A Monotonic Higher-Order Semantic Path Ordering

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Abstract. There is an increasing use of (first- and higher-order) rewrite rules in many programming languages and logical systems. The *recursive path ordering* (RPO) is a well-known tool for proving termination of such rewrite rules in the first-order case. However, RPO has some weaknesses. For instance, since it is a simplification ordering, it can only handle simply terminating systems. Several techniques have been developed for overcoming these weaknesses of RPO. A very recent such technique is the *monotonic semantic path ordering* (MSPO), a simple and easily automatable ordering which generalizes other more ad-hoc methods.
Another recent extension of RPO is its higher-order version HORPO. HORPO is an ordering on terms of a typed lambda-calculus generated by a signature of higher-order function symbols. Although many interesting examples can be proved terminating using HORPO, it inherits the weaknesses of the first-order RPO.
Therefore, there is an obvious need for higher-order termination orderings without these weaknesses. Here we define the first such ordering, the *monotonic higher-order semantic path ordering* (MHOSPO), which is still automatable like MSPO. We give evidence of its power by means of several natural and non-trivial examples which cannot be handled by HORPO.

1 Introduction

There is an increasing use of higher-order rewrite rules in many programming languages and logical systems. As in the first-order case, termination is a fundamental property of most applications of higher-order rewriting. Thus, there exists a need to develop for the higher-order case the kind of semi-automated termination proof techniques that are available for the first-order case.

There have been several attempts at designing methods for proving strong normalization of higher-order rewrite rules based on ordering comparisons. These orderings are either quite weak [LSS92,JR98], or need an important user interaction [PS95]. Recently, in [JR99], the recursive path ordering (RPO) [Der82] —the most popular ordering-based termination proof method for first-order rewriting— has been extended to a higher-order setting by defining a higher-order recursive path ordering (HORPO) on terms following a typing discipline

including ML-like polymorphism. This ordering is powerful enough to deal with many non-trivial examples and can be automated. Besides, all aforementioned previous methods operate on terms in η-long β-normal form, hence apply only to the higher-order rewriting "à la Nipkow" [MN98], based on higher-order pattern matching modulo $\beta\eta$. HORPO is the first method which operates on arbitrary higher-order terms, therefore applying to the other kind of rewriting, based on plain pattern matching, where β-reduction is considered as any other rewrite rule. Furthermore, HORPO can operate as well on terms in η-long β-normal form, and hence it provides a termination proof method for both kinds of higher-order rewriting (see also [vR01] for a particular version of HORPO dealing η-long β-normal forms).

However, HORPO inherits the same weaknesses that RPO has in the first-order case. RPO is a *simplification ordering* (a monotonic ordering including the subterm relation), which extends a precedence on function symbols to an ordering on terms. It is simple and easy to use, but unfortunately, it turns out, in many cases, to be a weak termination proving tool. First, there are many term rewrite systems (TRSs) that are terminating but are not contained in any simplification ordering, i.e. they are not *simply terminating*. Second, in many cases the head symbol, the one that is compared with the precedence, does not provide enough information to prove the termination of the TRS. Therefore, since HORPO follows the same structure and the same use of a precedence as in RPO (in fact, it reduces to RPO when restricted to first-order terms), it is easy to expect that similar weaknesses will appear when proving termination of higher-order rewriting.

To avoid these weaknesses, in the first-order case, many different so-called *transformation methods* have been developed. By transforming the TRS into a set of ordering constraints, the *dependency pair* method [AG00] has become a successful general technique for proving termination of (non-simply terminating) TRSs.

As an alternative to transformation methods, more powerful term orderings like the *semantic path ordering* (SPO) ([KL80]) can be used. SPO generalizes RPO by replacing the precedence on function symbols by any (well-founded) underlying (quasi-)ordering involving the whole term and not only its head symbol. Although the simplicity of the presentation is kept, this makes the ordering much more powerful. Unfortunately, SPO is not so useful in practice, since, although it is well-founded, it is not, in general, monotonic. Hence, in order to ensure termination, apart from checking that the rules of the rewrite system are included in the ordering, in addition the monotonicity for contexts of the rewrite rules has to be proved.

In [BFR00], a monotonic version of SPO, called MSPO, has been presented. MSPO overcomes the weaknesses of RPO, it is automatable and it is shown to generalize other existing transformation methods.

Due to the fact that RPO and SPO share the same "path ordering nature", our aim is to obtain for SPO and MSPO the same kind of extensions to the higher-order case as it was done for RPO.

In this paper we present the *higher-order semantic path ordering* (HOSPO) which operates on terms of a typed lambda-calculus generated by a signature of higher-order function symbols. As done for HORPO in [JR99], HOSPO is proved well-founded by Tait and Girard's computability predicate proof technique. Then a monotonic version of HOSPO, called MHOSPO, is obtained, which provides an automatable powerful method for proving termination of higher-order rewriting on arbitrary higher-order terms union β-reduction. To illustrate this power several non-trivial examples are shown to be terminating.

In this work we do not consider η-reductions, although all results can be extended to include them at the expense of some easy, but technical, complications. For the same reason we have not included ML-like polymorphism.

Besides its own interest as a termination method for higher-order rewriting, the extension of HORPO to HOSPO and the definition of MHOSPO on top of HOSPO is also interesting for other reasons. On the one hand, it shows the stability of the definition of HORPO, since it extends to HOSPO in the same way as RPO extends to SPO. On the other hand, it shows the stability of the definition of MSPO, since MHOSPO is obtained from HOSPO in the same way as MSPO is obtained from SPO. This gives some intuition of why term orderings provide a more adequate framework for defining general termination proving methods than other techniques.

Formal definitions and basic tools are introduced in Section 2. In Section 3 an example motivating the need of extending HORPO is given. In Section 4 we present and study the higher-order semantic path ordering. Section 5 introduces MHOSPO. The method is applied to two examples in Section 6. Some conclusions and possible extensions are given in Section 7. The reader is expected to be familiar with the basics of term rewrite systems [DJ90] and typed lambda calculi [Bar92]. Due to the lack of room we have provided almost no proofs. All them can be found in [BR01].

2 Preliminaries

2.1 Types, Signatures and Terms

We consider terms of a simply typed lambda-calculus generated by a signature of higher-order function symbols.

The set of types $\mathbb{T}$ is generated from the set $V_{\mathbb{T}}$ of *type variables* (considered as sorts) by the constructor $\rightarrow$ for *functional types* in the usual way. Types are called *functional* when they are headed by the $\rightarrow$ symbol, and *basic* when they are a type variable. As usual, $\rightarrow$ associates to the right. In the following, we use α, β for type variables and $\sigma, \tau, \rho, \theta$ for arbitrary types.

Let $\equiv$ be the congruence on types generated by equating all type variables in $V_{\mathbb{T}}$. Note that two types are equivalent iff they have the same arrow skeleton, i.e. $\sigma \equiv \tau$ iff replacing all type variables in σ and τ by the same type variable α we obtain two identical types.

A signature $\mathcal{F}$ is a set of function symbols which are meant to be algebraic operators, equipped with a fixed number n of arguments (called the *arity*) of

respective types $\sigma_1 \in \mathbb{T}, \ldots, \sigma_n \in \mathbb{T}$, and an output type $\sigma \in \mathbb{T}$. A *type declaration* for a function symbol f will be written as $f : \sigma_1 \times \ldots \times \sigma_n \to \sigma$. Type declarations are not types, although they are used for typing purposes. Note, however, that $\sigma_1 \to \ldots \to \sigma_n \to \sigma$ is a type if $f : \sigma_1 \times \ldots \times \sigma_n \to \sigma$ is a type declaration. We will use the letters f, g, h to denote function symbols.

Given a signature $\mathcal{F}$ and a denumerable set $\mathcal{X}$ of variables, the set of *raw algebraic λ-terms* is defined by $\mathcal{T} := \mathcal{X} \mid (\lambda\mathcal{X} : \mathbb{T}.\mathcal{T}) \mid @(\mathcal{T}, \mathcal{T}) \mid \mathcal{F}(\mathcal{T}, \ldots, \mathcal{T})$. Terms of the form $\lambda x : \sigma.u$ are called *abstractions*, while the other terms are said to be *neutral*. For sake of brevity, we will often omit types. $@(u, v)$ denotes the application of u to v. The application operator is allowed to have a variable arity. We call a *partial left-flattening* of the term $@(@(\ldots @(t_1, t_2) \ldots, t_{n-1}), t_n)$, any term of the form $@(@(\ldots @(t_1, t_2) \ldots, t_i), t_{i+1}, \ldots, t_n)$. As a matter of convenience, we may write $@(u, v_1, \ldots, v_n)$ for $@(@(\ldots @(u, v_1) \ldots), v_n)$, assuming $n \geq 1$.

We denote by $\mathcal{V}ar(t)$ the set of free variables of t. We may assume for convenience (and without further notice) that bound variables in a term are all different, and are different from the free ones. The *subterm* of t at position p is denoted by $t|_p$, and we write $t \trianglerighteq t|_p$. The result of replacing $t|_p$ at position p in t by u is denoted by $t[u]_p$. We use $t[u]$ to indicate that u is a subterm of t, and simply $t[\,]_p$ for a term with a hole, also called a context. The notation $\overline{s}$ will be ambiguously used to denote a list, or a multiset, or a set of terms $s_1, \ldots, s_n$.

2.2 Typing Rules

Typing rules restrict the set of terms by constraining them to follow a precise discipline. Environments are sets of pairs written $x : \sigma$, where x is a variable and σ is a type. Our typing judgments are written as $\Gamma \vdash M : \sigma$ if the term M can be proved to have the type σ in the environment Γ:

Variables:
$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma}$$

Functions:
$$\frac{f : \sigma_1 \times \ldots \times \sigma_n \to \sigma \in \mathcal{F} \qquad \Gamma \vdash t_1 : \sigma'_1 \equiv \sigma_1 \ \ldots \ \Gamma \vdash t_n : \sigma'_n \equiv \sigma_n}{\Gamma \vdash f(t_1, \ldots, t_n) : \sigma}$$

Abstraction:
$$\frac{\Gamma \cup \{x : \sigma\} \vdash t : \tau}{\Gamma \vdash (\lambda x : \sigma.t) : \sigma \to \tau}$$

Application:
$$\frac{\Gamma \vdash s : \sigma \to \tau \quad \Gamma \vdash t : \sigma' \equiv \sigma}{\Gamma \vdash @(s, t) : \tau}$$

A term M has type σ in the environment Γ if $\Gamma \vdash M : \sigma$ is provable in the above inference system. A term M is typable in the environment Γ if there exists a unique type σ such that M has type σ in the environment Γ. A term M is typable if it is typable in some environment Γ. Note again that function symbols are uncurried, hence must come along with all their arguments.

The reason to use $\equiv$ is that having a larger set of typable terms allow us to increase the power of the ordering we will define (see the end of the proof of example 3).

Substitutions are supposed to be well-typed. We use the letter γ for substitutions and postfix notation for their application, e.g. $t\gamma$. Substitutions behave as endomorphisms defined on free variables (avoiding captures).

2.3 Higher-order rewrite rules

The rewrite relation which is considered in this paper is the union of the one induced by a set of higher-order rewrite rules and the β-reduction relation both working modulo α-conversion.

We use $\longrightarrow_\beta$ for the β-reduction rule: $@(\lambda x.v, u) \longrightarrow_\beta v\{x \mapsto u\}$. The simply typed λ-calculus is confluent and terminating with respect to β-reductions.

As said, for simplicity reasons, in this work we do not consider η-reductions, although all results can be extended to include them.

A higher-order *term rewrite system* is a set of rewrite rules $R = \{\Gamma \vdash l_i \to r_i\}_i$, where l_i and r_i are higher-order terms such that l_i and r_i have the same type σ_i in the environment Γ. Note that usually the terms l_i and r_i will be typable in the system without using the type equivalence $\equiv$, i.e. they will be typable by the system replacing $\equiv$ by syntactic equality $=$ on types.

Given a term rewriting system R, a term s rewrites to a term t at position p with the rule $\Gamma \vdash l \to r$ and the substitution γ, written $s \xrightarrow[l\to r]{p} t$, or simply $s \to_R t$, if $s|_p = l\gamma$ and $t = s[r\gamma]_p$ (modulo α-conversion). We denote by $\xrightarrow[R]{*}$ the reflexive, transitive closure of the rewrite relation $\xrightarrow[R]{}$. We are actually interested in the relation $\longrightarrow_{R\beta} = \longrightarrow_R \cup \longrightarrow_\beta$.

Given a rewrite relation $\longrightarrow$, a term s is strongly normalizing if there is no infinite sequence of rewrites issuing from s. The rewrite relation itself is *strongly normalizing*, or *terminating*, if all terms are strongly normalizing.

2.4 Orderings and quasi-orderings

We will make intensive use of well-founded orderings for proving strong normalization properties. We will use the vocabulary of rewrite systems for orderings and quasi-orderings (see e.g. [DJ90]). For our purpose, a *(strict) ordering*, always denoted by $>$ or $\succ$ (possibly with subscripts), is an irreflexive and transitive relation. An ordering $>$ is *monotonic* if $s > t$ implies $f(\ldots s \ldots) > f(\ldots t \ldots)$; it is *stable* if $s > t$ implies $s\gamma > t\gamma$ for all substitutions γ; and it is *well-founded* if there are no infinite sequences $t_1 > t_2 > \ldots$ An ordering is said to be higher-order when it operates on higher-order terms and is α-compatible: it does not distinguish between α-convertible terms.

A *quasi-ordering*, always denoted by $\succeq$ or $\geq$, is a transitive and reflexive binary relation. Its inverse is denoted by $\preceq$. Its *strict part* $\succ$ is the strict ordering $\succeq \setminus \preceq$ (i.e, $s \succ t$ iff $s \succeq t$ and $s \not\preceq t$). Its *equivalence* $\sim$ is $\succeq \cap \preceq$. Note that $\succeq$ is the disjoint union of $\succ$ and $\sim$.

A quasi-ordering $\succeq$ is *well-founded* if $\succ$ is. It is *stable* if $\succ$ is and $s\sigma \succeq t\sigma$ whenever $s \succeq t$. $\succeq$ is *quasi-monotonic* if $f(\ldots, s, \ldots) \succeq f(\ldots, t, \ldots)$ whenever

$s \succeq t$. A quasi-ordering is said to be higher-order when it operates on higher-order terms and its equivalence includes α-conversion. Note that if $\succ$ is a higher-order ordering $\succ \cup =_\alpha$ is a higher-order quasi-ordering whose strict part is $\succ$.

Assume $>_1, \ldots, >_n$ are orderings on sets $S_1, \ldots, S_n$. Then its *lexicographic combination* $(>_1, \ldots, >_n)_{lex}$ is an ordering on $S_1 \times \ldots \times S_n$. We write $>_{lex}$ if all sets and orderings are equal. Similarly the lexicographic combination of quasi-orderings $\succeq_1, \ldots, \succeq_n$ is denoted by $(\succeq_1, \ldots, \succeq_n)_{lex}$ and defined as $s(\succeq_1, \ldots, \succeq_n)_{lex} t$ iff either $s \succ_i t$ for some i and $s \succeq_j t$ for all $j < i$, or $s \succeq_i t$ for all i. If all $>_i$, respectively $\succeq_i$, are well-founded then its lexicographic combination also is. Same happens for stability and α-compatibility.

Assume $\succ$ is an ordering on a set S. Then its *multiset extension*, denoted by $\succ\!\!\succ$, is an ordering on the set of multisets of elements of S, defined as the transitive closure of: $M \cup \{s\} \succ\!\!\succ M \cup \{t_1, \ldots, t_n\}$ if $s \succ t_i \; \forall i \in [1..n]$ (using $\cup$ for multiset union). If $\succ$ is well-founded, stable and α-compatible then $\succ\!\!\succ$ also is.

Definition 1. *A higher-order reduction ordering is a well-founded, monotonic and stable higher-order ordering $>$, such that $\longrightarrow_\beta \subseteq >$.*

A higher-order quasi-rewrite ordering is a quasi-monotonic and stable higher-order quasi-ordering $\succeq$, such that $\longrightarrow_\beta \subseteq \succeq$. A higher-order quasi-reduction ordering is in addition well-founded.

Reduction orderings allow us to show that the relation $\longrightarrow_R \cup \longrightarrow_\beta$ is terminating by simply comparing the left-hand and right-hand sides of each rule in R:

Theorem 1. *Let $R = \{\Gamma \vdash l_i \to r_i\}_{i \in I}$ be a higher-order rewrite system such that $l_i > r_i$ for every $i \in I$. Then the relation $\longrightarrow_R \cup \longrightarrow_\beta$ is strongly normalizing.*

2.5 The Higher-Order Recursive Path Ordering (HORPO)

We present here a restricted version of HORPO (with no status for function symbols) which is enough for our purposes. The HORPO is based on a quasi-ordering $\geq_\mathcal{F}$, called *precedence*, on the set of function symbols $\mathcal{F}$ whose strict part $>_\mathcal{F}$ is well-founded and whose equivalence is denoted by $=_\mathcal{F}$. HORPO compares terms of equivalent type by using the head symbol wrt. the precedence and/or the arguments recursively following a similar structure as RPO in the first-order case.

Additionally, in HORPO, in order to make the ordering more powerful, when comparing two terms s and t, when s is headed by an algebraic function symbol, we cannot only use the arguments of s but also any term in the so called *computable closure* of s, which mainly allows us to introduce abstractions in some cases. The intuition for doing so comes from the strong normalization proof. In that proof, it is crucial to show the computability (in the sense of Tait and Girard's strong normalization proof technique) of the right-hand side term t by

using the left-hand side arguments of s, which are assumed to be computable, and the head function symbol. Therefore, instead of using directly the arguments of s, we can use any term obtained applying computability preserving operations to its arguments.

To ease the reading, here we give a subset of the possible computability preserving operations given in [JR99] which is enough for our purposes and gives the flavor of the method.

Definition 2. *Given a term $s = f(s_1, \ldots, s_n)$, we define its computable closure $\mathcal{CC}(s)$ as $\mathcal{CC}(s, \emptyset)$, where $\mathcal{CC}(s, \mathcal{V})$ with $\mathcal{V} \cap \mathcal{V}ar(s) = \emptyset$, is the smallest set of well-typed terms containing the arguments $s_1, \ldots, s_n$, all variables in $\mathcal{V}$ and closed under the following two operations:*

1. *precedence: $h(\overline{u}) \in \mathcal{CC}(s, \mathcal{V})$ if $f >_{\mathcal{F}} h$ and $\overline{u} \in \mathcal{CC}(s, \mathcal{V})$.*
2. *abstraction: $\lambda x.u \in \mathcal{CC}(s, \mathcal{V})$ if $x \notin \mathcal{V}ar(s) \cup \mathcal{V}$ and $u \in \mathcal{CC}(s, \mathcal{V} \cup \{x\})$.*

Now we give the definition of HORPO, denoted by $\succ_{horpo}$, adapted from [JR99], where $\succeq_{horpo}$ means the union of $\succ_{horpo}$ and α-conversion.

$$s : \sigma \underset{horpo}{\succ} t : \tau \text{ iff } \sigma \equiv \tau \text{ and}$$

1. $s = f(\overline{s})$ with $f \in \mathcal{F}$, and $u \succeq_{horpo} t$ for some $u \in \overline{s}$
2. $s = f(\overline{s})$ with $f \in \mathcal{F}$ and $t = g(\overline{t})$ with $f >_{\mathcal{F}} g$, and
   for all $t_i \in \overline{t}$ either $s \succ_{horpo} t_i$ or $u \succeq_{horpo} t_i$ for some $u \in \mathcal{CC}(s)$
3. $s = f(\overline{s})$ and $t = g(\overline{t})$, $f =_{\mathcal{F}} g \in \mathcal{F}$ and $\{\overline{s}\} \gg\!\!\succ_{horpo} \{\overline{t}\}$
4. $s = f(\overline{s})$ with $f \in \mathcal{F}$, $t = @(\overline{t})$ is some partial left-flattening of t,
   and for all $t_i \in \overline{t}$ either $s \succ_{horpo} t_i$ or $u \succeq_{horpo} t_i$ for some $u \in \mathcal{CC}(s)$
5. $s = @(s_1, s_2)$, $t = @(t_1, t_2)$ and $\{s_1, s_2\} \gg\!\!\succ_{horpo} \{t_1, t_2\}$
6. $s = \lambda x.u$ and $t = \lambda x.v$, and $u \succ_{horpo} v$
7. $s = @(\lambda x.u, v)$ and $u\{x \mapsto v\} \succeq_{horpo} t$

The definition of HORPO is used as a recursive algorithm to check whether a term s is greater than a term t. In case 6 we apply an α-conversion on the heading lambda if necessary. The following theorem states that HORPO is a higher-order reduction ordering, which means that if we show that the left hand side of each rule in a rewriting system is greater than the right hand side then we can conclude that the system is terminating for higher-order rewriting.

Theorem 2. *The transitive closure of $\succ_{horpo}$ is a higher-order reduction ordering.*

3 A motivating example

In this section we present an example which on the one hand will help us to show how HORPO is used, and on the other hand will exhibit its weakness, since the example cannot be proved terminating although it is, and the need to improve in the direction of semantic orderings.

The example defines the *prefix sum* of a list of natural numbers, i.e. the list with the sum of all prefixes of the given list, using the *map* function. We do not include the rules defining the + symbol.

Example 1. Prefix sum. Let $V_{\mathbb{T}} = \{Nat, List\}$,
$\mathcal{X} = \{\ x : Nat, xs : List, F : Nat \to Nat\ \}$ and
$\mathcal{F} = \{\ [] : List,\ cons : Nat \times List \to List,\ + : Nat \times Nat \to Nat,$
$map : (Nat \to Nat) \times List \to List,\ ps : List \to List \qquad \}.$

$$map(F, []) \to [] \tag{1}$$
$$map(F, cons(x, xs)) \to cons(@(F, x), map(F, xs)) \tag{2}$$

$$ps([]) \to [] \tag{3}$$
$$ps(cons(x, xs)) \to cons(x, ps(map(\lambda y.x + y, xs))) \tag{4}$$

Now we try to prove that this system is included in HORPO. Rules 1 and 3 hold applying case 1 of the definition of HORPO. For the second rule we need to show that $map(F, cons(x, xs)) : List \succ_{horpo} cons(@(F, x), map(F, xs)) : List$. In this case the only possibility is to have $map >_{\mathcal{F}} cons$ and apply case 2. Then we check recursively (1) $map(F, cons(x, xs)) : List \succ_{horpo} @(F, x) : Nat$ and (2) $map(F, cons(x, xs)) : List \succ_{horpo} map(F, xs) : List$. For (1) we apply case 4, since $F \in \mathcal{C}(map(F, cons(x, xs)))$ and $map(F, cons(x, xs)) : List \succ_{horpo} x : Nat$, applying twice case 1. Finally, to prove (2) we apply case 3 which holds since $cons(x, xs) : List \succ_{horpo} xs : List$ by case 1.

For the last rule, we can only apply case 2 taking $ps >_{\mathcal{F}} cons$. This requires that $ps(cons(x, xs)) : List \succ_{horpo} x : Nat$, which holds applying twice case 1, and also $ps(cons(x, xs)) : List \succ_{horpo} ps(map(\lambda y.x + y, xs)) : List$. For the latter we apply case 3, which requires $cons(x, xs) : List \succ_{horpo} map(\lambda y.x + y, xs) : List$. To prove this we need $cons >_{\mathcal{F}} map$ and apply case 2, showing that $\lambda y.x + y \in \mathcal{C}(cons(x, xs))$ taking $cons >_{\mathcal{F}} +$, and $cons(x, xs) : List \succ_{horpo} xs : List$ using case 1.

Unfortunately, to show that the second rule is in HORPO we need to take $map >_{\mathcal{F}} cons$ and to show the last rule we need $cons >_{\mathcal{F}} map$, which of course cannot be the case if $>_{\mathcal{F}}$ is well-founded. Note that the considered system cannot be proved either using the definition of HORPO in [JR99], hence the problem is not due to the simplified version of HORPO we are considering here.

Now, if we look at the example, the intuition behind its termination comes from the fact that in all recursive calls the size of the list parameter decrease. This somehow means that this parameter should play an important role when comparing the terms, since the use of the head symbol is not enough. Generalizing path orderings to using more information about the term than only the head symbol is done by means of *semantic path orderings* [KL80].

4 The Higher-Order Semantic Path Ordering (HOSPO)

We present now HOSPO, which generalizes HORPO by using a well-founded stable higher-order quasi-ordering $\succeq_Q$, which does not need to include β-reduction, instead of the precedence $\geq_{\mathcal{F}}$. We also adapt in the same way the computable closure. Note that, although we can fully adapt the definition of the computable closure given in [JR99], for simplicity reasons, we will provide only an adapted version of the restricted computable closure given in section 2.5.

Definition 3. *Given a term $s = f(s_1, \ldots, s_n)$, we define its computable closure $\mathcal{CC}(s)$ as $\mathcal{CC}(s, \emptyset)$, where $\mathcal{CC}(s, \mathcal{V})$ with $\mathcal{V} \cap \mathcal{V}ar(s) = \emptyset$, is the smallest set of well-typed terms containing all variables in $\mathcal{V}$ and all terms in $\{s_1, \ldots, s_n\}$, and closed under the following operations:*

1. *quasi-ordering: $h(\overline{u}) \in \mathcal{CC}(s, \mathcal{V})$ if $f(\overline{s}) \succ_Q h(\overline{u})$ and $\overline{u} \in \mathcal{CC}(s, \mathcal{V})$.*
2. *abstraction: $\lambda x.u \in \mathcal{CC}(s, \mathcal{V})$ if $x \notin \mathcal{V}ar(s) \cup \mathcal{V}$ and $u \in \mathcal{CC}(s, \mathcal{V} \cup \{x\})$.*

Now we give the definition of HOSPO, where $\succeq_{hospo}$ means the union of $\succ_{hospo}$ and α-conversion.

Definition 4. *$s : \sigma \succ_{hospo} t : \tau$ iff $\sigma \equiv \tau$ and*

1. *$s = f(\overline{s})$ with $f \in \mathcal{F}$, $u \succeq_{hospo} t$ for some $u \in \overline{s}$.*
2. *$s = f(\overline{s})$ and $t = g(\overline{t})$ with $f, g \in \mathcal{F}$, $s \succ_Q t$, and for all $t_i \in \overline{t}$ either $s \succ_{hospo} t_i$ or $u \succeq_{hospo} t_i$ for some $u \in \mathcal{CC}(s)$.*
3. *$s = f(\overline{s})$ and $t = g(\overline{t})$ with $f, g \in \mathcal{F}$, $s \succeq_Q t$ and $\{\overline{s}\} \gg_{hospo} \{\overline{t}\}$*
4. *$s = f(\overline{s})$, $f \in \mathcal{F}$, $@(\overline{t})$ is some partial left-flattening of t, and for all $t_i \in \overline{t}$ either $s \succ_{hospo} t_i$ or $u \succeq_{hospo} t_i$ for some $u \in \mathcal{CC}(s)$.*
5. *$s = @(s_1, s_2)$, $t = @(t_1, t_2)$, $\{s_1, s_2\} \gg_{hospo} \{t_1, t_2\}$*
6. *$s = \lambda x.u$, $t = \lambda x.v$, $u \succ_{hospo} v$*
7. *$s = @(\lambda x.u, v)$ and $u\{x \mapsto v\} \succeq_{hospo} t$*

In case 6 we apply an α-conversion on the heading lambda if necessary. Note that $\succeq_Q$ is only used in cases 2 and 3, where the head symbol of both terms is in $\mathcal{F}$. On the other hand, case 7 captures β-reduction at top position, and since, in general, HOSPO is not monotonic, it may be the case that HOSPO does not include β-reduction at any position.

The resulting ordering is shown to be well-defined by comparing pairs of terms $\langle t, s\rangle$ in the well-founded ordering $(\rhd, \rightarrow_\beta \cup \rhd)_{lex}$.

Lemma 1. *$\succ_{hospo}$ is stable under substitutions and α-compatible.*

For stability under substitutions, first we show that $u \in \mathcal{CC}(s)$ implies that $u\gamma \in \mathcal{CC}(s\gamma)$ for every substitution γ. Then the proof is done by induction on the pair $\langle t, s\rangle$ wrt. $(\rhd, \rightarrow_\beta \cup \rhd)_{lex}$, distinguishing cases according to the definition of $\succ_{hospo}$ and using the stability of $\succeq_Q$. For α-compatibility we follow the same pattern.

To prove well-foundedness of the ordering we follow the Tait and Girard's computability predicate proof method as for HORPO in [JR99]. We denote by $[\![\sigma]\!]$ the computability predicate of type σ.

Definition 5. *The family of type interpretations* $\{[\![\sigma]\!]\}_{\sigma\in\mathcal{T}_S}$ *is the family of subsets of the set of typed terms whose elements are the least sets satisfying the following properties:*

1. *If* σ *is a basic type, then* $s:\sigma\in[\![\sigma]\!]$ *iff* $\forall t:\tau$ *s.t.* $s\succ_{hospo} t$, $t\in[\![\tau]\!]$.
2. *If* $s:\sigma=\tau\to\rho$ *then* $s\in[\![\sigma]\!]$ *iff* $@(s,t)\in[\![\rho]\!]$ *for every* $t\in[\![\tau']\!]$, *with* $\tau'\equiv\tau$.

The above definition is based on a lexicographic combination of an induction on the size of the type and a fixpoint computation for basic types. The existence of a least fixpoint is ensured by the monotonicity of the underlying family of functionals (indexed by the set of basic types) with respect to set inclusion (for each basic type). Note that for basic types this definition can be seen as a closure wrt. case 1, taking as initial set for each basic type the set of minimal, wrt. $\succ_{hospo}$, terms (which includes the variables). A typed term s of type σ is said to be computable if $s\in[\![\sigma]\!]$. A vector $\overline{s}$ of terms is computable iff so are all its components.

Property 1 *Computability properties.*

1. *Every computable term is strongly normalizing.*
2. *If* s *is computable and* $s\succ_{hospo} t$ *then* t *is computable.*
3. *A neutral term* s *is computable iff* t *is computable for every* t *s.t.* $s\succ_{hospo} t$.
4. *If* $\overline{t}$ *is a vector of at least two computable terms s.t.* $@(\overline{t})$ *is a typed term, then* $@(\overline{t})$ *is computable.*
5. $\lambda x:\sigma.u$ *is computable iff* $u\{x\mapsto w\}$ *is computable for every computable term* $w:\alpha\equiv\sigma$.

Note that variables are computable as a consequence of Property 1.3. The precise assumption of the following property comes from the ordering used in the proof by induction of Lemma 2 and gives some intuition about the definition of the computable closure.

Property 2 *Assume* $\overline{s}:\overline{\tau}$ *is computable, as well as every term* $h(\overline{u})$ *with* $\overline{u}$ *computable and* $f(\overline{s})\succ_Q h(\overline{u})$. *Then every term in* $\mathcal{C}(f(\overline{s}))$ *is computable.*

Lemma 2. *Let* $f:\overline{\sigma}\to\tau\ \in\mathcal{F}$ *and* $\overline{t}:\overline{\tau}\equiv\overline{\sigma}$ *be a set of terms. If* $\overline{t}$ *is computable, then* $f(\overline{t})$ *is computable.*

The proof is done by induction on the ordering $\langle\succ_Q,\gg_{hospo}\rangle$ operating on pairs $\langle f(\overline{t}),\overline{t}\rangle$. This ordering is well-founded since we are assuming that $\overline{t}$ is computable and hence strongly normalizing by Property 1.1. Note that in the assumption of Property 2 we are only using the first component of the induction ordering. By using both components, we can improve the computable closure, as done in [JR99], adding new cases (see [BR01] for details).

Lemma 3. $\succ_{hospo}$ *is well-founded.*

The proof is done by showing that $t\gamma$ is computable for every typed term t and computable substitution γ, by induction on the size of t and using Property 1.5 and Lemma 2. Note that for the empty substitution γ we have that all typed terms are computable and hence strongly normalizing by Property 1. 1.

5 A monotonic Higher Order semantic path ordering.

In this section we present a monotonic version of HOSPO, called MHOSPO, and show how it can be used in practice.

To define MHOSPO we need an additional quasi-ordering $\succeq_I$ as ingredient. This quasi-ordering is used to ensure monotonicity, and due to this we need to require some properties on it.

Definition 6. *We say that $\succeq_I$ is* quasi-monotonic on $\succeq_Q$ *(or $\succeq_Q$ is* quasi-monotonic wrt. $\succeq_I$*) if*

$$s \succeq_I t \quad \textit{implies} \quad f(\ldots s \ldots) \succeq_Q f(\ldots t \ldots)$$

for all terms s and t and all function symbols f.

A pair $\langle \succeq_I, \succeq_Q \rangle$ is called a higher-order quasi-reduction pair *if $\succeq_I$ is a higher-order quasi-rewrite ordering, $\succeq_Q$ is a well-founded stable higher-order quasi-ordering, and $\succeq_I$ is quasi-monotonic on $\succeq_Q$.*

Now we give the definition of MHOSPO.

Definition 7. *Let $\langle \succeq_I, \succeq_Q \rangle$ be a higher-order quasi-reduction pair.*

$$s \succ_{mhospo} t \textit{ iff } s \succeq_I t \textit{ and } s \succ_{hospo} t.$$

Theorem 3. *The transitive closure of $\succ_{mhospo}$ is a higher-order reduction ordering.*

Well-foundedness follows from the fact that $\succ_{mhospo} \subseteq \succ_{hospo}$ and $\succ_{hospo}$ is well-founded. Stability and α-compatibility follow respectively from the stability and α-compatibility of $\succeq_I$ and $\succ_{hospo}$. Monotonicity follows directly from the fact that $\succeq_I$ is quasi-monotonic on $\succeq_Q$ and includes β-reduction, and cases 3, 5 and 6 of $\succ_{hospo}$. Finally, to prove that $\longrightarrow_\beta \subseteq \succ_{mhospo}$, we use monotonicity, case 7 of HOSPO and the fact that $\succeq_I$ includes β-reduction.

Note that HORPO is a particular case of MHOSPO, which is obtained by taking $\succeq_I$ as $s \succeq_I t$ for all s and t, which has an empty strict part (and trivially fulfils all required properties), and $\succeq_Q$ as a precedence.

In order to make MHOSPO useful in practice we need general methods to obtain quasi-reduction pairs $\langle \succeq_I, \succeq_Q \rangle$. We will first provide possible candidates for the quasi-ordering $\succeq_I$ and then show how to obtain a $\succeq_Q$ forming a pair.

5.1 Building $\succeq_I$

We consider $\succeq_I$ obtained by combining an interpretation I on terms with some higher-order quasi-reduction ordering $\succeq_B$, called the *basic* quasi-ordering, i.e. $s : \sigma \succeq_I t : \tau$ if and only if $I(s : \sigma) \succeq_B I(t : \tau)$ (note that hence we also have that $s : \sigma \succ_I t : \tau$ if and only if $I(s : \sigma) \succ_B I(t : \tau)$). An obvious candidate for $\succeq_B$ is (the transitive closure of) HORPO union α-conversion. For the interpretation I, as a general property, we require the preservation of the

typability of terms and the quasi-monotonicity, stability, α-compatibility and the inclusion of β-reduction of the basic quasi-ordering $\succeq_B$. Note that, since $\succeq_B$ is well-founded, the obtained $\succeq_I$ is also well-founded and hence it is a quasi-reduction ordering, although $\succeq_I$ is only required to be a quasi-rewrite ordering. The reason for adding the well-foundedness property to $\succeq_I$ in this construction will become clear later when building the quasi-reduction pairs $\langle \succeq_I, \succeq_Q \rangle$.

Below some suitable such interpretations obtained by adapting usual interpretations for the first-order case are provided. We consider interpretations that are mappings from typed terms to typed terms. As a matter of simplicity we have considered only interpretations to terms in the same given signature and set of types. Note that we can enlarge our signature (or the set of types) if new symbols (or types) are needed in the interpretation. On the other hand if we consider interpretations to terms in a new signature and set of types then we not only need to interpret the terms but also the types.

Each symbol $f : \sigma_1 \times ... \times \sigma_n \to \sigma$ can be interpreted either by (1) a projection on a single argument of an equivalent type to the output type of f, denoted by the pair $(f(x_1, \ldots, x_n), x_i)$ with $x_i : \tau$ and $\sigma \equiv \tau$, or else by (2) a function symbol f_I, with an equivalent output type to f, applied to a sequence obtained from the arguments of f preserving the typability of the term, denoted by the pair $(f(x_1, \ldots, x_n), f_I(x_{i_1}, \ldots, x_{i_k}))$, for some $k \geq 0$, $i_1, \ldots, i_k \in \{1, ..., n\}$ and $f_I : \sigma'_{i_1} \times ... \times \sigma'_{i_k} \to \sigma'$, with $\sigma'_{i_j} \equiv \sigma_{i_j}$ for all $j \in \{1, ..., k\}$ and $\sigma' \equiv \sigma$. In order to include β-reduction we consider I to be the identity for λ and @. Additionally, we consider I to be the identity for variables (although it can be any bijection). We assume that there is only one pair for each symbol. Usually the identity pairs will be omitted. Thus the interpretation I of a term is obtained, as usual, by using the pairs on the top symbol once the arguments have been recursively interpreted.

It is easy to show that this interpretations preserve typability and the quasi-monotonicity, stability, α-compatibility and the inclusion of the β-reduction of $\succeq_B$.

Example 2. Following the example 1, consider the interpretation I defined by the pairs $(map(x_1, x_2), x_2)$ and $(cons(x_1, x_2), cons_I(x_2))$, where $cons_I : List \to List$, then we have that:

1. $I(map(F, cons(x, xs))) = cons_I(xs)$
2. $I(cons(@(F, x), map(F, xs))) = cons_I(xs)$
3. $I(ps(cons(x, xs))) = ps(cons_I(xs))$
4. $I(cons(x, ps(map(\lambda y.x + y, xs)))) = cons_I(ps(xs))$
5. $I(@(F, x)) = @(F, x)$ and $I(\lambda y.x + y) = \lambda y.x + y$

5.2 Building $\succeq_Q$

Now we show how higher-order quasi-reduction pairs $\langle \succeq_I, \succeq_Q \rangle$ can be obtained. This can be done in a general way, as for MSPO in [BFR00], but for simplicity reasons we will just provide the most usual examples of quasi-orderings $\succeq_Q$ that work in practice.

The simplest case is consider that $\succeq_Q$ and $\succeq_I$ coincide, provided that $\succeq_I$ is a quasi-reduction ordering (which is the case with the $\succeq_I$ we have defined in the previous section).

A more elaborated case for $\succeq_Q$ is to combine lexicographically a precedence and the quasi-ordering $\succeq_I$, that is $\succeq_Q$ is defined as

$$(\succeq_{\mathcal{P}}, \succeq_I)_{lex}$$

where $\succeq_{\mathcal{P}}$ is a (well-founded) precedence, which can be completely different from the precedence used in the HORPO inside $\succeq_I$. Note that the precedence is used on terms by just comparing their head symbols. We can also add another precedence as third component. Since $\succeq_I$ has been built to be well-founded and the precedences are well-founded, its lexicographic combination also is. Quasi-monotonicity of $\succeq_I$ on $\succeq_Q$ can be easily shown using the quasi-monotonicity of $\succeq_I$.

Finally, using an idea coming from the dependency pair method [AG00], instead of using directly $\succeq_I$ inside $\succeq_Q$, we can apply first a *renaming* of the head symbol, in case it is a function symbol, before using $\succeq_I$. This renaming allow us to apply different interpretations when a function symbol occurs on the head than when it occurs inside the term. Therefore thanks to the renaming sometimes the proof can be easier. Since this renaming is not needed in our examples we refer to [BR01] for details.

6 Examples

Let us illustrate the use and the power of MHOSPO, by means of several examples. Due to the lack of room, only in the first example we will give some details of the checking (see [BR01] for further details).

Example 3. We recall example 1.

$$map(F, []) \to [] \tag{1}$$
$$map(F, cons(x, xs)) \to cons(@(F, x), map(F, xs)) \tag{2}$$

$$ps([]) \to [] \tag{3}$$
$$ps(cons(x, xs)) \to cons(x, ps(map(\lambda y.x + y, xs))) \tag{4}$$

Termination of this TRS can be proved with $\succ_{mhospo}$ taking $\succeq_Q$ as $(\succeq_{\mathcal{P}}, \succeq_I)_{lex}$. The precedence $\succeq_{\mathcal{P}}$ is defined by $ps \succ_{\mathcal{P}} map \succ_{\mathcal{P}} cons$ and $ps \succ_{\mathcal{P}} +$; and we define $\succeq_I$ by combining $\succ_{horpo}$, with the precedence $ps \succ_{\mathcal{F}} cons_I$, as basic quasi-ordering and the interpretation function I defined by the pairs $(map(x_1, x_2), x_2)$ and $(cons(x_1, x_2), cons_I(x_2))$, where $cons_I : List \to List$. Note that with this interpretation for the *map* function only the List parameter is considered and the List is interpreted as the amount of *cons* it has, which represents its size.

In order to prove that all rules are included in the obtained MHOSPO, we need to check both that $l \succeq_I r$ and $l \succ_{hospo} r$ for all rules $l \to r$ in the system.

Fist of all recall that, since $List$ and Nat are type variables (considered as sorts) we have $List \equiv Nat$, which will be used many times along the checking.

We start showing that all rules are included in $\succeq_I$. Rules 1 and 2 have the same interpretation for both sides. For rule 3 we trivially have $I(ps([])) = ps([]) \succ_{horpo} [] = I([])$. For rule 4, $I(ps(cons(x, xs))) = ps(cons_I(xs)) \succ_{horpo} cons_I(ps(xs)) = I(cons(x, ps(map(\lambda y.x + y, xs))))$, using $ps \succ_{\mathcal{F}} cons_I$.

Rules 1 and 3 are included in HOSPO by case 1. For rule 2 we show that $map(F, cons(x, xs)) \succ_{hospo} cons(@(F, x), map(F, xs))$. We apply first case 2, since $map(F, cons(x, xs)) \succ_Q cons(@(F, x), map(F, xs))$ by the first component of $\succeq_Q$ as $map \succ_{\mathcal{P}} cons$. Then we need to check recursively that (1) $map(F, cons(x, xs)) \succ_{hospo} @(F, x)$ and that (2) $map(F, cons(x, xs)) \succ_{hospo} map(F, xs)$. For (1) we apply case 4 since $F \in \mathcal{C}(map(F, cons(x, xs)))$ and $map(F, cons(x, xs)) \succ_{hospo} x$ applying case 1 twice. For (2) we apply again case 2 since $map(F, cons(x, xs)) \succ_Q map(F, xs)$, by the second component of $\succeq_Q$ as $I(map(F, cons(x, xs))) = cons_I(xs) \succ_{horpo} xs = I(map(F, xs))$, and the recursive checking hold in the same way as before.

For the last rule we need $ps(cons(x, xs)) \succ_{hospo} cons(x, ps(map(\lambda y.x + y, xs)))$. We apply case 2 using the precedence component. For the recursive checking we only develop $ps(cons(x, xs)) \succ_{hospo} ps(map(\lambda y.x + y, xs))$. Then we apply again case 2 twice in the first case using the second component of $\succeq_Q$ and in the second case using the precedence. Finally we show (apart from the easy checking for xs) that $\lambda y.cons(x, xs) + y \in \mathcal{C}(ps(cons(x, xs)))$ using the closure with case 2 and then case 1 with the precedence, and conclude with $\lambda y.cons(x, xs)+y \succ_{hospo} \lambda y.x+y$, which holds easily (note that $\lambda y.cons(x, xs)+y$ is well-typed since $Nat \equiv List$).

The last part of the proof of the previous example may look hard to automate, since we have to pick a term from the computable closure of $ps(cons(x, xs))$ and then check that it is greater than or equal to $\lambda y.x + y$. In practice, we look for such a term in the computable closure in a goal-directed way: since $\lambda y.x + y$ is headed by lambda and then $+$ we apply case 2 and case 1, and then, when we reach x, we check whether some argument of $ps(cons(x, xs))$ (which belongs to the computable closure by definition) is greater than or equal to x. Therefore, since $cons(x, xs) \succ_{hospo} x$, by monotonicity we conclude, without any additional checking, that $\lambda y.cons(x, xs) + y \succ_{hospo} \lambda y.x + y$

Let us present another example which can be proved by MHOSPO and not by HORPO. We only provide the ingredients of MHOSPO which are needed to prove that the rules are included in the ordering.

Example 4. Quick sort. Let $V_{\mathbb{T}} = \{Bool, Nat, List\}$,
$\mathcal{X} = \{x, y : Nat; xs, ys : List; p : Nat \to Bool\}$ and
$\mathcal{F} = \{\ 0 : Nat,\ s : Nat \to Nat,\ le, gr : Nat \times Nat \to Bool,$
$True, False : Bool,\ if : Bool \times List \times List \to List,$
$[] : List,\ cons : Nat \times List \to List,\ ++ : List \times List \to List,$
$filter : (Nat \to Bool) \times List \to List,\ qsort : List \to List \quad \}$

$$
\begin{array}{ll}
if(True, xs, ys) \to xs & [] +\!\!+ xs \to xs \\
if(False, xs, ys) \to ys & cons(x, xs) +\!\!+ ys \to cons(x, xs +\!\!+ ys)
\end{array}
$$

$$
\begin{array}{ll}
le(0, y) \to True & gr(0, y) \to False \\
le(s(x), 0) \to False & gr(s(x), 0) \to True \\
le(s(x), s(y)) \to le(x, y) & gr(s(x), s(y)) \to gr(x, y)
\end{array}
$$

$$
\begin{array}{l}
filter(p, []) \to [] \\
filter(p, cons(x, xs)) \to if(@(p, x), cons(x, filter(p, xs)), filter(p, xs))
\end{array}
$$

$$
\begin{array}{l}
qsort([]) \to [] \\
qsort(cons(x, xs)) \to qsort(filter(\lambda z.le(z, x), xs)) +\!\!+ cons(x, []) +\!\!+ \\
\qquad\qquad\qquad\quad qsort(filter(\lambda z.gr(z, x), xs))
\end{array}
$$

Termination of this TRS can be proved with $\succ_{mhospo}$ taking $\succeq_Q$ as $(\succeq_{\mathcal{P}}, \succeq_I)_{lex}$. The precedence $\succeq_{\mathcal{P}}$ is defined by $0 \succ_{\mathcal{P}} True, False$; $+\!\!+ \succ_{\mathcal{P}} cons =_{\mathcal{P}} filter \succ_{\mathcal{P}} if$ and $qsort \succ_{\mathcal{P}} filter, +\!\!+, cons, le, gr, []$, and we define $\succeq_I$ by combining $\succ_{horpo}$ as basic ordering and the interpretation I defined by the pairs $(filter(x_1, x_2), filter_I(x_2))$ and $(if(x_1, x_2, x_3), if_I(x_2, x_3))$ where $filter_I : List \to List$ and $if_I : List \times List \to List$. As precedence for $\succ_{horpo}$ we take $0 \succ_{\mathcal{F}} True, False$; $qsort \succ_{\mathcal{F}} +\!\!+, cons, []$ and $+\!\!+ \succ_{\mathcal{F}} cons =_{\mathcal{F}} filter_I \succ_{\mathcal{F}} if_I$.

Let us finally mention that there are also some natural examples that come from the disjoint union of a first-order TRS, which can be proved terminating by MSPO (or by the dependency pair method), and a higher-order TRS, which can be proved terminating by HORPO. Although its components can be proved terminating separately, no method can ensure the termination of the whole system. Using MHOSPO, we can somehow combine the proofs (used in MSPO and HORPO) and show its termination.

7 Conclusions

In this paper we have presented a new ordering-based method for proving termination of higher-order rewriting. The method properly extends both MSPO (in the first-order case) and HORPO (in the higher-order case). The method can be automated and its power has been shown by means of several examples which could not be handled by the previous methods.

Finally let us mention some work already in progress and some future work we plan to do.

1. We are currently working on a constraint-based termination proof method, in the light of the dependency pair method [AG00], for the higher-order case. Using the ideas of [BFR00], the constraints are extracted from the definition

of MHOSPO. Here we show the result of applying this method to Example 1:

$$\mathcal{I}: map(F,[]) \succeq_I [] \wedge map(F,cons(x,xs)) \succeq_I cons(@(F,x),map(F,xs)) \wedge ps([]) \succeq_I [] \wedge ps(cons(x,xs)) \succeq_I cons(x,ps(map(\lambda y.x+y,xs)))$$

$$\mathcal{Q}: MAP(F,cons(x,xs)) \succ_I MAP(F,xs) \wedge PS(cons(x,xs)) \succ_I PS(map(\lambda y.x+y,xs))$$

where $\mathcal{I}$ is the constraint coming from $\succeq_I$ and $\mathcal{Q}$ the constraint coming from $\succeq_Q$, considering that $\succeq_Q$ is defined as the lexicographic combination of a precedence (obtained by a simple cycle analysis) and $\succeq_I$ after applying a renaming of head symbols.
Solving these constraints requires on the one hand, as in the first-order case, to automatically generate the adequate quasi-ordering $\succeq_I$ (note that we are using the same kind of interpretations as in the first-order case) and on the other hand to adapt the notion of dependency graph for the higher-order case.

2. We can adapt, in the same way as it can be done for HORPO [JR01], the method to be applicable to higher-order rewriting "à la Nipkow" [MN98], i.e. rewriting on terms in η-long β-normal form.
3. We will study other possible interpretations to build $\succeq_I$ using functionals in a similar way as in [Pol96], but with two relevant differences. First due to the fact that we are building a quasi-ordering we can use *weakly monotonic functionals* instead of *strict functionals*. Second, since we are not working on terms in η-long β-normal forms, we have to study whether we need to define $\succeq_I$ by first obtaining η-long β-normal form and then interpreting the normalized terms by using functionals. Note that if we normalize first, $\succeq_I$ will include trivially β-reduction.
4. We want to add to HOSPO more powerful cases to deal with terms headed by lambdas, which is, by now, the main weakness of the ordering, as well as some other improvements that have already been added to the initial version of HORPO [JR01].
5. We want to analyze the relationship between our method and a recent constraint-based method developed in [Pie01] for proving termination of higher-order logic programs.

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