}_{emb}$ generated by the rewrite system $emb = \{f(\ldots x \ldots) \rightarrow x \mid \forall f \in \mathcal{F}\}$.

If R is a TRS with left and right hand sides in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ then it is (E-)terminating over $\mathcal{T}(\mathcal{F})$ iff there is an (E-compatible) well-founded monotonic ordering $\succ$ s.t. $l\sigma \succ r\sigma$ for all $l \rightarrow r \in R$ and ground substitution σ with range in $\mathcal{T}(\mathcal{F})$. Therefore if $\succ$ is an (E-compatible) reduction ordering it suffices to prove $l \succ r$ for all rules $l \rightarrow r$ in R. In fact, if R is (E-)terminating then $\rightarrow^{+}_{R}$ is itself an (E-compatible) reduction ordering. If R is (E-)terminating over $\mathcal{T}(\mathcal{F})$ (and $\mathcal{F}$ contains at least one constant symbol) then R is (E-)terminating over $\mathcal{T}(\mathcal{F} \cup \mathcal{F}')$ for any set of function symbols $\mathcal{F}'$. This means that we can, in general, speak about termination without mentioning over which signature.

From now on we suppose that we have n relations $\succ_1, \ldots, \succ_n$ over the sets $\mathcal{T}(\mathcal{F}_1), \ldots, \mathcal{T}(\mathcal{F}_n)$ respectively, where all $\mathcal{F}_i$ are disjoint and contain at least one constant symbol.

3 Extension of reduction orderings and modularity

In this section we extend general orderings on terms and study modularity of termination of TRS's. The ground case is considered first. Then we analyze the applicability of our ordering to terms with variables and termination of TRS's. The following mapping N_i eliminates from terms in $\mathcal{T}(\mathcal{F})$ those function symbols that are not in $\mathcal{F}_i$ and is the basis of our extension ordering $\succ_e$.

Definition 1. N_i is the mapping from terms in $\mathcal{T}(\mathcal{F})$ to multisets of terms in $\mathcal{T}(\mathcal{F}_i)$ recursively defined as:

$$N_i(f(t_1, \ldots, t_m)) = \begin{cases} \{f(t'_1, \ldots, t'_m) \mid t'_j \in N_i(t_j) \cup Z_i(t_j)\} & \text{if } f \in \mathcal{F}_i \\ N_i(t_1) \cup \ldots \cup N_i(t_m) & \text{otherwise} \end{cases}$$

where 0_i is some (fixed) constant symbol in $\mathcal{F}_i$ and $Z_i(t) = \{0_i\}$ if $t \notin \mathcal{T}(\mathcal{F}_i)$ and $Z_i(t) = \emptyset$ otherwise.

Proposition 2. *For all $i, j : 1 \ldots n$ and any $t \in \mathcal{T}(\mathcal{F}_i)$, if $j \neq i$ then $N_j(t) = \emptyset$ and if $j = i$ then $N_j(t) = \{t\}$. Furthermore, for all $s, t \in \mathcal{T}(\mathcal{F})$ and $f \in \mathcal{F}$, if $s \notin \mathcal{T}(\mathcal{F}_i)$ or both s and t are in $\mathcal{T}(\mathcal{F}_i)$ then $N_i(s) = N_i(t)$ implies $N_i(f(\ldots s \ldots)) \supseteq N_i(f(\ldots t \ldots))$.*

Definition 3. The relation $\succ_e$ on $\mathcal{T}(\mathcal{F})$ extending relations $\succ_i$ for all $i : 1 \ldots n$ is defined as: $\quad s \succ_e t \quad$ iff

$$N_1(s) = N_1(t) \wedge \ldots \wedge N_{i-1}(s) = N_{i-1}(t) \wedge N_i(s) \succ\!\succ_i N_i(t) \text{ for some } i : 1 \ldots n$$

Of course, the roles of all $\succ_i$ (and N_i) are symmetric here, and can be exchanged, obtaining in fact different $\succ_e$.

The following lemmas show that $\succ_e$ preserves several properties of the $\succ_i$ and extends them.

Lemma 4. *$\succ_e$ extends $\succ_i$ for all $i : 1 \ldots n$.*

Proof. By proposition 2, for all $i, j : 1 \ldots n$ and $s, t \in \mathcal{T}(\mathcal{F}_i)$ we have $N_j(s) = N_j(t) = \emptyset$ if $j < i$, and $N_j(s) = \{s\}$ and $N_j(t) = \{t\}$ otherwise. Then $s \succ_i t$ implies $s \succ_e t$. □

Lemma 5. *If $\succ_i$ is an ordering for all $i : 1 \ldots n$ then $\succ_e$ is an ordering.*

Proof. Irreflexivity: since $N_i(s) = N_i(s)$, by irreflexivity of $\succ_i$ (and hence of $\succ\!\succ_i$) for all $i : 1 \ldots n$, we have $N_i(s) \not\succ\!\succ_i N_i(s)$ and therefore $s \not\succ_e s$.

Transitivity: suppose $s \succ_e t$ and $t \succ_e u$ with $N_1(s) = N_1(t), \ldots, N_{j-1}(s) = N_{j-1}(t)$, $N_j(s) \succ\!\succ_j N_j(t)$ and $N_1(t) = N_1(u), \ldots, N_{k-1}(t) = N_{k-1}(u)$, $N_k(t) \succ\!\succ_k N_k(u)$, for some $j, k : 1 \ldots n$. Then, by transitivity of $\succ_j$ and $\succ_k$ (and hence of $\succ\!\succ_j$ and $\succ\!\succ_k$), we have $N_1(s) = N_1(u), \ldots, N_{m-1}(s) = N_{m-1}(u)$, $N_m(s) \succ\!\succ_k N_m(u)$, for $m = min(j, k)$. □

Lemma 6. *If $\succ_i$ is well-founded for all $i : 1 \ldots n$ then $\succ_e$ is well-founded.*

Proof. Supose that there exists an infinite sequence $t_1 \succ_t t_2 \succ_t \ldots$. By well-foundedness of $\succ_1$ and hence of $\succ\!\succ_1$, there must be some t_j in the sequence s.t. $N_1(t_j) = N_1(t_k)$ for all $k > j$. Let $t_1^1 \succ_t t_2^1 \succ_t \ldots$ the infinite subsequence starting from t_j. Repeating this reasoning n times it follows that there must be an infinite subsequence $t_1^n \succ_t t_2^n \succ_t \ldots$ s.t. $N_i(t_1^n) = N_i(t_2^n) = \ldots$ for all $i : 1 \ldots n$, which contradicts the existence of such an infinite decreasing sequence. □

Although $\succ_e$ is not monotonic in general it fulfills the following property which we call *quasi-monotonicity* and which, as we will see, is enough to ensure termination of TRS's.

Lemma 7. *Let s and t be terms in $\mathcal{T}(\mathcal{F})$. If $\succ_i$ is monotonic and $s \notin \mathcal{T}(\mathcal{F}_i)$ or both s and t are in $\mathcal{T}(\mathcal{F}_i)$ for all $i : 1 \ldots n$ then $s \succ_e t$ implies $f(\ldots s \ldots) \succ_e f(\ldots t \ldots)$, i.e. $\succ_e$ is* quasi-monotonic.

Proof. Assume $N_1(s) = N_1(t) \wedge \ldots \wedge N_{j-1}(s) = N_{j-1}(t) \wedge N_j(s) \succ\!\succ_j N_j(t)$, for some $j : 1 \ldots n$. We have to show $f(\ldots s \ldots) \succ_e f(\ldots t \ldots)$. By proposition 2, since $t \in \mathcal{T}(\mathcal{F}_i)$ whenever $s \in \mathcal{T}(\mathcal{F}_i)$ for all $i : 1 \ldots n$, we have $N_1(f(\ldots s \ldots)) \succeq\!\succeq_1 N_1(f(\ldots t \ldots)) \wedge \ldots \wedge N_{j-1}(f(\ldots s \ldots)) \succeq\!\succeq_{j-1} N_{j-1}(f(\ldots t \ldots))$. Assume $N_1(f(\ldots s \ldots)) = N_1(f(\ldots t \ldots)) \wedge \ldots \wedge N_{j-1}(f(\ldots s \ldots)) = N_{j-1}(f(\ldots t \ldots))$ (otherwise the lemma already holds):

- If $f \notin \mathcal{F}_j$ then $N_j(f(\ldots s \ldots)) = N_j(s) \cup X$ and $N_j(f(\ldots t \ldots)) = N_j(t) \cup X$ for some X obtained from the transformations of the other top level arguments, and therefore $N_j(f(\ldots s \ldots)) \succ\!\succ_j N_j(f(\ldots t \ldots))$.
- If $f \in \mathcal{F}_j$ then we have $N_j(f(\ldots s \ldots)) = \{f(\ldots s' \ldots) \mid s' \in N_j(s) \cup Z_j(s)\}$ and $N_j(f(\ldots t \ldots)) = \{f(\ldots t' \ldots) \mid t' \in N_j(t) \cup Z_j(t)\}$. Since $t \in \mathcal{T}(\mathcal{F}_j)$ whenever $s \in \mathcal{T}(\mathcal{F}_j)$, either $Z_j(s) = Z_j(t) = \emptyset$ or $Z_j(s) \neq \emptyset$. Therefore, by monotonicity of $\succ_j$, $N_j(f(\ldots s \ldots)) \succ\!\succ_j N_j(f(\ldots t \ldots))$.

Since in both cases we have $N_j(f(\ldots s \ldots)) \succ\!\succ_j N_j(f(\ldots t \ldots))$, it holds that $f(\ldots s \ldots) \succ_e f(\ldots t \ldots)$. □

Now we are going to lift $\succ_e$ to terms with variables ($s \succ_e t$ iff $s\sigma \succ_e t\sigma$ as in section 2) in order to obtain modularity results. Let R be $R_1 \uplus \ldots \uplus R_n$ and let $\succ_i$ be reduction orderings on $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$ containing the terminating rewrite relations $\rightarrow^+_{R_i}$ respectively.

A possible way to prove termination of R over $\mathcal{T}(\mathcal{F})$ (see e.g. [2]) is to show that there exists a well-founded ordering $\succ$ on $\mathcal{T}(\mathcal{F})$ s.t.: (1) $l\sigma \succ r\sigma$ for all rules $l \rightarrow r \in R$ and all substitutions σ with range in $\mathcal{T}(\mathcal{F})$, and (2) $s \rightarrow_R t$ and $s \succ t$ implies $f(\ldots s \ldots) \succ f(\ldots t \ldots)$ for all $s, t \in \mathcal{T}(\mathcal{F})$ and $f \in \mathcal{F}$. From these two properties we can derive that $s \succ t$ whenever $s \rightarrow_R t$, for all terms $s, t \in \mathcal{T}(\mathcal{F})$, i.e. R is terminating over $\mathcal{T}(\mathcal{F})$. Property (2) is fulfilled by $\succ_e$:

Lemma 8. *Let s and t be terms in $\mathcal{T}(\mathcal{F})$. Then $s \rightarrow_R t$ and $s \succ_e t$ implies $f(\ldots s \ldots) \succ_e f(\ldots t \ldots)$ for all $f \in \mathcal{F}$.*

However, property (1) does not hold in general for $\succ_e$. This is not surprising since this result would imply the termination of the direct sum of any two terminating rewriting systems, which is well-known to be false, as shown by Toyama's counter example ([17]). If we apply our method to this example, indeed the resulting ordering does not fulfil the property:

Example 1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be $\{f, 1, 0_1\}$ and $\{g, 0_2\}$ repectively and let $\succ_1$ and $\succ_2$ be the orderings generated by the TRS's $R_1 = \{f(1, 0_1, x) \rightarrow f(x, x, x)\}$ and $R_2 = \{g(x, y) \rightarrow x, g(x, y) \rightarrow y\}$ respectively. Then with the substitution $\{x \mapsto g(1, 0_1)\}$ we have:
$N_1(f(1, 0_1, g(1, 0_1))) = \{f(1, 0_1, 1), f(1, 0_1, 0_1), f(1, 0_1, 0_1)\}$ and
$N_1(f(g(1, 0_1), g(1, 0_1), g(1, 0_1))) = \{f(1, 1, 1), \ldots, f(1, 0_1, 1), f(1, 0_1, 0_1),$
$f(1, 0_1, 0_1), \ldots, f(0_1, 0_1, 0_1)\}$, which implies $f(g(1, 0_1), g(1, 0_1), g(1, 0_1)) \succ_e$
$f(1, 0_1, g(1, 0_1))$, contradicting the rule of R_1. □

In fact, the problem is that although $\succ_e$ extends each $\succ_i$ at the ground level, this becomes false at the variable level. Nevertheless, by imposing some syntactic conditions on the TRS's, we can prove that $\succ_e$ includes the rewrite relation $\rightarrow_R$ on $\mathcal{T}(\mathcal{F})$. We consider the case where all rules $l \rightarrow r$ in R are *right linear*: all variables in $\mathcal{V}ars(r)$ occur only once in r, i.e. r is *linear*.

Proposition 9. *Let R be a right-linear TRS over a signature $\mathcal{F}$ and let 0 be a new constant symbol (not occurring in R). Then R is terminating iff $R'' = R \cup R'$ is terminating, where $R' = \{f(\ldots 0 \ldots) \rightarrow 0 \mid f \in \mathcal{F}\}$.*

Note that this property does not hold for non right linear TRS:

Example 2. The rewrite system $R = \{f(x, x) \rightarrow f(g(x), h(x)\}$ is terminating. But $R \cup R'$ is not: $f(0, 0) \rightarrow_{R''} f(g(0), h(0)) \rightarrow_{R''} f(0, h(0)) \rightarrow_{R''} f(0, 0) \rightarrow''_R \ldots$ □

By proposition 9 we can consider that if R_i is right linear and terminating then there is a reduction ordering $\succ_i$ on $\mathcal{T}(\mathcal{F}_i)$ including $\rightarrow^+_{R_i}$, where $\mathcal{F}_i$ contains a constant symbol, which is taken as 0_i, s.t. $f(\ldots 0_i \ldots) \succ_i 0_i$.

Lemma 10. *Let s and t be terms in $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$ for some $i : 1 \ldots n$. If $\succ_i$ is stable under substitutions and t is linear then $s \succ_i t$ implies $s\sigma \succ_e t\sigma$ for all substitutions σ with range in $\mathcal{T}(\mathcal{F})$, i.e. $s \succ_e t$.*

Proof. (Hint) Use the fact that if $t \in \mathcal{T}(\mathcal{F}_i, \mathcal{X})$ for some $i : 1 \ldots n$ and t is not a variable then $N_i(t\sigma) \supseteq \{t\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \mathcal{V}ars(t)\}$ for all ground substitutions σ with range in $\mathcal{T}(\mathcal{F})$. Moreover if t is linear then $max_{\succ_i}(N_i(t\sigma)) \subseteq \{t\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \mathcal{V}ars(t)\}$ for all ground substitutions σ with range in $\mathcal{T}(\mathcal{F})$. $\square$

This lemma leads to a constructive proof of modularity of termination for direct sums of *right linear* TRS's:

Theorem 11. *If $R_1, \ldots, R_n$ are repectively right linear terminating TRS's over $\mathcal{T}(\mathcal{F}_1), \ldots, \mathcal{T}(\mathcal{F}_n)$ then $R = R_1 \uplus \ldots \uplus R_n$ is terminating.*

Proof. If $R_1, \ldots, R_n$ are terminating then $\rightarrow_{R_1} \ldots \rightarrow_{R_n}$ are repsctively included in some reduction orderings $\succ_1 \ldots \succ_n$. Then, by lemmas 6, 5, 8 and 10 the extension ordering $\succ_e$ of $\succ_1 \ldots \succ_2$ is a well-founded ordering s.t. $s \succ_e t$ for all ground terms $s, t \in \mathcal{T}(\mathcal{F})$ with $s \rightarrow_R t$, i.e. $R_1 \uplus \ldots \uplus R_n$ is terminating. $\square$

4 Extension of Simplification Orderings and Modularity

Definition 12. A TRS R (with left and right hand sides in $\mathcal{T}(\mathcal{F})$) is *simply terminating* over $\mathcal{T}(\mathcal{F})$ iff there exists some simplification ordering $\succ$ s.t. $l\sigma \succ r\sigma$ for all $l \rightarrow r$ in R and substitutions σ with range in $\mathcal{T}(\mathcal{F})$. We say that R is simply terminating (in general) if R is simply terminating over $\mathcal{T}(\mathcal{F} \cup \mathcal{F}')$ for any set of new function symbols $\mathcal{F}'$.

At the end of this section we will see that if a TRS (with left and right hand sides in $\mathcal{T}(\mathcal{F})$) is simply terminating over $\mathcal{T}(\mathcal{F})$ then it is simply terminating in general.

We present a new extension ordering which extends any collection of simplification orderings at the ground level and at the variable level. Therefore, the modularity of simple termination follows.

4.1 A transformation on terms

Definition 13. Let S_i be a mapping from terms in $\mathcal{T}(\mathcal{F})$ to terms in $\mathcal{T}(\mathcal{F}_i)$ recursively defined as:

$$S_i(f(t_1, \ldots, t_m)) = \begin{cases} 0_i & \text{if } f \notin \mathcal{F}_i \\ f(S_i(t_1), \ldots, S_i(t_m)) & \text{otherwise} \end{cases}$$

where 0_i is some (fixed) constant symbol in $\mathcal{F}_i$.

Let S be a mapping from terms in $\mathcal{T}(\mathcal{F})$ to terms in $\cup_{i=1}^{n}\mathcal{T}(\mathcal{F}_i)$ defined as:

$$S(t) = S_i(t) \qquad \text{if } \mathcal{H}ead(t) \in \mathcal{F}_i$$

Let S_E be a mapping from terms in $\mathcal{T}(\mathcal{F})$ to multisets of terms in $\cup_{i=1}^{n}\mathcal{T}(\mathcal{F}_i)$ defined as:

$$S_E(t) = \{S(t') \mid t \succeq_{emb} t'\}$$

Proposition 14. *Let $t_1, \ldots, t_p$ be ground terms in $\mathcal{T}(\mathcal{F})$ and let f be a function symbol in $\mathcal{F}$. Then $S_E(f(t_1, \ldots, t_p)) = S_E(t_1) \cup \ldots \cup S_E(t_p) \cup \{S(f(t'_1, \ldots, t'_p)) \mid t_j \succeq_{emb} t'_j\}$.*

Definition 15. Let s, t be terms in $\mathcal{T}(\mathcal{F})$. Then
$s = f(s_1, \ldots, s_p) \succ_s g(t_1, \ldots, t_q) = t$ iff

- $max_{\succ_u}(S_E(s)) \succ\!\succ_u max_{\succ_u}(S_E(t))$ or
- $max_{\succ_u}(S_E(s)) = max_{\succ_u}(S_E(t))$ and $f = g$ and $\{s_1, \ldots, s_p\} \succ\!\succ_s \{t_1, \ldots t_p\}$

where the relation $\succ_u$ is defined as $\cup_{i=1}^{n} \succ_i$.

Note that, since all $\succ_i$ are orderings on $\mathcal{T}(\mathcal{F}_i)$, if $s \succ_u t$ then $s, t \in \mathcal{T}(\mathcal{F}_i)$ and $s \succ_i t$ for some $i : 1 \ldots n$, which implies that $\succ_u$ is an ordering whenever all $\succ_i$ also are.

Lemma 16. *Let all $\succ_i$ be orderings on $\mathcal{T}(\mathcal{F}_i)$. Then*
(i) $\succ_s$ is an ordering on $\mathcal{T}(\mathcal{F})$
(ii) if $\succ_i$ is monotonic and fulfills the subterm property then $\succ_s$ extends $\succ_i$
(iii) if all $\succ_i$ have the subterm property then $\succ_s$ has the subterm property on $\mathcal{T}(\mathcal{F})$.

Lemma 17. *If $\succ_i$ are monotonic orderings on $\mathcal{T}(\mathcal{F}_i)$ for all $i : 1 \ldots n$ then $\succ_s$ is monotonic on $\mathcal{T}(\mathcal{F})$.*

Proof. (Hint) Use the fact that $max_{\succ_u}(S_E(s)) \underline{\succ\!\succ}_u max_{\succ_u}(S_E(t))$ implies $max_{\succ_u}(S_E(f(\ldots s \ldots))) \underline{\succ\!\succ}_u max_{\succ_u}(S_E(f(\ldots t \ldots)))$ for all $s, t \in \mathcal{T}(\mathcal{F})$ and $f \in \mathcal{F}$. $\square$

From the previous lemmas, we can derive the following theorem:

Theorem 18. *If $\succ_i$ are simplification orderings on $\mathcal{T}(\mathcal{F}_i)$ for all $i : 1 \ldots n$. Then $\succ_s$ is a simplification ordering on $\mathcal{T}(\mathcal{F})$ extending $\succ_i$ for all $i : 1 \ldots n$.*

As in the previous section now $\succ_s$ is lifted to terms with variables. Let R_i be simply terminating TRS's over $\mathcal{T}(\mathcal{F}_i)$ for all $i : 1 \ldots n$, and let R be $R_1 \uplus \ldots \uplus R_n$.

Lemma 19. *Let s and t be terms in $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$ for some $i : 1 \ldots n$. If $\succ_i$ is a simplification ordering and $s\sigma' \succ_i t\sigma'$ for all substitutions σ' with range in $\mathcal{T}(\mathcal{F}_i)$, then $s\sigma \succ_s t\sigma$ for all ground substitutions with range in $\mathcal{T}(\mathcal{F})$, i.e. $s \succ_s t$.*

Proof. (Hint) Use the fact that if s (similarly t) is not a variable then we have $max_{\succ_u}(S_E(s\sigma)) = max_{\succ_u}(\cup_{x \in \mathcal{V}ars(s)} S_E(x\sigma) \cup \{s\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \mathcal{V}ars(s)\})$ for all substitutions σ with range in $\mathcal{T}(\mathcal{F})$. □

Theorem 20. *If $R_1, \ldots, R_n$ are repectively simply terminating TRS's over the sets $\mathcal{T}(\mathcal{F}_1), \ldots, \mathcal{T}(\mathcal{F}_n)$ then $R = R_1 \uplus \ldots \uplus R_n$ is simply terminating over $\mathcal{T}(\mathcal{F})$.*

Proof. If R_i is simply terminating over $\mathcal{T}(\mathcal{F}_i)$ for all $i : 1 \ldots n$ then, by definiton, for each R_i, there exists some simplification ordering $\succ_i$ on $\mathcal{T}(\mathcal{F}_i)$, s.t. $l\sigma \succ_i r\sigma$ for all rules $l \rightarrow r \in R_I$ and all substitution σ with range in $\mathcal{T}(\mathcal{F}_i)$. Therefore, by theorem 18 and lemma 19 it follows that $\succ_s$ is a simplification ordering on $\mathcal{T}(\mathcal{F})$, s.t. $l\sigma \succ_s r\sigma$ for all rule $l \rightarrow r \in R$ and substitutions σ with range in $\mathcal{T}(\mathcal{F})$, i.e. R is simply terminating over $\mathcal{T}(\mathcal{F})$. □

Note that this also implies that if R is simply terminating over $\mathcal{T}(\mathcal{F})$ then R is simply terminating over $\mathcal{T}(\mathcal{F} \cup \mathcal{F}')$ for any $\mathcal{F}'$, and hence that simple termination is modular.

5 Extension of Total Orderings and Modularity

In this section we consider *total termination* (which implies simple termination). For our purposes, the orderings $\succ_e$ and $\succ_s$ (defined in section 3 and 4) can not be used since they do not preserve totality of the initial orderings $\succ_i$. However, it is quite easy to adapt $\succ_e$ (the transformation N_i is replaced by T_i) to obtain an extension ordering $\succ_t$, which also preserves totality (and monotonicity). Moreover, the resulting extension ordering $\succ_t$ needs weaker sufficient conditions to extend orderings on terms with variables. Therefore, we obtain better results on modularity of total termination. The proofs are similar to the ones used for $\succ_e$.

Definition 21. Let T_i be a mapping from terms in $\mathcal{T}(\mathcal{F})$ to multisets of terms in $\mathcal{T}(\mathcal{F}_i)$ recursively defined as:

$$T_i(f(t_1, \ldots, t_m)) = \begin{cases} \{f(t'_1, \ldots, t'_m) \mid t'_j \in \{0_i\} \cup T_i(t_j)\} & \text{if } f \in \mathcal{F}_i \\ T_i(t_1) \cup \ldots \cup T_i(t_m) & \text{otherwise} \end{cases}$$

where 0_i is some (fixed) minimal wrt. $\succ_i$ constant symbol in $\mathcal{F}_i$. Note that if $\succ_i$ is a total ordering then there is only one minimal constant, and if $\succ_i$ is also monotonic then 0_i is the minimal term wrt. $\succ_i$ in $\mathcal{T}(\mathcal{F}_i)$.

Definition 22. Let $\succ_r$ be an arbitrary relation on $\mathcal{T}(\mathcal{F})$ and let s and t be terms in $\mathcal{T}(\mathcal{F})$. Then $s \succ_t t$ iff

- $T_1(s) = T_1(t) \wedge \ldots \wedge T_{i-1}(s) = T_{i-1}(t) \wedge T_i(s) \succ\!\!\succ_i T_i(t)$ for some $i : 1 \ldots n$ or
- $T_i(s) = T_i(t)$ for all $i : 1 \ldots n$ and $s \succ_r t$

Theorem 23. *If $\succ_i$ are monotonic well-founded orderings total on $\mathcal{T}(\mathcal{F}_i)$ for all $i : 1 \ldots n$. Then $\succ_t$ is a monotonic well-founded ordering total on $\mathcal{T}(\mathcal{F})$ extending $\succ_i$ for all $i : 1 \ldots n$.*

A term t is *conservative* wrt. another term s if $\mathcal{M}vars(s) \supseteq \mathcal{M}vars(t)$. A TRS R is conservative if r is conservative wrt. l, for all $l \rightarrow r$ in R.

Lemma 24. *Let s and t be terms in $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$ for some $i : 1 \ldots n$ and let $\succ_i$ be a monotonic ordering total on $\mathcal{T}(\mathcal{F}_i)$. If $i = 1$ or t is conservative wrt. s then $s\sigma' \succ_i t\sigma'$ for all substitutions σ' with range in $\mathcal{T}(\mathcal{F}_i)$ implies $s\sigma \succ_t t\sigma$ for all substitutions σ with range in $\mathcal{T}(\mathcal{F})$, i.e. $s \succ_t t$.*

Proof. (Hint) Use the fact that due to the monotonicity and totality of the orderings $\succ_i$ we have that the maximal terms (wrt. $\succ_i$) in $T_i(s\sigma)$ and $T_i(t\sigma)$ are respectively $s\sigma'$ and $t\sigma'$ for some substitution σ', s.t. $x\sigma'$ is the maximal term (wrt. $\succ_i$) in $T_i(x\sigma)$ for all $x \in \mathcal{V}ars(s)$. □

Again, from theorem 23 and lemma 24, we derive the following result on modularity of termination:

Definition 25. A TRS R (with left and right hand sides in $\mathcal{T}(\mathcal{F}, \mathcal{X})$) is *totally terminating* iff there exists some well-founded monotonic ordering $\succ$ total on $\mathcal{T}(\mathcal{F})$ s.t. $l\sigma \succ r\sigma$ for all $l \rightarrow r$ in R and substitutions σ with range in $\mathcal{T}(\mathcal{F})$.

Theorem 26. *Let $R_1, \ldots, R_n$ be totally terminating TRS's. If $R_2, \ldots, R_n$ are conservative then $R = R_1 \uplus \ldots \uplus R_n$ is totally terminating.*

6 E-compatible Extension Orderings and Modularity

Here we will study how to deal with the extension orderings when we have E-compatible orderings. We will apply the same techniques as in previous sections to obtain easily some results for certain equational theories E, more precisely for *permutative* theories:

Definition 27. An equation $s = t$, with $s = f(s_1, \ldots, s_m)$ and $t = f(t_1, \ldots, t_m)$, is *permutative* if $\{s_1, \ldots, s_m\} = \{t_1, \ldots, t_m\}$. An equational theory E is permutative if it has a presentation in which all the axioms are permutative.

In the following the same extension orderings $\succ_e$, $\succ_s$ and $\succ_t$ defined as in the previous sections, but replacing $=$ by $=_E$, allow us to prove the same results as before, but for E-termination. Note that in fact we have each $\succ_i$ being E_i-compatible for some equational theory E_i on $\mathcal{T}(\mathcal{F}_i)$, but since we consider only disjoint signatures it is equivalent to speak about E-compatibility for all $\succ_i$ where $E = E_1 \cup \ldots \cup E_n$.

Lemma 28. *If all $\succ_i$ are E-compatible orderings then $\succ_e$, $\succ_s$ and $\succ_t$ are E-compatible orderings for any permutative theory E.*

Proof. (Hint) Show that the transformations N_i, S_i and T_i are compatible with E. □

All other lemmas of the previous sections can be straightforwardly adapted thanks to the E-compatibility of all orderings. This leads to the following results:

Theorem 29. *Let $R_1, \ldots, R_n$ be E-terminating TRS's over $\mathcal{T}(\mathcal{F}_1), \ldots, \mathcal{T}(\mathcal{F}_n)$ repectively, let R be $R_1 \uplus \ldots \uplus R_n$ and let E be a permutative theory. Then*
(i) if all R_i are right linear then R is E-terminating;
(ii) if all R_i are simply E-terminating then R is simply E-terminating;
(iii) if all R_i are totally E-terminating and $R_2, \ldots, R_n$ are conservative then R is simply E-terminating.

7 Enlarging the signature

Here we present a new extension ordering, which allows us to extend an ordering to deal with new function symbols. This extension ordering is based on a precedence on the set of function symbols. Although when dealing with terms with variables this extension ordering imposes strong syntactic conditions on the compared terms, in the case of total extensions it has a better behaviour.

These results can sometimes be applied to prove the termination of *hierarchical unions* [3, 14]: $R_1 \cup R_2$ where the defined symbols of R_2, denoted $\mathcal{D}_2$ do not occur in R_1. We can consider simply that we are enlarging the signature $\mathcal{F}_1$ with the new symbols of $\mathcal{F}_2$ which are at least $\mathcal{D}_2$. Then $R_1 \cup R_2$ is terminating if it is included in one of the extension ordering we will define below. We first show how to extend an ordering $\succ$ on $\mathcal{T}(\mathcal{F})$, with $\mathcal{F}$ finite, to an ordering $\succ_p$ on $\mathcal{T}(\mathcal{F}')$, where $\mathcal{F}' = \mathcal{F} \cup \mathcal{F}_0$ and $\mathcal{F}_0$ is a set of new function symbols.

Definition 30. Let P be a mapping from terms in $\mathcal{T}(\mathcal{F}', \mathcal{X})$ to multisets of terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ recursively defined as:

$$
\begin{aligned}
P(x) &= x \qquad \text{if } x \text{ is a variable} \\
P(f(t_1, \ldots, t_m)) &= \begin{cases} \{0\} & \text{if } m = 0 \text{ and } f \notin \mathcal{F} \\ \{f(t'_1, \ldots, t'_m) \mid t'_i \in P(t_i)\} & \text{if } f \in \mathcal{F} \\ P(t_1) \cup \ldots \cup P(t_m) & \text{otherwise} \end{cases}
\end{aligned}
$$

where 0 is some (fixed) minimal (wrt. $\succ$) representative constant symbol in $\mathcal{F}$.

In the following definition, $\succ_{\mathcal{F}}$ is a precedence on $\mathcal{F}'$ (although, in fact, relations in $\succ_{\mathcal{F}}$ between symbols in $\mathcal{F}$ are not needed, since if $P(s) = P(t)$ and $f = \mathcal{H}ead(s) \neq \mathcal{H}ead(t) = g$ then $f \in \mathcal{F}' - \mathcal{F}$ or $g \in \mathcal{F}' - \mathcal{F}$). We also associate a status *mul* or *lex* to each function symbol. We assume that all symbols have status *mul* when $\succ$ is not a simplification ordering.

Definition 31. Let s and t be tems in $\mathcal{T}(\mathcal{F}')$. Then $s = f(s_1, \ldots, s_m) \succ_p g(t_1, \ldots, t_n) = t$ iff $P(s) \succ P(t)$ or $P(s) = P(t)$ for all $i : 1 \ldots n$ and

– $s_i \succeq_p t$, for some $i : 1 \ldots m$
– $f \succ_{\mathcal{F}} g$ and $s \succ_p t_j$, for all $j : 1 \ldots n$
– $f = g$ and $Stat(f) = mul$ and $\{s_1, \ldots, s_m\} \succ\!\!\succ_p \{t_1, \ldots, t_n\}$
– $f = g$ and $Stat(f) = lex$ and $\langle s_1, \ldots, s_m \rangle \succ_p^{lex} \langle t_1, \ldots, t_n \rangle$ and $s \succ t_i$ for all $i : 1 \ldots m$.

Lemma 32.
1. If $\succ$ is an ordering on $\mathcal{T}(\mathcal{F})$ and $\succ_{\mathcal{F}}$ is an ordering on $\mathcal{F}'$ then $\succ_p$ is an ordering extending $\succ$ on $\mathcal{T}(\mathcal{F}')$, preserving monotonicity, well-foundedness and the subterm property of $\succ$.
2. If $\succ$ is stable under substitutions, t is linear and $s \succ t$ then $s\sigma \succ_p t\sigma$ for all substitutions σ with range in $\mathcal{T}(\mathcal{F}')$;
3. If $\succ$ is stable under substitutions then $\succ_p$ is stable under substitutions when comparing linear terms;
4. If $\succ$ is total on $\mathcal{T}(\mathcal{F})$ and $Stat(f) = lex$ for all symbols $f \in \mathcal{F}'$ and $\succ_{\mathcal{F}}$ is a total precedence (in fact, as said, symbols in $\mathcal{F}$ are not need to be comparable) then $\succ_p$ is total on $\mathcal{T}(\mathcal{F}')$;
5. If $\succ$ is a reduction ordering total on $\mathcal{T}(\mathcal{F})$ then $\succ_p$ is a reduction ordering total on $\mathcal{T}(\mathcal{F}')$ containing $\succ$.

Proof. (Hint) Most of the properties are proved as in previous sections. Only well-foundedness and stability under substitutions require the following additional remarks: (i) if $s, t \in \mathcal{T}(\mathcal{F}')$ with $P(s) = P(t)$ and t is embedded in s (i.e. $s \succ_{emb} t$) then $s \succ_p t$; (ii) if $P(t) = \{t_1, \ldots, t_n\}$ then $P(t\sigma) = P(t_1\sigma) \cup \ldots \cup P(t_k\sigma)$. □

These results improve the ones given in [1], where only enlargements with new constants where considered. Finally, note that in section 5 we could have used this path extension instead of using the ordering $\succ_r$ as last component.

All results in this section can be obtained for E-compatible ordering for some equational theories E, e.g. containing associativity and commutativity axioms, provided that the new function symbols are free (or commutative) and that in the precedence the new function symbols are smaller than the old ones (in fact it is only requiered the new symbols to be smaller than the associative old ones). Then a slight modification of the ordering defined here is E-compatible (see [15] for details).

8 Improvements and conclusions

It seems possible to improve the results of this paper by defining other extension orderings. For instance, we can easily relax the conditions of section 3 by considering $\succ_2$ as an ordering on $\mathcal{T}(\mathcal{F})$, i.e. it is an ordering on $\mathcal{T}(\mathcal{F}_2, \mathcal{X})$ stable under substitutions with range in $\mathcal{T}(\mathcal{F})$. Then the extension ordering: $s \succ_c t$ iff $N_1(s) \succ\!\!\succ_1 N_1(t)$ or $N_1(s) = N_1(t)$ and $s \succ_2 t$ allows us to prove that if R_1 is right linear and R_2 is conservative and both are terminating then $R_1 \uplus R_2$ is terminating. Therefore we believe that it is possible to prove most of the known results in combination of rewrite systems using extension ordering (and perhaps to improve them, e.g. to more general equational theories). But this requires a deeper study of transformations (note that in this paper we already use four different kind of transformations) and their combination.

Also the applicability of the method to modularity of termination of *constructor-sharing* or *composable* TRS's ([8, 13, 6, 10]) and conditional term rewrite systems ([12]) has to be further investigated.

On the other hand, extension orderings can also be used to prove the termination of the union of particular TRS's. For instance, we can build an extension ordering for proving that $R \uplus \{g(x,y) \rightarrow x\}$ is terminating for any terminating TRS R. In fact this is true for any single projection rule $g(\ldots x \ldots) \rightarrow x$. This gives a very simple proof for the fact that adding pairing to (e.g. simply) typed lambda calculus preserves strong normalization.

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